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Higher Dimensional Numerical Range of Generalized (Jordan) Product of Operators*

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Abstract Let H be a complex Hilbert space. Denote by $\mathcal{B}(H)$ and $\mathcal{B}_s(H)$ the algebra of all bounded linear operators on H and the Jordan algebra of all self-adjoint operators in $\mathcal{B}(H)$, respectively. In this paper, the authors first give some elementary properties about higher dimensional numerical range of generalized (Jordan) product of operators in $\mathcal{B}(H)$. Based on these results, all surjective maps on $\mathcal{B}(H)$ (respectively, $\mathcal{B}_s(H)$) that preserve higher dimensional numerical range of generalized (Jordan) product of operators are completely characterized. Particularly, they give complete characterizations of all surjective maps preserving higher dimensional numerical range of operator products, Jordan semi-triple products and Jordan products on $\mathcal{B}(H)$ and $\mathcal{B}_s(H)$, respectively.

Keywords k-Dimensional numerical range, Generalized product, Generalized Jordan product
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1 Introduction

Let H be a complex Hilbert space and $\mathcal{B}(H)$ be the algebra of all bounded linear operators on H. For $A \in \mathcal{B}(H)$ and a positive integer k with $k < \dim H$, recall that the k-dimensional numerical range of A is defined by

$$W_k(A) = \Big\{\frac{1}{k}\sum_{i=1}^{k}\langle Ax_i, x_i\rangle : x_1, \cdots, x_k \in H \text{ are orthonormal unit vectors}\Big\}.$$

The k-dimensional numerical range is a natural generalization of the classical numerical range, which is defined by the set $W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$. Obviously, $W_1(A) = W(A)$. The k-dimensional numerical ranges are also useful tools in studying matrices and operators, and have important applications in many research fields, for example, the authors in [14] obtained

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some quantum uncertainty relations in quantum information theory by using numerical range of self-adjoint operators. The readers can see [7, 12–14, 26] for more applications.

Let $\mathcal{A}$ be a subset of $\mathcal{B}(H)$. We say that a map $\phi : \mathcal{A} \to \mathcal{A}$ preserves the k-numerical range of the operator product $\star$ if $W_k(\phi(A) \star \phi(B)) = W_k(A \star B)$ holds for all $A, B \in \mathcal{A}$. The problem of characterizing maps on operator algebras preserving the k-dimensional numerical range of different kinds of products, such as the usual product AB, Jordan semi-triple product ABA and Jordan product $AB + BA$, has been studied extensively (see [5, 11, 15, 23] and the references therein).

Note that there are many generalizations of various operator products. In [3, 20], generalized product and generalized Jordan product were proposed.

Definition 1.1 *Fix a positive integer l and a finite sequence $(i_1, i_2, \cdots, i_m)$ such that $\{i_1, i_2, \cdots, i_m\} = \{1, 2, \cdots, l\}$ and there is an i_p not equal to i_q for all other q. The generalized product and generalized Jordan product for operators $T_1, \cdots, T_l \in \mathcal{B}(H)$ are defined respectively by*

$$T_1 \circ \cdots \circ T_l = T_{i_1} \cdots T_{i_m} \quad \text{and} \quad T_1 * \cdots * T_l = T_{i_1} \cdots T_{i_m} + T_{i_m} \cdots T_{i_1}.$$

Evidently, the generalized products cover the usual product T_1T_2 and Jordan semi-triple product $T_1T_2T_1$, and the generalized Jordan products cover the usual Jordan product $T_1T_2 + T_2T_1$. About the study of generalized (Jordan) products, the readers may see [1, 3, 17, 27] and the references therein. Li and Sze in [20] discussed maps preserving the numerical range of the generalized products on $\mathcal{B}(H)$ and $\mathcal{B}_s(H)$ (the Jordan algebra of all self-adjoint operators in $\mathcal{B}(H)$), respectively, and gave their complete characterizations.

The main goal of this paper is to try to characterize maps preserving k-dimensional numerical range of the generalized (Jordan) product on $\mathcal{B}(H)$ and $\mathcal{B}_s(H)$, respectively. This paper is organized as follows. In Section 2, we give some useful properties of the k-numerical range and some characterizations of bounded linear operators by the k-numerical range of generalized product and generalized Jordan product. Section 3 is devoted to discussing maps on respectively $\mathcal{B}_s(H)$ and $\mathcal{B}(H)$ preserving the k-numerical range of generalized product (Theorems 3.1–3.4). In Section 4, all maps preserving the k-numerical range of generalized Jordan product on $\mathcal{B}_s(H)$ and $\mathcal{B}(H)$ are characterized, respectively (Theorems 4.1–4.2). Particularly, we give complete characterizations of all surjective maps preserving the k-numerical range of the usual operator products, Jordan semi-triple products and Jordan products on $\mathcal{B}(H)$ and $\mathcal{B}_s(H)$, respectively (Corollaries 3.1–3.4 and 4.1–4.2).

We close this section by fixing some notions and notations. Let H be a complex Hilbert space. Denote by $\mathcal{B}(H)$ and $\mathcal{B}_s(H)$ the algebra of all bounded linear operators on H and the Jordan algebra of all self-adjoint operators in $\mathcal{B}(H)$, respectively. Assume $A \in \mathcal{B}(H)$. $\ker A$, $\operatorname{ran} A$, $\operatorname{rank} A$ and $\operatorname{tr} A$ stand for the kernel, the range, the rank and the trace of A, respectively.

Denote by $\sigma(A)$, $\sigma_p(A)$, $\sigma_{ap}(A)$ and $\partial\sigma(A)$ the spectrum, the point spectrum, the approximate point spectrum of A and the boundary of $\sigma(A)$, respectively. For any nonzero $x, y \in H$, denote by $x \otimes y$ the rank one operator $z \mapsto \langle z, y\rangle x$, and by $[x, y]$ the subspace spanned by x, y. For a subset $M \subseteq H$, $M^\perp = \{x \in H : \langle x, y\rangle = 0 \text{ for all } y \in M\}$. We will use $\mathrm{cl}(X)$ and $\mathrm{conv}(X)$ to denote the closure and the convex hull of a set X, respectively.

2 Properties of the k-Dimensional Numerical Range

In this section, we give some properties about the k-dimensional numerical range of bounded linear operators.

The first proposition summarizes some basic properties of the k-dimensional numerical ranges of operators which will be needed in this paper.

Proposition 2.1 (cf. [4, 6, 9–10]) *Let H be a complex Hilbert space and k be a positive integer with $k < \dim H$. Assume that $A \in \mathcal{B}(H)$.*

(1) *If the operator $U \in \mathcal{B}(H)$ is unitary, then $W_k(A) = W_k(UAU^*)$; if the bounded operator U on H is conjugate unitary, then $W_k(UAU^*) = W_k(A^*) = W_k(A)^*$, where $W_k(A)^* = \{\overline{\lambda} : \lambda \in W_k(A)\}$.*

(2) $W_k(A) = \{\lambda\}$ *if and only if* $A = \lambda I$.

(3) $W_k(A) \subset \mathbb{R}$ (*the real field*) *if and only if A is self-adjoint.*

(4) *If $\dim H = n < \infty$ and A is self-adjoint with eigenvalues $\alpha_1 \ge \cdots \ge \alpha_n$, then* $W_k(A) = \left[\frac{\alpha_{n-k+1}+\cdots+\alpha_n}{k}, \frac{\alpha_1+\cdots+\alpha_k}{k}\right]$.

(5) *If $\dim H = n < \infty$, then* $(n-k)W_{n-k}(A) = \{\mathrm{tr}\, A\} - kW_k(A)$.

(6) *If $A = x \otimes y$ with some $x, y \in H$, then $W_k(x \otimes y)$ is an ellipse with the foci $\left\{0, \frac{\langle x,y\rangle}{k}\right\}$ and the minor axis $\frac{\sqrt{\|x\otimes y\|^2 - |\langle x,y\rangle|^2}}{k}$. In particular, if $x \otimes y$ is normal, then $W_k(x \otimes y)$ is a line segment with endpoints $\left\{0, \frac{\langle x,y\rangle}{k}\right\}$; if $\langle x, y\rangle = 0$, then $W_k(x \otimes y)$ is a circle with center at the origin and diameter $\frac{1}{k}\|x\|\|y\|$.*

(7) *For any $x \in H$, the center of the rectangular box from the vertical and horizontal support lines of $W_k(Ax \otimes x + x \otimes xA)$ is $\frac{1}{k}\langle Ax, x\rangle$.*

For the k-dimensional numerical range of finite rank operators in infinite dimensional Hilbert spaces, we have the following result.

Proposition 2.2 (cf. [22, Proposition 2]) *Let H be an infinite dimensional complex Hilbert space and k be a positive integer. Assume that $A \in \mathcal{B}(H)$ is a finite rank operator. Then there exists a subspace $H_1 \subset H$ with $k + \dim(\ker A \cap \ker A^*)^\perp \le \dim H_1 < \infty$ such that $A = A_1 \oplus 0$ with respect to the space decomposition $H = H_1 \oplus H_1^\perp$ and $W_k(A) = W_k(A_1)$.*

By Proposition 2.1(5), Proposition 2.2 and [19], the following proposition is obvious.

Proposition 2.3 *Let H be a complex Hilbert space with $\dim H = n \geq 2$ and k be a positive integer with $k < n$. Assume that H has a space decomposition $H = H_1 \oplus H_2$ and $A \in \mathcal{B}(H_1)$.*

(1) *If $\dim H_1 = 1$ and $B \in \mathcal{B}(H_2)$ with $\operatorname{rank} B < \infty$, then*

$$kW_k(A \oplus B) = \operatorname{conv}\{kW_k(B), W(A) + (k-1)W_{k-1}(B)\}.$$

(2) *If $\dim H_1 = 2$, then*

$$kW_k(A \oplus 0) = \begin{cases} W(A), & \text{if } k = 1, n = 2; \\ \operatorname{conv}\{\{0\}, W(A)\}, & \text{if } k = 1, n > 2; \\ \operatorname{conv}\{\{0\}, W(A), \{\operatorname{tr} A\}\}, & \text{if } 2 \leq k \leq n-2; \\ \operatorname{conv}\{W(A), \{\operatorname{tr} A\}\}, & \text{if } 2 \leq k = n-1. \end{cases}$$

(3) *If $\dim H_1 = 3$, then*

$$kW_k(A \oplus 0) = \begin{cases} W(A), & \text{if } k = 1, n = 3; \\ \operatorname{conv}\{\{0\}, W(A)\}, & \text{if } k = 1, n \geq 4; \\ \{\operatorname{tr} A\} - W(A), & \text{if } k = 2, n = 3; \\ \operatorname{conv}\{W(A), \{\operatorname{tr} A\} - W(A)\}, & \text{if } k = 2, n = 4; \\ \operatorname{conv}\{\{0\}, W(A), \{\operatorname{tr} A\} - W(A)\}, & \text{if } k = 2, n \geq 5; \\ \operatorname{conv}\{\{0\}, W(A), \{\operatorname{tr} A\} - W(A), \{\operatorname{tr} A\}\}, & \text{if } 3 \leq k \leq n-3; \\ \operatorname{conv}\{W(A), \{\operatorname{tr} A\} - W(A), \{\operatorname{tr} A\}\}, & \text{if } 3 \leq k = n-2; \\ \operatorname{conv}\{\{\operatorname{tr} A\} - W(A), \{\operatorname{tr} A\}\}, & \text{if } 3 \leq k = n-1. \end{cases}$$

The following proposition gives a characterization of rank one projections by the k-dimensional numerical range.

Proposition 2.4 *Let H be a complex Hilbert space, k be a positive integer with $2 \leq k < \dim H$, and let m be an even number. For $A \in \mathcal{B}(H)$, A is a rank one projection if and only if $W_k(A) = W_k(A^m) = \left[0, \frac{1}{k}\right]$.*

Proof By Proposition 2.1(4) and Proposition 2.2, the "only if" part is obvious.

For the "if" part, by Proposition 2.1(3), we see that $A \in \mathcal{B}_s(H)$. Since $0 \in W_k(A^m)$ and m is an even number, there exist orthonormal unit vectors $e_1, \cdots, e_k \in H$ such that

$$0 = \sum_{j=1}^{k} \langle A^m e_j, e_j \rangle = \sum_{j=1}^{k} \langle A^{\frac{m}{2}} e_j, A^{\frac{m}{2}} e_j \rangle = \sum_{j=1}^{k} \|A^{\frac{m}{2}} e_j\|^2,$$

which gives $[e_1, \cdots, e_k] \subseteq \ker A^{\frac{m}{2}} = \ker A$.

Next, for any unit vector $x \in H$, take orthonormal unit vectors $x_2, \cdots, x_k \in [e_1, \cdots, e_k] \cap [x]^{\perp}$. By the definition of $W_k(A)$, one gets

$$\langle Ax, x \rangle = \langle Ax, x \rangle + \sum_{j=2}^{k} \langle Ax_j, x_j \rangle \in kW_k(A) = [0, 1],$$

which implies

$$0 \le A \le I. \tag{2.1}$$

We claim that

$$\text{there exists a unit vector } x_0 \in H \text{ such that } \langle Ax_0, x_0\rangle = 1. \tag{2.2}$$

Otherwise, $\langle Ax, x\rangle < 1$ for all unit vectors $x \in H$. Note that (2.1) holds. For any unit vector $x \in H$, if $A^{\frac{m-1}{2}}x \neq 0$, then

$$\begin{aligned}\langle A^m x, x\rangle &= \langle AA^{\frac{m-1}{2}}x, A^{\frac{m-1}{2}}x\rangle < \|A^{\frac{m-1}{2}}x\|^2 = \langle A^{m-1}x, x\rangle \\ &= \langle AA^{\frac{m-2}{2}}x, A^{\frac{m-2}{2}}x\rangle < \|A^{\frac{m-2}{2}}x\|^2 = \langle A^{m-2}x, x\rangle = \cdots < \langle Ax, x\rangle;\end{aligned}$$

if $A^{\frac{m-1}{2}}x = 0$, then

$$\langle A^m x, x\rangle = \langle A^{\frac{m+1}{2}}A^{\frac{m-1}{2}}x, x\rangle = 0 = \langle Ax, x\rangle.$$

So, for any orthonormal vectors $x_1, \cdots, x_k \in H$, we have $\sum\limits_{j=1}^{k}\langle A^m x_j, x_j\rangle < \sum\limits_{j=1}^{k}\langle Ax_j, x_j\rangle \le 1$, a contradiction to $\frac{1}{k} \in W_k(A^m)$.

Now, (2.1)–(2.2) imply $Ax_0 = x_0$. Thus, with respect to the space decomposition $H = [x_0] \oplus [x_0]^\perp$, one has $A = 1 \oplus A_1$ with $A_1 \in \mathcal{B}_s([x_0]^\perp)$. For any unit vector $y \in [x_0]^\perp$, take orthonormal vectors $y_3, \cdots, y_k \in [x_0, y]^\perp$. Now, by the assumption $W_k(A) = \left[0, \frac{1}{k}\right]$ and (2.1), we infer that

$$1 \ge \langle Ax_0, x_0\rangle + \langle A(0 \oplus y), (0 \oplus y)\rangle + \sum_{j=3}^{k}\langle A(0 \oplus y_j), (0 \oplus y_j)\rangle \ge 1 + \langle A_1 y, y\rangle \ge 1,$$

which implies $\langle A_1 y, y\rangle = 0$ for all $y \in [x_0]^\perp$. It follows that $A_1 = 0$, and so $A = x_0 \otimes x_0$, as desired.

Proposition 2.5 (cf. [22, Proposition 5]) *Let H be a complex Hilbert space and k be a positive integer with $k < \dim H$. If $A, B \in \mathcal{B}(H)$ are rank one operators and $W_k(A) = W_k(B)$, then* $\operatorname{tr} A = \operatorname{tr} B$.

For any $T_1, \cdots, T_l \in \mathcal{B}(H)$, write $T_{i_p} = B$ and $T_{i_q} = A$ for all other q in Definition 1.1. Then $T_1 \circ \cdots \circ T_l = A^r BA^s$ and $T_1 * \cdots * T_l = A^r BA^s + A^s BA^r$ for some $r, s \in \mathbb{R}$, which are the special cases of generalized product and generalized Jordan product.

The following several propositions give characterizations about the k-dimensional numerical range of the products $A^r BA^s$ and $A^r BA^s + A^s BA^r$.

Proposition 2.6 *Let H be a complex Hilbert space with $\dim H \ge 2$, k be a positive integer with $k < \dim H$, and $r, s \ge 0$ be integers with $(r, s) \neq (0, 0)$. Assume that $B \in \mathcal{B}(H)$. Then the following statements are equivalent*:

(1) *B is a rank one operator.*

(2) *For any $A \in \mathcal{B}(H)$, $W_k(A^rBA^s)$ is an ellipse with 0 as a focus.*

Proof (1)⇒(2) If B is a rank one operator, then $\operatorname{rank}(A^rBA^s) \leq 1$ for all $A \in \mathcal{B}(H)$. By Proposition 2.1(6), $W_k(A^rBA^s)$ is an ellipse which has 0 as a focus.

(2)⇒(1) Suppose, on the contrary, that $\operatorname{rank}(B) \geq 2$. Then there exist $x, y \in H$ such that Bx, By are orthonormal unit vectors. The proof is similar to that of [20, Lemma 2.2] only by some slight modifications. For the sake of completeness, we give a sketch of the proof.

Case 1 $r = 0$.

Write $C = x \otimes Bx - y \otimes By$. C has an operator matrix representation of the form $C_1 \oplus 0$, where $C_1 \in \mathcal{M}_l(\mathbb{C})$ with $2 \leq l \leq 4$. Let $D = D_1 \oplus 0$ with $D_1 = \operatorname{diag}(1, \cdots, l)$. Then $C_1 + \nu D_1$ has distinct eigenvalues except for finite many $\nu \in \mathbb{R}$, that is, there exists an invertible matrix $P \in \mathcal{M}_l(\mathbb{C})$ such that $P^{-1}(C_1 + \nu D_1)P = \operatorname{diag}\{b_1, \cdots, b_l\}$ is a diagonal matrix. Take $A_\nu = P\operatorname{diag}\{\sqrt[s]{b_1}, \cdots, \sqrt[s]{b_l}\}P^{-1} \oplus 0$, and so $A_\nu^s = (C_1 + \nu D_1) \oplus 0 = C + \nu D$. Thus, except for finitely many $\nu \in \mathbb{R}$, we have $W_k(BA_\nu^s) = W_k(BC + \nu BD)$. Since $\operatorname{rank}(C), \operatorname{rank}(D) < \infty$, we infer that $W_k(BC + \nu BD)$ is a closed set. Also note that the map $\nu \mapsto W_k(BC + \nu BD)$ is continuous. As $W_k(BA_\nu^s)$ is an ellipse with 0 as a focus for any sufficiently small $\nu \in \mathbb{R}$, we get a contradiction to $W_k(BC) = W_k(Bx \otimes Bx - By \otimes By) = \big[-\frac{1}{k}, \frac{1}{k}\big]$.

Case 2 $s = 0$.

Let B^t be the transpose of B with respect to an arbitrarily given orthonormal basis of H. Note that $W_k(A^rB) = W_k((A^rB)^t) = W_k(B^t(A^t)^r)$. So, by the same argument as that of Case 1, one can show $\operatorname{rank}(B) = 1$.

Case 3 $rs > 0$.

Let $H_1 = [x, y, Bx, By]$ and $B_1 = B|_{H_1} \in \mathcal{B}(H_1)$. It is easily checked that there exists a 2-dimensional subspace H_{11} of H_1 such that $B_{11} = B_1|_{H_{11}}$ is invertible. Take an orthonormal basis $\{u, v\}$ of H_{11} such that $B_{11} = au \otimes u + cv \otimes v + bu \otimes v$, where $a, b, c \in \mathbb{C}$ and $ac \neq 0$. Let $A = A_{11} \oplus 0 = (\alpha u \otimes u + \beta v \otimes v) \oplus 0$ with $\alpha, \beta \in \mathbb{C}$, $\alpha^{r+s}a = 1$ and $\beta^{r+s}c = -1$. Then $A^rBA^s = E \oplus 0$ with $E = u \otimes u - v \otimes v + \alpha^r\beta^s bu \otimes v$ and $W(E)$ is an elliptical disk with foci $\{-1, 1\}$. Note that $0, \operatorname{tr} E \in W(E)$. By Proposition 2.3(2), one gets $W_k(A^rBA^s) = W_k(E \oplus 0) = \frac{1}{k}W(E)$, a contradiction.

Hence B is of rank one. The proof is finished.

Proposition 2.7 *Let H be a complex Hilbert space with $\dim H \geq 2$, and $r, s \geq 0$ be integers with $(r, s) \neq (0, 0)$. Assume that $A \in \mathcal{B}(H)$ and $W(A) = [0, 1]$. Then the following statements are equivalent*:

(1) *$A = x \otimes x$ for some unit vector $x \in H$.*

(2) *For any $B \in \mathcal{B}_s(H)$, if $2W(B) = W(A^rBA^s + A^sBA^r) = [0, 2]$, then*

$$\{C \in \mathcal{B}_s(H) : B^rCB^s + B^sCB^r = 0\} \subseteq \{C \in \mathcal{B}_s(H) : A^rCA^s + A^sCA^r = 0\}.$$

Proof Assume that $A \in \mathcal{B}(H)$ and $W(A) = [0,1]$.

(1)⇒(2) If $A = x \otimes x$ for some unit vector $x \in H$, then, with respect to the space decomposition $H = [x] \oplus [x]^\perp$, we have $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. For any $B \in \mathcal{B}_s(H)$, write $B = \left(\begin{smallmatrix} b_1 & b_2 \\ \overline{b}_2 & B_3 \end{smallmatrix}\right) \in \mathcal{B}_s(H)$.

If $rs > 0$ and $2W(B) = W(A^rBA^s + A^sBA^r) = W\left(\left(\begin{smallmatrix} 2b_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right) = [0,2]$, then $b_1 = 1$ and $b_2 = 0$. For any $C = \left(\begin{smallmatrix} c_1 & c_2 \\ \overline{c}_2 & C_3 \end{smallmatrix}\right) \in \mathcal{B}_s(H)$, one has

$$B^rCB^s + B^sCB^r = \begin{pmatrix} 2c_1 & c_2(B_3^s + B_3^r) \\ (B_3^r + B_3^s)\overline{c}_2 & B_3^rC_3B_3^s + B_3^sC_3B_3^r \end{pmatrix}.$$

If $B^rCB^s + B^sCB^r = 0$, then $c_1 = 0$, and so $A^rCA^s + A^sCA^r = \left(\begin{smallmatrix} 2c_1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = 0$.

If $rs = 0$, without loss of generality, we assume $s = 0$. In this case, if

$$2W(B) = W(A^rB + BA^r) = W\left(\begin{pmatrix} 2b_1 & b_2 \\ \overline{b}_2 & 0 \end{pmatrix}\right) = [0,2],$$

noting that $\{b_1 \pm \sqrt{b_1^2 + |b_2|^2}\} \subseteq \sigma(A^rB + BA^r)$, then $b_2 = 0$ and $b_1 = 1$. Now, for any $C = \left(\begin{smallmatrix} c_1 & c_2 \\ \overline{c}_2 & C_3 \end{smallmatrix}\right) \in \mathcal{B}_s(H)$, if $B^rC + CB^r = \left(\begin{smallmatrix} 2c_1 & c_2 + c_2B_3^r \\ B_3^r\overline{c}_2 + \overline{c}_2 & B_3^rC_3 + C_3B_3^r \end{smallmatrix}\right) = 0$ and $B \geq 0$, then $c_1 = c_2 = 0$. So $A^rC + CA^r = \left(\begin{smallmatrix} 2c_1 & c_2 \\ \overline{c}_2 & 0 \end{smallmatrix}\right) = 0$.

(2)⇒(1) Since $W(A) = [0,1]$, we have $A \in \mathcal{B}_s(H)$ by Proposition 2.1(3) and $1 \in \sigma_p(A)$ by [26, Corollary II.2.4]. Let $x \in H$ be the eigenvector for 1. So, under the space decomposition $H = [x] \oplus [x]^\perp$, we have $A = 1 \oplus A_2$. Taking $B = 1 \oplus 0$, and then $A^rBA^s + A^sBA^r = 2 \oplus 0$. This means $W(A^rBA^s + A^sBA^r) = 2W(B) = [0,2]$. For $C = 0 \oplus I_2$ with $I_2 \in \mathcal{B}([x]^\perp)$ the identity operator, it is obvious that $B^rCB^s + B^sCB^r = 0$. By the assumption, $A^rCA^s + A^sCA^r = 0 \oplus A_2^{r+s} = 0$. Note that $A_2 \in \mathcal{B}_s([x]^\perp)$. It follows that $A_2 = 0$, as desired.

Proposition 2.8 *Let H be a complex Hilbert space with $\dim H \geq 2$, and k, r, s be positive integers with $k < \dim H$. Assume that $A \in \mathcal{B}(H)$ and $W_k(A^{r+s+1}) = \left[0, \frac{1}{k}\right]$. Then the following statements are equivalent*:

(1) *There exists a space decomposition $H = [x] \oplus [x]^\perp$ such that $A = \lambda \oplus A_1$, where $x \in H$ is a unit vector, $\lambda \in \mathbb{C}$ with $\lambda^{r+s+1} = 1$ and $A_1^r = A_1^s = 0$.*

(2) *For any $B \in \mathcal{B}(H)$, $W_k(A^rBA^s + A^sBA^r)$ is a closed line segment with 0 as an endpoint.*

Proof Assume that $A \in \mathcal{B}(H)$ and $W_k(A^{r+s+1}) = \left[0, \frac{1}{k}\right]$.

(1)⇒(2) If there exists some unit vector $x \in H$ such that $A = \lambda \oplus A_1$ with respect to the space decomposition $H = [x] \oplus [x]^\perp$, where $\lambda \in \mathbb{C}$ with $\lambda^{r+s+1} = 1$ and $A_1^r = A_1^s = 0$, then for any $B \in \mathcal{B}(H)$, we have

$$W_k(A^rBA^s + A^sBA^r) = W_k(2\lambda^{r+s}\langle Bx, x\rangle x \otimes x) = 2\lambda^{r+s}\langle Bx, x\rangle \left[0, \frac{1}{k}\right].$$

So (2) holds.

(2)⇒(1) We will prove the statement (1) by considering two cases.

Case 1 $\partial\sigma(A)$ has at most one non-zero element.

If $\partial\sigma(A)$ has no non-zero elements, then $\sigma(A)=\{0\}$. This means $\sigma(A^{r+s+1})=\sigma(A)^{r+s+1}=\{0\}$. In addition, as $W_k(A^{r+s+1})=\big[0,\frac{1}{k}\big]$, one has that A^{r+s+1} is a normal operator, and so $W(A^{r+s+1})=\operatorname{conv}\sigma(A^{r+s+1})=\{0\}$. Hence $A^{r+s+1}\in\mathbb{C}I$, a contradiction.

Thus, $\partial\sigma(A)$ has one non-zero element λ. Then $\sigma(A)=\{\lambda\}$ or $\{0,\lambda\}$. If $\sigma(A)=\{\lambda\}$, a similar argument as above yields a contradiction. If $\sigma(A)=\{0,\lambda\}$, then $\sigma(A^{r+s+1})=\sigma(A)^{r+s+1}=\{0,\lambda^{r+s+1}\}$. Since A^{r+s+1} is a normal operator, by [8, Proposition XI.1.4], $\{0,\lambda^{r+s+1}\}\subseteq\sigma_p(A^{r+s+1})$. This and [8, Exercises XI.1.3] give $\{0,\lambda^{r+s+1}\}\subseteq\sigma_p(A)^{r+s+1}$. By keeping in mind that $\sigma(A)=\{0,\lambda\}$, we obtain $\sigma(A)=\sigma_p(A)=\{0,\lambda\}$, and so $\sigma(A^*)=\sigma_p(A^*)=\{0,\overline{\lambda}\}$. Take unit vectors $x,y\in H$ such that

$$Ax=\lambda x \quad\text{and}\quad A^*y=\overline{\lambda}y.$$

Letting $B=x\otimes y$ and by the assumption, one has $W_k(A^rBA^s+A^sBA^r)=W_k(2\lambda^{r+s}x\otimes y)$ is a closed line segment. Then Proposition 2.1(6) implies $y=\mathrm{e}^{\mathrm{i}\theta}x$ for some $\theta\in[0,2\pi]$, and so

$$A^*x=\overline{\lambda}x.$$

If $\dim\ker(A-\lambda I)\geq 2$, there are two orthonormal unit vectors $e_1,e_2\in\ker(A-\lambda I)$ such that

$$Ae_1=\lambda e_1,\quad A^*e_1=\overline{\lambda}e_1,\quad Ae_2=\lambda e_2,\quad A^*e_2=\overline{\lambda}e_2.$$

By taking $B=e_1\otimes e_2$, one obtains $W_k(A^rBA^s+A^sBA^r)=W_k(2\lambda^{r+s}e_1\otimes e_2)$. By Proposition 2.1(6) again, $W_k(2\lambda^{r+s}e_1\otimes e_2)$ is a circle, which contradicts the assumption.

If $\dim\ker(A-\lambda I)=1$, then with respect to the space decomposition $H=[x]\oplus[x]^{\perp}$, we have $A=\lambda\oplus A_1$ with $A_1\in\mathcal{B}([x]^{\perp})$ and $\sigma(A_1)=\{0\}$. For any $B=b_1\oplus B_1\in\mathcal{B}(H)$ with $b_1\in\mathbb{C}$ and $\operatorname{rank}B_1=1$, we have

$$A^rBA^s+A^sBA^r=2b_1\lambda^{r+s}\oplus C_1\quad\text{with } C_1=A_1^rB_1A_1^s+A_1^sB_1A_1^r.$$

By Proposition 2.3(1), we infer that

$$kW_k(A^rBA^s+A^sBA^r)=\operatorname{conv}\{kW_k(C_1),2b_1\lambda^{r+s}+(k-1)W_{k-1}(C_1)\}$$

is a closed line segment for any $b_1\in\mathbb{C}$. If $C_1\neq 0$, by taking $b_1=0$, we obtain that $W_k(C_1)$ and $W_{k-1}(C_1)$ are collinear segments, and say that the argument of $W_k(C_1)$ is θ. Again, by taking $b_1=\frac{\mathrm{e}^{\mathrm{i}(\theta+\frac{\pi}{2})}}{2\lambda^{r+s}}$, this yields a contradiction to the fact that $W_k(A^rBA^s+A^sBA^r)$ is a line segment. This implies

$$C_1=A_1^rB_1A_1^s+A_1^sB_1A_1^r=0\quad\text{for any } B_1\in\mathcal{B}([x]^{\perp})\text{ with }\operatorname{rank}B_1=1.\tag{2.3}$$

Next, without loss of generality, we always assume $r \geq s$ in the sequel. We first claim

$$A_1^r = 0.$$

If not, there exists some $y_0 \in [x]^\perp$ such that $A_1^{*r} y_0 \neq 0$, and then $A_1^{*s} y_0 \neq 0$. For any $y \in [x]^\perp$, by taking $B_1 = y \otimes y_0$ in (2.3), one gets that $A_1^r y$ and $A_1^s y$ are linearly dependent. So, either there exists some $\lambda \in \mathbb{C}$ such that

$$A_1^r = \lambda A_1^s, \tag{2.4}$$

or

$$\begin{aligned} &A_1^r = x_0 \otimes y_1 \quad \text{and} \quad A_1^s = x_0 \otimes y_2 \\ &\text{with } x_0 \in [x]^\perp \text{ and linearly independent } y_1, y_2 \in [x]^\perp. \end{aligned} \tag{2.5}$$

If (2.4) holds, (2.3) becomes $A_1^s B_1 A_1^s = 0$ for all rank one operators B_1. It follows that $A_1^s = A_1^r = 0$, a contradiction. If (2.5) holds, by taking $B_1 = y_1 \otimes x_0$ in (2.3), one has $x_0 \otimes (\langle B_1^* y_1, x_0 \rangle y_2 + \langle B_1^* y_2, x_0 \rangle y_1) = 0$, which yields a contradiction again. Hence $A_1^r = 0$. Now we will prove

$$A_1^s = 0.$$

Otherwise, there exist vectors $z_1, z_2 \in [x]^\perp$ such that $\|A_1^{*s} z_1\| = 2\|A_1^s z_2\| \neq 0$. Taking $B = \left(\begin{smallmatrix} b_1 & x\otimes z_1 \\ z_2 \otimes x & 0 \end{smallmatrix}\right)$ and by the assumption, we have that

$$W_k(A^r B A^s + A^s B A^r) = W_k\left(\begin{pmatrix} 2\lambda^{r+s} b_1 & \lambda^r x \otimes A_1^{*s} z_1 \\ \lambda^r A_1^s z_2 \otimes x & 0 \end{pmatrix} \right)$$

is a line segment which has 0 as an endpoint. So, there exists some $\delta \in [0, 2\pi)$ such that $\mathrm{e}^{\mathrm{i}\delta}(A^r B A^s + A^s B A^r) \in \mathcal{B}_s(H)$, and thus $\mathrm{e}^{\mathrm{i}\delta} \lambda^r x \otimes A_1^{*s} z_1 = (\mathrm{e}^{\mathrm{i}\delta} \lambda^r A_1^s z_2 \otimes x)^* = \mathrm{e}^{-\mathrm{i}\delta} \overline{\lambda}^r x \otimes A_1^s z_2$. This entails that $\|A_1^{*s} z_1\| = \|A_1^s z_2\|$, a contradiction.

Hence, we have proved that, with respect to the space decomposition $H = [x] \oplus [x]^\perp$, $A = \lambda \oplus A_1$ with $A_1^r = A_1^s = 0$. Also note that $W_k(A^{r+s+1}) = \left[0, \frac{1}{k}\right]$. It follows that $\lambda^{r+s+1} = 1$.

Case 2 $\partial\sigma(A)$ has at least two non-zero elements.

In this case, take $\lambda_1, \lambda_2 \in \partial\sigma(A) \backslash \{0\}$ with $\lambda_1 \neq \lambda_2$. Obviously, $\lambda_1, \lambda_2 \in \sigma_{ap}(A)$ and $\overline{\lambda}_1, \overline{\lambda}_2 \in \sigma_{ap}(A^*)$. For λ_1, there exist $\{x_n \in H : \|x_n\| = 1\}$ and $\{y_n \in H : \|y_n\| = 1\}$ such that

$$\|A x_n - \lambda_1 x_n\| \to 0 \quad \text{and} \quad \|A^* y_n - \overline{\lambda}_1 y_n\| \to 0.$$

For any n, write $B_n = x_n \otimes y_n$. Then

$$A^r B_n A^s + A^s B_n A^r = 2\lambda_1^{r+s} x_n \otimes y_n + C_n$$

and $W_k(A^rB_nA^s + A^sB_nA^r)$ is a line segment which has 0 as an endpoint by the assumption. A simple calculation yields $\|C_n\| \to 0$, and so $W_k(2\lambda_1^{r+s}x_n \otimes y_n)$ converges to a line segment which has 0 as an endpoint. This means

$$\lim_n \langle x_n, y_n\rangle = 1. \tag{2.6}$$

For any n, let $y_n = \alpha_n x_n + \beta_n z_n$, where $z_n \in H$ is a unit vector with $x_n \perp z_n$. It is clear that $\lim_n |\alpha_n| = 1$ and $\lim_n |\beta_n| = 0$. Note that

$$\begin{aligned}\|\alpha_n A^* x_n - \alpha_n\overline{\lambda}_1 x_n\| &= \|\alpha_n A^* x_n + \beta_n A^* z_n - \beta_n A^* z_n - (\overline{\lambda}_1 y_n - \overline{\lambda}_1\beta_n z_n)\| \\ &= \|A^* y_n - \beta_n A^* z_n - \overline{\lambda}_1 y_n + \overline{\lambda}_1\beta_n z_n\| \\ &\le \|A^* y_n - \overline{\lambda}_1 y_n\| + |\beta_n|\|A^*\| + |\overline{\lambda}_1\beta_n|.\end{aligned}$$

This implies

$$\|A^* x_n - \overline{\lambda}_1 x_n\| \le \frac{1}{|\alpha_n|}(\|A^* y_n - \overline{\lambda}_1 y_n\| + |\beta_n|\|A^*\| + |\overline{\lambda}_1\beta_n|) \to 0$$

whenever $n \to \infty$. So

$$\|A^* x_n - \overline{\lambda}_1 x_n\| \to 0, \quad n \to \infty.$$

Similarly, for λ_2, one can show

$$\lim_n \|Au_n - \lambda_2 u_n\| = 0 \Rightarrow \lim_n \|A^* u_n - \overline{\lambda}_2 u_n\| = 0 \quad \text{for some unit vectors } \{u_n\} \subset H.$$

Thus, we obtain

$$\lambda_1 \lim_n \langle x_n, u_n\rangle = \lim_n \langle Ax_n, u_n\rangle = \lim_n \langle x_n, A^* u_n\rangle = \lim_n \langle x_n, \overline{\lambda}_2 u_n\rangle = \lambda_2 \lim_n \langle x_n, u_n\rangle.$$

It follows that

$$\lim_n \langle x_n, u_n\rangle = 0.$$

Finally, take $B_n = x_n \otimes u_n$, and then

$$A^rB_nA^s + A^sB_nA^r = (\lambda_1^r\lambda_2^s + \lambda_1^s\lambda_2^r)x_n \otimes u_n + D_n$$

with $\|D_n\| \to 0$. By the same discussion as in (2.6), one sees that $\lim_n \langle x_n, u_n\rangle = 1$, a contradiction.

Combining Case 1 and Case 2, we complete the proof.

Proposition 2.9 *Let H be a complex Hilbert space with $\dim H \ge 3$ and k, r be positive integers with $k < \dim H$. Assume that $A \in \mathcal{B}(H)$ and $W_k(A^{r+1}) = \left[0, \frac{1}{k}\right]$. Then the following statements are equivalent*:

(1) *$A = \lambda x \otimes x$ for some unit vector $x \in H$ and $\lambda \in \mathbb{C}$ with $\lambda^{r+1} = 1$.*

(2) *For any $B \in \mathcal{B}(H)$, $W_k(B^rA + AB^r)$ is a closed ellipse.*

Proof We first prove that, for any 3×3 matrix

$$C=\begin{pmatrix}2c_1 & c_2 & c_3\\ c_4 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

with any $c_1,c_2,c_3,c_4\in\mathbb{C}$, $W_k(C\oplus 0)$ is a closed ellipse. In fact, it is easy to calculate that

$$\begin{aligned}&\sigma(C)=\Big\{c_1+\sqrt{c_1^2+c_2c_4},c_1-\sqrt{c_1^2+c_2c_4},0\Big\}=\{\lambda_1,\lambda_2,\lambda_3\},\\ &s=\operatorname{tr}(C^*C)-\sum_{i=1}^{3}|\lambda_i|^2=2|c_1|^2+|c_2|^2+|c_3|^2+|c_4|^2-2|c_1^2+c_2c_4|>0,\\ &\operatorname{tr}(C)+\frac{1}{s}\Big(\sum_{i=1}^{3}|\lambda_i|^2\lambda_i-\operatorname{tr}(C^*C^2)\Big)=0=\lambda_3\end{aligned}$$

and

$$(|\lambda_1|+|\lambda_2|)^2-|\lambda_1-\lambda_2|^2=2|c_1|^2-2|c_1^2+c_2c_4|+2|c_2c_4|\le s.$$

By [18, Theorem 2.4], $W(C)$ is an ellipse with foci $\big\{c_1+\sqrt{c_1^2+c_2c_4},c_1-\sqrt{c_1^2+c_2c_4}\big\}$. In addition, the center of the ellipse is $c_1=\frac{1}{2}\operatorname{tr}C$. Also note that $0,\operatorname{tr}(C)\in W(C)$. So

$$W(C)=\{\operatorname{tr}C\}-W(C).$$

Now, by Proposition 2.3(3), one obtains that $kW_k(C\oplus 0)=W(C)$ is a closed ellipse.

(1)⇒(2) Assume that $A=\lambda x\otimes x$ for some unit vector $x\in H$ and $\lambda\in\mathbb{C}$ with $\lambda^{r+1}=1$. For any $B\in\mathcal{B}(H)$, there exist unit vectors $y,z\in H$ with $\{x,y,z\}$ orthonormal such that $B^rx=\alpha x+\beta y$ and $B^{*r}x=\overline{\alpha}x+\gamma y+\mu z$, where $\alpha,\beta,\gamma,\mu\in\mathbb{C}$. Then, with respect to the space decomposition $H=[x,y,z]\oplus[x,y,z]^\perp$, we have

$$B^rA+AB^r=\lambda\begin{pmatrix}2\alpha & \overline{\gamma} & \overline{\mu}\\ \beta & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\oplus 0.$$

So $W_k(B^rA+AB^r)$ is a closed ellipse.

(2)⇒(1) Assume that $A\in\mathcal{B}(H)$ with $W_k(A^{r+1})=\big[0,\frac{1}{k}\big]$, and for any $B\in\mathcal{B}(H)$, $W_k(B^rA+AB^r)$ is a closed ellipse. Since $W_k(A^{r+1})=\big[0,\frac{1}{k}\big]$ is closed, we see that $A^{r+1}\in\mathcal{B}_s(H)$ and $W(A^{r+1})$ is closed by [2, Theorem 3.1] if $\dim H=\infty$. Write $W(A^{r+1})=[\lambda_1,\lambda_2]$ with $\lambda_1<\lambda_2\in\mathbb{R}$. By [26, Corollary II.2.4], $\lambda_1,\lambda_2\in\sigma_p(A^{r+1})$. As $\sigma_p(A^{r+1})=\sigma_p(A)^{r+1}$, there exist $\mu_1\ne\mu_2\in\sigma_p(A)$ such that $\mu_1^{r+1}=\lambda_1$ and $\mu_2^{r+1}=\lambda_2$. We also may assume $\mu_1\ne 0$. Taking unit vectors $e_1,e_2\in H$ such that $Ae_1=\mu_1e_1$ and $Ae_2=\mu_2e_2$. Then $A^{r+1}e_1=\lambda_1e_1$ and $A^{r+1}e_2=\lambda_2e_2$. Thus, we have

$$\lambda_1\langle e_1,e_2\rangle=\langle A^{r+1}e_1,e_2\rangle=\langle e_1,\lambda_2e_2\rangle=\lambda_2\langle e_1,e_2\rangle,$$

which implies $\langle e_1, e_2\rangle = 0$. Under the space decomposition $H = [e_1, e_2] \oplus [e_1, e_2]^{\perp}$, we get

$$A = \begin{pmatrix} \mu_1 & 0 & A_1 \\ 0 & \mu_2 & A_2 \\ 0 & 0 & A_3 \end{pmatrix}.$$

Next, we will show $A_1 = 0$. Suppose, on the contrary, that $A_1 \neq 0$. Write $A_1 = a_1 e_1 \otimes f$ with $a_1 \in \mathbb{C}\backslash\{0\}$ and $f \in [e_1, e_2]^{\perp}$. Taking

$$B = \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with any $b_1, b_2 \in \mathbb{C}$, and then

$$AB + BA = \begin{pmatrix} 2\mu_1 b_1 & 0 & a_1 b_1 e_1 \otimes f \\ (\mu_1+\mu_2) b_2 & 0 & a_1 b_2 e_2 \otimes f \\ 0 & 0 & 0 \end{pmatrix}.$$

With respect to the decomposition $H = [e_1, e_2, f] \oplus [e_1, e_2, f]^{\perp}$, one gets

$$AB + BA = C \oplus 0 \quad \text{with} \quad C = \begin{pmatrix} 2\mu_1 b_1 & 0 & a_1 b_1 \\ (\mu_1+\mu_2) b_2 & 0 & a_1 b_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

For C, it is easily checked that

$$\begin{aligned} \sigma(C) &= \{0, 0, 2\mu_1 b_1\} = \{\lambda_1, \lambda_2, \lambda_3\}, \\ s &= \operatorname{tr}(C^*C) - \sum_{i=1}^{3} |\lambda_i|^2 = |a_1 b_1|^2 + |a_1 b_2|^2 + |(\mu_1+\mu_2) b_2|^2 \end{aligned}$$

and

$$\begin{aligned} t &= \operatorname{tr}(C) + \frac{1}{s}\Big(\sum_{i=1}^{3} |\lambda_i|^2 \lambda_i - \operatorname{tr}(C^* C^2)\Big) \\ &= 2\mu_1 b_1 - \frac{2\mu_1 b_1 |(\mu_1+\mu_2) b_2|^2 + 2\mu_1 b_1 |a_1 b_1|^2 + (\mu_1+\mu_2) b_1 |a_1 b_2|^2}{|a_1 b_1|^2 + |a_1 b_2|^2 + |(\mu_1+\mu_2) b_2|^2}. \end{aligned}$$

By taking $b_1 b_2 \neq 0$ and $|b_1|^2 \neq \frac{-1}{2|a_1|^2}\big(2|(\mu_1+\mu_2)b_2|^2 + \big(1 + \frac{\mu_2}{\mu_1}\big)|a_1 b_2|^2\big)$, we infer that $t \notin \sigma(C)$. This and [18, Theorem 2.4] yield that

$$W(C) \text{ is not an ellipse.}$$

As 0 and $\operatorname{tr} C$ in $W(C)$, Proposition 2.3(3) becomes

$$kW_k(C \oplus 0) = \begin{cases} W(C), & \text{if } k = 1, n \geq 3; \\ \{\operatorname{tr} C\} - W(C), & \text{if } 2 \leq k = n-1; \\ \operatorname{conv}\{W(C), \{\operatorname{tr} C\} - W(C)\}, & \text{if } 2 \leq k \leq n-2. \end{cases}$$

If $\mathrm{conv}\{W(C), \{\mathrm{tr}\, C\} - W(C)\}$ is an ellipse represented as $\mathcal{S}$, then $\partial W(C)$ contains the elliptic arc of length greater than zero. It follows from [16, Lemma 2.1] that $\mathcal{S} \subseteq W(C)$. So $W(C) = \mathcal{S}$ is an ellipse, a contradiction. Thus, $\mathrm{conv}\{W(C), \{\mathrm{tr}\, C\} - W(C)\}$ is not an ellipse. Hence

$$W_k(AB + BA) = W_k(C \oplus 0) \quad \text{is not an ellipse.} \tag{2.7}$$

With respect to the decomposition $H = [e_1, e_2, f] \oplus [e_1, e_2, f]^{\perp}$, let

$$D = D_1 \oplus 0 \quad \text{with } D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

For B the above, B can be written as

$$B = B_1 \oplus 0 \quad \text{with } B_1 = \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Except for finitely many $\nu \in \mathbb{R}$, $B_1 + \nu D_1$ has distinct eigenvalues, and then there exists invertible matrix P such that $P^{-1}(B_1+\nu D_1)P$ is a diagonal matrix. Let $B_\nu = P(P^{-1}(B_1+\nu D_1)P)^{\frac{1}{r}} P^{-1} \oplus 0$, and then $B_\nu^r = B + \nu D$. Actually, we have $W_k(AB_\nu^r + B_\nu^r A) = W_k(AB + BA + \nu(AD + DA))$ is an ellipse except for finitely many $\nu \in \mathbb{R}$. Letting $\nu \to 0$, we obtain that $W_k(AB + BA)$ is an ellipse, a contradiction to (2.7). Hence $A_1 = 0$.

Thus, A can be represented as $A = \mu_1 \oplus A'$ for some $A' \in \mathcal{B}([e_1]^{\perp})$ with respect to the space decomposition $H = [e_1] \oplus [e_1]^{\perp}$. If $A' = 0$, then $A = \mu_1 e_1 \otimes e_1$. Let $\lambda = \mu_1$. It follows from the fact $W_k(A^{r+1}) = \left[0, \frac{1}{k}\right]$ that $\lambda^{r+1} = 1$, completing the proof. So, it remains to show that $A' = 0$. Otherwise, there exists $e_3 \in [e_1]^{\perp}$ such that $\langle A' e_3, e_3 \rangle \neq 0$. Under the space decomposition $H = [e_1, e_3] \oplus [e_1, e_3]^{\perp}$, we have

$$A = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & a_2 & A_3 \\ 0 & A_4 & A_5 \end{pmatrix}$$

with $a_2 \neq 0$. Take $B = b_1^{\frac{1}{r}} e_1 \otimes e_1 + b_2^{\frac{1}{r}} e_3 \otimes e_3$ with $b_1, b_2 \in \mathbb{C}$, and then

$$AB^r + B^r A = 2\mu_1 b_1 \oplus E \quad \text{with } E = \begin{pmatrix} 2a_2 b_2 & b_2 A_3 \\ b_2 A_4 & 0 \end{pmatrix}.$$

By Proposition 2.3(1), one obtains

$$kW_k(AB^r + B^r A) = kW_k(2\mu_1 b_1 \oplus E) = \mathrm{conv}\{kW_k(E), 2\mu_1 b_1 + (k-1)W_{k-1}(E)\}.$$

Thus, we can choose a suitable $b_1 \in \mathbb{C}$ such that $W_k(AB^r + B^r A)$ is not an ellipse, a contradiction.

3 Maps Preserving the k-Dimensional Numerical Range of Generalized Products

In this section, we are devoted to discussing maps preserving the k-dimensional numerical range of generalized products on $\mathcal{B}_s(H)$ and $\mathcal{B}(H)$, respectively.

For the special generalized product B^rAB^s, we can obtain the following Theorems 3.1–3.2, which are also of independent interest.

Theorem 3.1 *Let H be a complex Hilbert space with $\dim H \geq 3$, k be a positive integer with $k < \dim H$ and $r, s \geq 0$ be integers with $(r,s) \neq (0,0)$. Suppose that $\phi : \mathcal{B}_s(H) \to \mathcal{B}_s(H)$ is a surjective map. Then ϕ satisfies*

$$W_k(A^rBA^s) = W_k(\phi(A)^r\phi(B)\phi(A)^s) \quad \text{for all } A, B \in \mathcal{B}_s(H)$$

if and only if one of the following statements holds:

(1) *$r \neq s$, there exists some $\lambda \in \mathbb{R}$ with $\lambda^{r+s+1} = 1$ and a unitary operator $U \in \mathcal{B}(H)$ such that $\phi(A) = \lambda UAU^*$ for all $A \in \mathcal{B}_s(H)$.*

(2) *$r = s$, there exists some $\lambda \in \mathbb{R}$ with $\lambda^{r+s+1} = 1$ and a unitary or conjugate unitary operator U on H such that $\phi(A) = \lambda UAU^*$ for all $A \in \mathcal{B}_s(H)$.*

Theorem 3.2 *Let H be a complex Hilbert space with $\dim H \geq 3$, k be a positive integer with $k < \dim H$ and $r, s \geq 0$ be integers with $(r,s) \neq (0,0)$. Suppose that $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a surjective map. Then ϕ satisfies*

$$W_k(A^rBA^s) = W_k(\phi(A)^r\phi(B)\phi(A)^s) \quad \text{for all } A, B \in \mathcal{B}(H)$$

if and only if one of the following statements holds:

(1) *$r \neq s$, there exists some $\lambda \in \mathbb{C}$ with $\lambda^{r+s+1} = 1$ and a unitary operator $U \in \mathcal{B}(H)$ such that $\phi(A) = \lambda UAU^*$ for all $A \in \mathcal{B}(H)$.*

(2) *$r = s$, there exists some $\lambda \in \mathbb{C}$ with $\lambda^{r+s+1} = 1$ such that either $\phi(A) = \lambda UAU^*$ for all $A \in \mathcal{B}(H)$, where U is a unitary operator, or $\phi(A) = \lambda UA^*U^*$ for all $A \in \mathcal{B}(H)$, where U is a conjugate unitary operator.*

Particularly, for the usual operator product and the Jordan semi-triple product, the following corollaries are immediate.

Corollary 3.1 *Let H be a complex Hilbert space with $\dim H \geq 3$ and k be a positive integer with $k < \dim H$. Suppose that $\phi : \mathcal{B}_s(H) \to \mathcal{B}_s(H)$ is a surjective map. Then ϕ satisfies*

$$W_k(AB) = W_k(\phi(A)\phi(B)) \quad \text{for all } A, B \in \mathcal{B}_s(H)$$

if and only if there exists some $\lambda \in \{-1, 1\}$ and a unitary operator $U \in \mathcal{B}(H)$ such that $\phi(A) = \lambda UAU^*$ for all $A \in \mathcal{B}_s(H)$.

Corollary 3.2 *Let H be a complex Hilbert space with* $\dim H \geq 3$ *and k be a positive integer with $k < \dim H$. Suppose that $\phi : \mathcal{B}_s(H) \to \mathcal{B}_s(H)$ is a surjective map. Then ϕ satisfies*

$$W_k(ABA) = W_k(\phi(A)\phi(B)\phi(A)) \quad \text{for all } A, B \in \mathcal{B}_s(H)$$

if and only if there exists a unitary or conjugate unitary operator U on H such that $\phi(A) = UAU^*$ for all $A \in \mathcal{B}_s(H)$.

Corollary 3.3 *Let H be a complex Hilbert space with* $\dim H \geq 3$ *and k be a positive integer with $k < \dim H$. Then a surjective map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ satisfies*

$$W_k(AB) = W_k(\phi(A)\phi(B)) \quad \text{for all } A, B \in \mathcal{B}(H)$$

if and only if there exists some $\lambda \in \{-1, 1\}$ and a unitary operator $U \in \mathcal{B}(H)$ such that $\phi(A) = \lambda UAU^*$ for all $A \in \mathcal{B}(H)$.

Corollary 3.4 *Let H be a complex Hilbert space with* $\dim H \geq 3$ *and k be a positive integer with $k < \dim H$. Then a surjective map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ satisfies*

$$W_k(ABA) = W_k(\phi(A)\phi(B)\phi(A)) \quad \text{for all } A, B \in \mathcal{B}(H)$$

if and only if there exists some $\lambda \in \mathbb{C}$ with $\lambda^3 = 1$ such that either $\phi(A) = \lambda UAU^*$ for all $A \in \mathcal{B}(H)$, where U is a unitary operator, or $\phi(A) = \lambda UA^*U^*$ for all $A \in \mathcal{B}(H)$, where U is a conjugate unitary operator.

Now, based on Theorems 3.1–3.2, we can give a complete characterization of all maps preserving the k-dimensional numerical range of generalized products.

Theorem 3.3 *Let H be a complex Hilbert space with* $\dim H \geq 3$ *and k be a positive integer with $k < \dim H$. Assume that $\phi : \mathcal{B}_s(H) \to \mathcal{B}_s(H)$ is a surjective map and the generalized product is defined in Definition* 1.1. *Then ϕ satisfies*

$$W_k(A_1 \circ \cdots \circ A_l) = W_k(\phi(A_1) \circ \cdots \circ \phi(A_l)) \quad \text{for all } A_1, \cdots, A_l \in \mathcal{B}_s(H)$$

if and only if one of the following statements holds:

(1) *$m \neq 2p - 1$ or $(i_1, \cdots, i_m) \neq (i_m, \cdots, i_1)$, there exists some $\lambda \in \mathbb{R}$ with $\lambda^m = 1$ and a unitary operator $U \in \mathcal{B}(H)$ such that $\phi(A) = \lambda UAU^*$ for all $A \in \mathcal{B}_s(H)$.*

(2) *$m = 2p - 1$ and $(i_1, \cdots, i_m) = (i_m, \cdots, i_1)$, there exists some $\lambda \in \mathbb{R}$ with $\lambda^m = 1$ and a unitary or conjugate unitary operator U on H such that $\phi(A) = \lambda UAU^*$ for all $A \in \mathcal{B}_s(H)$.*

Theorem 3.4 *Let H be a complex Hilbert space with* $\dim H \geq 3$ *and k be a positive integer with $k < \dim H$. Assume that $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a surjective map and the generalized product is defined in Definition* 1.1. *Then ϕ satisfies*

$$W_k(A_1 \circ \cdots \circ A_l) = W_k(\phi(A_1) \circ \cdots \circ \phi(A_l)) \quad \text{for all } A_1, \cdots, A_l \in \mathcal{B}(H)$$

if and only if one of the following statements holds:

(1) $m \neq 2p-1$ *or* $(i_1, \cdots, i_m) \neq (i_m, \cdots, i_1)$, *there exists a fixed scalar* $\lambda \in \mathbb{C}$ *with* $\lambda^m = 1$ *and a unitary operator* $U \in \mathcal{B}(H)$ *such that* $\phi(A) = \lambda UAU^*$ *for all* $A \in \mathcal{B}(H)$.

(2) $m = 2p-1$ *and* $(i_1, \cdots, i_m) = (i_m, \cdots, i_1)$, *there exists a fixed scalar* $\lambda \in \mathbb{C}$ *with* $\lambda^m = 1$ *such that either* $\phi(A) = \lambda UAU^*$ *for all* $A \in \mathcal{B}(H)$, *where* $U \in \mathcal{B}(H)$ *is a unitary operator, or* $\phi(A) = \lambda UA^*U^*$ *for all* $A \in \mathcal{B}(H)$, *where* $U : H \to H$ *is a conjugate unitary operator.*

In the rest of this section, we give the proofs of the above main theorems.

Proof of Theorem 3.1 For the "if" part, it is straightforward by Proposition 2.1(1). We will prove the "only if" part by the following several claims. For the case $k = 1$, it holds by [20, Theorem 1.2]. So, we always assume $k \geq 2$ in the sequel.

Claim 1 ϕ is injective.

For any $A, B \in \mathcal{B}_s(H)$, suppose $\phi(A) = \phi(B)$. For any unit vector $x \in H$, one has

$$\begin{aligned} W_k((x\otimes x)^r A(x\otimes x)^s) &= W_k(\phi(x\otimes x)^r\phi(A)\phi(x\otimes x)^s) \\ &= W_k(\phi(x\otimes x)^r\phi(B)\phi(x\otimes x)^s) \\ &= W_k((x\otimes x)^r B(x\otimes x)^s). \end{aligned} \tag{3.1}$$

If $rs > 0$, (3.1) reduces to $\langle Ax, x\rangle W_k(x\otimes x) = \langle Bx, x\rangle W_k(x\otimes x)$, which implies $\langle Ax, x\rangle = \langle Bx, x\rangle$. If $rs = 0$, assuming $r = 0$, (3.1) reduces to $W_k(Ax\otimes x) = W_k(Bx\otimes x)$. Combining this with Proposition 2.5 yields $\langle Ax, x\rangle = \langle Bx, x\rangle$, too. It follows from the arbitrariness of x that $A = B$.

Claim 2 $\phi(0) = 0$.

For any unit vector $x \in H$, by the surjectivity of ϕ, there exists some $S_x \in \mathcal{B}_s(H)$ such that $\phi(S_x) = x\otimes x$. Then one gets

$$\{0\} = W_k(S_x^r 0 S_x^s) = W_k((x\otimes x)^r\phi(0)(x\otimes x)^s)$$

for any $x \in H$. By a similar discussion to that in the proof of Claim 1, one can show $\phi(0) = 0$.

Claim 3 There exists some $\lambda \in \mathbb{R}$ with $\lambda^{r+s+1} = 1$ such that $\phi(I) = \lambda I$.

By the assumption, one gets $\{1\} = W_k(I^{r+s+1}) = W_k(\phi(I)^{r+s+1})$, which, together with Proposition 2.1(2), implies $\phi(I)^{r+s+1} = I$. Note that $\phi(I) \in \mathcal{B}_s(H)$. If $r+s+1$ is odd, we have $\sigma(\phi(I)) = \{1\}$, which implies $\phi(I) = I$. If $r+s+1$ is even, we have $\sigma(\phi(I)) \subseteq \{-1, 1\}$. So there exists a projection $P \in \mathcal{B}_s(H)$ such that $\phi(I) = 2P - I$. If $P \notin \{0, I\}$, we can take $x_0 \in [\operatorname{ran} P]$ and $y_0 \in [\operatorname{ran} P]^\perp$. By the surjectivity of ϕ and Claim 2, there exists some $0 \neq S \in \mathcal{B}_s(H)$ such that $\phi(S) = x_0\otimes y_0 + y_0\otimes x_0$, and thus

$$W_k(S) = W_k(I^r S I^s) = W_k(\phi(I)^r(x_0\otimes y_0 + y_0\otimes x_0)\phi(I)^s).$$

Note that one of r and s is odd. Without loss of generality, we can assume that r is odd. Then, the above equation becomes

$$W_k(S) = W_k(x_0 \otimes y_0 - y_0 \otimes x_0) \subseteq \mathrm{i}\mathbb{R} \cap \mathbb{R},$$

and so $W_k(S) = \{0\}$. It follows from Proposition 2.1(2) that $S = 0$, a contradiction. Hence $\phi(I) = \pm I$ in the case $r + s + 1$ even.

Claim 4 There exists some $\lambda \in \mathbb{R}$ with $\lambda^{r+s+1} = 1$ and a unitary or conjugate unitary operator U on H such that $\phi(A) = \lambda UAU^*$ for all $A \in \mathcal{B}_s(H)$.

For any unit vector $x \in H$, by Claim 3, one has

$$\left[0, \frac{1}{k}\right] = W_k((x \otimes x)^{r+s+1}) = W_k(\phi(x \otimes x)^{r+s+1}),$$
$$\left[0, \frac{1}{k}\right] = W_k((x \otimes x)^r I (x \otimes x)^s) = \lambda W_k(\phi(x \otimes x)^r I \phi(x \otimes x)^s) = \frac{1}{\lambda^{r+s}} W_k(\phi(x \otimes x)^{r+s})$$

and

$$\left[0, \frac{1}{k}\right] = W_k(I^r x \otimes x I^s) = \lambda^{r+s} W_k(I^r \phi(x \otimes x) I^s) = \frac{1}{\lambda} W_k(\phi(x \otimes x)).$$

So

$$W_k\Big(\frac{1}{\lambda}\phi(x \otimes x)\Big) = W_k\Big(\Big(\frac{1}{\lambda}\phi(x \otimes x)\Big)^{r+s}\Big) = W_k\Big(\Big(\frac{1}{\lambda}\phi(x \otimes x)\Big)^{r+s+1}\Big) = \left[0, \frac{1}{k}\right],$$

which, together with Proposition 2.4, implies $\frac{1}{\lambda}\phi(x \otimes x) = y_x \otimes y_x$ for some unit vector $y_x \in H$, that is, for any unit vector $x \in H$,

$$\phi(x \otimes x) = \lambda y_x \otimes y_x \quad \text{for some unit vector } y_x \in H. \tag{3.2}$$

Next, take any unit vectors $x_1, x_2 \in H$. By (3.2), there exist some unit vectors $y_{x_1}, y_{x_2} \in H$ such that

$$\phi(x_1 \otimes x_1) = \lambda y_{x_1} \otimes y_{x_1} \quad \text{and} \quad \phi(x_2 \otimes x_2) = \lambda y_{x_2} \otimes y_{x_2}.$$

This and the fact that $\lambda^{r+s+1} = 1$ imply

$$W_k((x_1 \otimes x_1)^r (x_2 \otimes x_2)(x_1 \otimes x_1)^s) = W_k((y_{x_1} \otimes y_{x_1})^r (y_{x_2} \otimes y_{x_2})(y_{x_1} \otimes y_{x_1})^s).$$

If $rs \neq 0$, we arrive at $|\langle x_1, x_2\rangle|^2 W_k(x_1 \otimes x_1) = |\langle y_{x_1}, y_{x_2}\rangle|^2 W_k(y_{x_1} \otimes y_{x_1})$, which means $|\langle x_1, x_2\rangle| = |\langle y_{x_1}, y_{x_2}\rangle|$. If $rs = 0$, say $s = 0$, we get $\langle x_2, x_1\rangle W_k(x_1 \otimes x_2) = \langle y_{x_2}, y_{x_1}\rangle W_k(y_{x_1} \otimes y_{x_2})$, which, together with Proposition 2.5, implies $|\langle x_1, x_2\rangle| = |\langle y_{x_1}, y_{x_2}\rangle|$.

Now, noting that ϕ is a bijective map, by Wigner's Theorem (see [25]), there exist a unitary or conjugate unitary operator U on H and a complex number θ_x with $|\theta_x| = 1$ such that $y_x = \theta_x U x$ for all unit vectors $x \in H$. Thus, by (3.2), one obtains $\phi(x \otimes x) = \lambda Ux \otimes Ux$ for all unit vectors $x \in H$. Hence for any $A \in \mathcal{B}_s(H)$ and any unit vector $x \in H$, one has

$$W_k((x \otimes x)^r A (x \otimes x)^s) = \lambda^{r+s} W_k((Ux \otimes Ux)^r \phi(A)(Ux \otimes Ux)^s).$$

A similar argument to that in the proof of Claim 1 gives

$$\langle Ax, x\rangle = \lambda^{r+s}\langle \phi(A)Ux, Ux\rangle \quad \text{for all unit vectors } x \in H,$$

which implies $\phi(A) = \lambda UAU^*$ for all $A \in \mathcal{B}_s(H)$.

Claim 5 If $r \neq s$, it cannot occur that U is a conjugate unitary operator.

Suppose, on the contrary, that U is a conjugate unitary operator. In this case, by Proposition 2.1(1), one has

$$W_k(B^rAB^s) = W_k(\phi(B)^r\phi(A)\phi(B)^s) = W_k(B^sAB^r) \quad \text{for all } A, B \in \mathcal{B}_s(H).$$

In addition, we may assume that $r > s$.

Case 5.1 $2k \neq \dim H$.

Take $B = B_1 \oplus 0$ and $A = A_1 \oplus 0$, where

$$B_1 = \begin{pmatrix} \mu_1^{\frac{1}{r-s}} & 0 & \cdots & 0 \\ 0 & \mu_2^{\frac{1}{r-s}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_l^{\frac{1}{r-s}} \end{pmatrix},$$

$$A_1 = B_1^{-s} \begin{pmatrix} 0 & \mathrm{i}\frac{1}{\mu_1-\mu_2} & \cdots & \mathrm{i}\frac{1}{\mu_1-\mu_l} \\ -\mathrm{i}\frac{1}{\mu_1-\mu_2} & 0 & \cdots & \mathrm{i}\frac{1}{\mu_2-\mu_l} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathrm{i}\frac{1}{\mu_1-\mu_l} & -\mathrm{i}\frac{1}{\mu_2-\mu_l} & \ldots & 0 \end{pmatrix} B_1^{-s},$$

$\mu_1, \cdots, \mu_l \in \mathbb{R}\backslash\{0\}$, $\mu_i \neq \mu_j$ when $i \neq j$ and

$$l = \begin{cases} \dim H - k + 2, & \text{if } 2k > \dim H, \\ k + 2, & \text{if } 2k < \dim H. \end{cases}$$

Then

$$B^sAB^s = C \oplus 0 \quad \text{and} \quad B^{r-s} = D \oplus 0$$

with

$$C = \begin{pmatrix} 0 & \mathrm{i}\frac{1}{\mu_1-\mu_2} & \cdots & \mathrm{i}\frac{1}{\mu_1-\mu_l} \\ -\mathrm{i}\frac{1}{\mu_1-\mu_2} & 0 & \cdots & \mathrm{i}\frac{1}{\mu_2-\mu_l} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathrm{i}\frac{1}{\mu_1-\mu_l} & -\mathrm{i}\frac{1}{\mu_2-\mu_l} & \ldots & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_l \end{pmatrix}.$$

A direct calculation gives

$$DC - CD = \mathrm{i}\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}.$$

By Proposition 2.1(4) and Proposition 2.2, we get

$$W_k((DC-CD)\oplus 0) = \begin{cases} \dfrac{1}{k}[-(\dim H - k + 1), \dim H - k], & \text{if } 2k > \dim H, \\ \dfrac{1}{k}[-k, k+1], & \text{if } 2k < \dim H, \end{cases}$$

which means $W_k((DC-CD)\oplus 0) \neq -W_k((DC-CD)\oplus 0)$. It follows that

$$\begin{aligned} &2\mathrm{iIm}\, W_k(B^rAB^s) = 2\mathrm{iIm}\, W_k(DC\oplus 0) = W_k((DC-CD)\oplus 0) \\ \neq\; & -W_k((DC-CD)\oplus 0) = -2\mathrm{iIm}\, W_k(B^rAB^s). \end{aligned}$$

So $W_k(B^rAB^s) \neq W_k(B^rAB^s)^* = W_k(B^sAB^r)$, a contradiction.

Case 5.2 $2k = \dim H$.

In this case, take $B = B_1 \oplus 0$ and $A = A_1 \oplus 0$, where

$$B_1 = \begin{pmatrix} 3^{\frac{1}{r-s}} & 0 & 0 \\ 0 & 2^{\frac{1}{r-s}} & 0 \\ 0 & 0 & 4^{\frac{1}{r-s}} \end{pmatrix}, \quad A_1 = B_1^{-s}\begin{pmatrix} 0 & e^{\mathrm{i}\frac{\pi}{4}} & 1 \\ \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} B_1^{-s}.$$

Then $B^sAB^s = C\oplus 0$ and $B^{r-s} = D \oplus 0$ with

$$C = \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\frac{\pi}{4}} & 1 \\ \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

It is easy to see that $\det(\mathrm{Im}\, CD) \neq 0$, $\det(\mathrm{Re}\, CD) \neq 0$ and $\mathrm{tr}(\mathrm{Im}\, CD) = \mathrm{tr}(\mathrm{Re}\, CD) = 0$. These mean $0 \in \mathrm{Im}\, W(CD) = W(\mathrm{Im}\, CD) \neq -W(\mathrm{Im}\, CD)$ and $0 \in \mathrm{Re}\, W(CD) = W(\mathrm{Re}\, CD) \neq -W(\mathrm{Re}\, CD)$. By Proposition 2.3(3), and keeping in mind that $\mathrm{tr}\, CD = 0 \in W(CD)$, one obtains $kW_k(B^sAB^r) = kW_k(CD\oplus 0) = \mathrm{conv}\{W(CD) \cup (-W(CD))\}$. So $W_k(B^sAB^r) \neq W_k(B^sAB^r)^* = W_k(B^rAB^s)$, yielding another contradiction.

The proof of the theorem is completed.

Proof of Theorem 3.2 Still, we only need to show the "only if" part for the case $k \geq 2$.

Claim 1 ϕ is injective.

The proof is similar to that of Claim 1 in the proof of Theorem 3.1, and we omit it here.

Claim 2 For any $A \in \mathcal{B}(H)$ and any $\lambda \in \mathbb{C}$, we have $\phi(\lambda A) = \lambda\phi(A)$.

For any unit vector $x \in H$, there exists some $C \in \mathcal{B}(H)$ such that $\phi(C) = x \otimes x$. Then, for any $A \in \mathcal{B}(H)$ and any $\lambda \in \mathbb{C}$, by the assumption, we have

$$\begin{aligned} & W_k((x \otimes x)^r \phi(\lambda A)(x \otimes x)^s) \\ &= W_k(C^r \lambda A C^s) = \lambda W_k(C^r A C^s) \\ &= \lambda W_k((x \otimes x)^r \phi(A)(x \otimes x)^s) \\ &= W_k((x \otimes x)^r \lambda \phi(A)(x \otimes x)^s) \end{aligned}$$

for all unit vectors $x \in H$. Now, the claim is true by the same argument as in Claim 1 in the proof of Theorem 3.1.

Claim 3 $\phi(\mathcal{S}) = \mathcal{S}$, where $\mathcal{S} = \{\lambda x \otimes x : \lambda \in \mathbb{C}$ with $\lambda^{r+s+1} = 1, x \in H$ with $\|x\| = 1\}$.

Take any $A = \lambda x \otimes x \in \mathcal{S}$, where $x \in H$ is a unit vector and $\lambda \in \mathbb{C}$ with $\lambda^{r+s+1} = 1$. By Proposition 2.6, for any $B \in \mathcal{B}(H)$, $W_k(B^r A B^s) = W_k(\phi(B)^r \phi(A) \phi(B)^s)$ is an ellipse which has 0 as a focus. By Proposition 2.6 again and the surjectivity of ϕ, we obtain $\operatorname{rank}(\phi(A)) = 1$. Let $\phi(A) = u \otimes v$ with $u, v \in H$ and $\|v\| = 1$. Note that

$$\left[0, \frac{1}{k}\right] = W_k(A^{r+s+1}) = W_k(\phi(A)^{r+s+1}) = \langle u, v \rangle^{r+s} W_k(u \otimes v).$$

By Proposition 2.1(3), $u = \mu v$ for some $\mu \in \mathbb{C}$. So $\langle u, v \rangle^{r+s} W_k(u \otimes v) = \mu^{r+s+1} W_k(v \otimes v) = \left[0, \frac{1}{k}\right]$, which means $\mu^{r+s+1} = 1$ and $\phi(A) = \mu v \otimes v$, that is, $\phi(\mathcal{S}) \subseteq \mathcal{S}$. Since ϕ is bijective by Claim 1, $\phi(\mathcal{S}) = \mathcal{S}$.

Claim 4 ϕ is additive.

For any unit vector $x \in H$, by Claim 3, there is a rank one operator $C_x \in \mathcal{B}(H)$ such that $\phi(C_x) = x \otimes x$. Take any $A, B \in \mathcal{B}(H)$. We have $W_k(C_x^r A C_x^s) = W_k((x \otimes x)^r \phi(A)(x \otimes x)^s)$, which, together with Proposition 2.5, implies

$$\operatorname{tr}(C_x^r A C_x^s) = \operatorname{tr}((x \otimes x)^r \phi(A)(x \otimes x)^s).$$

Similarly, one can get

$$\operatorname{tr}(C_x^r B C_x^s) = \operatorname{tr}((x \otimes x)^r \phi(B)(x \otimes x)^s)$$

and

$$\operatorname{tr}(C_x^r (A+B) C_x^s) = \operatorname{tr}((x \otimes x)^r \phi(A+B)(x \otimes x)^s).$$

Combining the above equations gives

$$\operatorname{tr}((x \otimes x)^r (\phi(A) + \phi(B))(x \otimes x)^s) = \operatorname{tr}((x \otimes x)^r \phi(A+B)(x \otimes x)^s)$$

for all unit vectors $x \in H$. It follows that $\langle (\phi(A) + \phi(B))x, x \rangle = \langle \phi(A+B)x, x \rangle$ holds for all unit vectors $x \in H$, which implies $\phi(A+B) = \phi(A) + \phi(B)$.

Claim 5 There exists some $\lambda \in \mathbb{C}$ with $\lambda^{r+s+1} = 1$ such that either $\phi(A) = \lambda UAU^*$ for all $A \in \mathcal{B}(H)$, where U is a unitary operator, or $\phi(A) = \lambda UA^*U^*$ for all $A \in \mathcal{B}(H)$, where U is a conjugate unitary operator. Furthermore, the latter case cannot occur if $r \neq s$.

For any unit vector $x \in H$, by Claim 3, there exists a unit vector $y_x \in H$ and a scalar $\lambda_x \in \mathbb{C}$ with $\lambda_x^{r+s+1} = 1$ such that $\phi(\lambda_x y_x \otimes y_x) = x \otimes x$. As

$$W_k((\lambda_x y_x \otimes y_x)^r I(\lambda_x y_x \otimes y_x)^s) = W_k((x \otimes x)^r \phi(I)(x \otimes x)^s),$$

by Proposition 2.5, one has $\mathrm{tr}((\lambda_x y_x \otimes y_x)^r (\lambda_x y_x \otimes y_x)^s) = \mathrm{tr}((x \otimes x)^r \phi(I)(x \otimes x)^s)$, and so $\lambda_x^{r+s} = \frac{1}{\lambda_x} = \langle \phi(I)x, x\rangle$ holds for each unit vector $x \in H$. This means $W(\phi(I)) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Also note that $W(\phi(I))$ is convex. Thus, we obtain that $W(\phi(I))$ is a single point set. Applying Proposition 2.1(2) yields

$$\phi(I) = \lambda I \quad \text{for some } \lambda \in \mathbb{C}.$$

Since $\{1\} = W_k(I^{r+s+1}) = W_k(\phi(I)^{r+s+1})$, we have $\lambda^{r+s+1} = 1$. Hence, for any $A \in \mathcal{B}(H)$, we have

$$W_k(A) = W_k(I^r A I^s) = W_k(\phi(I)^r \phi(A) \phi(I)^s) = W_k(\lambda^{r+s}\phi(A)). \tag{3.3}$$

Define a map $\psi : \mathcal{B}(H) \to \mathcal{B}(H)$ by $\psi(\cdot) = \lambda^{r+s}\phi(\cdot)$. By Claims 1–2, Claim 4 and (3.3), ψ is a linear bijective map satisfying $W_k(A) = W_k(\psi(A))$ for all $A \in \mathcal{B}(H)$. Now, by [21] and Claim 3, one obtains that either

$$\phi(A) = \lambda UAU^* \quad \text{for all } A \in \mathcal{B}(H),$$

where $U \in \mathcal{B}(H)$ is a unitary operator, or

$$\phi(A) = \lambda UA^*U^* \quad \text{for all } A \in \mathcal{B}(H),$$

where $U \in \mathcal{B}(H)$ is a conjugate unitary operator.

Finally, to complete the proof, we still need to check that the second case cannot occur if $r \neq s$. In fact, if $r \neq s$ and $\phi(A) = \lambda UA^*U^*$ with U a conjugate unitary operator, then

$$W_k(B^r A B^s) = W_k(\phi(B)^r \phi(A) \phi(B)^s) = W_k(B^s A B^r) \quad \text{for all } A, B \in \mathcal{B}(H).$$

In particular, take $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \oplus 0$ and $B = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \oplus 0$ with respect to some decomposition $H = H_1 \oplus H_1^{\perp}$ with $\dim H_1 = 2$. If $rs > 0$, then

$$B^r A B^s = \begin{pmatrix} 0 & r \\ 0 & 1 \end{pmatrix} \oplus 0 \quad \text{and} \quad B^s A B^r = \begin{pmatrix} 0 & s \\ 0 & 1 \end{pmatrix} \oplus 0.$$

Note that $W\big(\left(\begin{smallmatrix} 0 & r \\ 0 & 1 \end{smallmatrix}\right)\big) \neq W\big(\left(\begin{smallmatrix} 0 & s \\ 0 & 1 \end{smallmatrix}\right)\big)$ and $0, 1 \in W\big(\left(\begin{smallmatrix} 0 & r \\ 0 & 1 \end{smallmatrix}\right)\big)$. By Proposition 2.3(2), one gets

$$W_k(B^r A B^s) = \frac{1}{k} W\Big(\begin{pmatrix} 0 & r \\ 0 & 1 \end{pmatrix}\Big) \neq \frac{1}{k} W\Big(\begin{pmatrix} 0 & s \\ 0 & 1 \end{pmatrix}\Big) = W_k(B^s A B^r),$$

a contradiction. If $rs=0$, say $r=0$, then

$$AB^s=\begin{pmatrix}0&0\\0&1\end{pmatrix}\oplus 0 \quad \text{and} \quad B^sA=\begin{pmatrix}0&s\\0&1\end{pmatrix}\oplus 0.$$

Note that $AB^s\in\mathcal{B}_s(H)$ and $B^sA\notin\mathcal{B}_s(H)$. This, together with Proposition 2.1(3), implies that $W_k(AB^s)\neq W_k(B^sA)$, yielding another contradiction.

Proof of Theorem 3.3 By Proposition 2.1(1), the "if" part is clear.

For the "only if" part, take any $A,B\in\mathcal{B}_s(H)$. Plug $A_{i_p}=B$ and $A_{i_q}=A$ for all other $q\neq p$. Then $A_1\circ\cdots\circ A_l=A^{p-1}BA^{m-p}$. Thus, by applying Theorem 3.1, there exists a unitary or conjugate unitary operator U on H and some $\lambda\in\mathbb{R}$ with $\lambda^m=1$ such that $\phi(A)=\lambda UAU^*$ for all $A\in\mathcal{B}_s(H)$; and in particular, the case that U is a conjugate unitary operator cannot occur if $m\neq 2p-1$.

To complete the proof, we still need to check that the case where U is a conjugate unitary operator cannot occur if $m=2p-1$ and $(i_1,\cdots,i_m)\neq(i_m,\cdots,i_1)$. Otherwise, assume that U is a conjugate unitary operator, $m=2p-1$ and $(i_1,\cdots,i_m)\neq(i_m,\cdots,i_1)$. Then, one can get

$$W_k(A_{i_1}\cdots A_{i_{2p-1}})=W_k(A_{i_{2p-1}}\cdots A_{i_1}) \quad \text{for all } A_{i_1},\cdots,A_{i_{2p-1}}\in\mathcal{B}_s(H).$$

Take any unit vector $x\in H$ and let $A_{i_p}=x\otimes x$. By Proposition 2.5, we have

$$\mathrm{tr}(A_{i_1}\cdots A_{i_{p-1}}x\otimes xA_{i_{p+1}}\cdots A_{i_{2p-1}})=\mathrm{tr}(A_{i_{2p-1}}\cdots A_{i_{p+1}}x\otimes xA_{i_{p-1}}\cdots A_{i_1}),$$

that is,

$$\langle A_{i_{p+1}}\cdots A_{i_{2p-1}}A_{i_1}\cdots A_{i_{p-1}}x,x\rangle=\langle A_{i_{p-1}}\cdots A_{i_1}A_{i_{2p-1}}\cdots A_{i_{p+1}}x,x\rangle,$$

which implies

$$A_{i_{p+1}}\cdots A_{i_{2p-1}}A_{i_1}\cdots A_{i_{p-1}}=A_{i_{p-1}}\cdots A_{i_1}A_{i_{2p-1}}\cdots A_{i_{p+1}} \quad \text{for all } A_{i_1},\cdots,A_{i_{2p-1}}\in\mathcal{B}_s(H).$$

Let $(i_{p+1},\cdots,i_{2p-1},i_1,\cdots,i_{p-1})=(j_1,\cdots,j_{2p-2})$. Clearly, $(j_1,\cdots,j_{2p-2})\neq(j_{2p-2},\cdots,j_1)$. Let $t\in\left[1,\frac{2p-1}{2}\right]$ be the smallest integer such that $j_t\neq j_{2p-1-t}$. Take any $\lambda>0$, and take $A_{j_t}=D\oplus 0$ with $D=\left(\begin{smallmatrix}\lambda&0\\0&1\end{smallmatrix}\right)$ and $A_{j_q}=S\oplus 0$, where S is a fixed 2×2 self-adjoint invertible matrix for all $j_q\neq j_t$. By a direct calculation, it is easily seen that there exists some $\lambda>0$ such that $A_{j_t}\cdots A_{j_{2p-1-t}}\neq A_{j_{2p-1-t}}\cdots A_{j_t}$. So $A_{j_1}\cdots A_{j_{2p-2}}\neq A_{j_{2p-2}}\cdots A_{j_1}$, that is, $A_{i_{p+1}}\cdots A_{i_{2p-1}}A_{i_1}\cdots A_{i_{p-1}}\neq A_{i_{p-1}}\cdots A_{i_1}A_{i_{2p-1}}\cdots A_{i_{p+1}}$. Thus, we get a contradiction. So $(i_1,\cdots,i_{2p-1})=(i_{2p-1},\cdots,i_1)$.

Proof of Theorem 3.4 Still, we only need to check the "only if" part. To do this, for any $A,B\in\mathcal{B}(H)$, by taking $A_{i_p}=B$ and $A_{i_q}=A$ for all other $q\neq p$, and applying Theorem 3.2, there exists some $\lambda\in\mathbb{C}$ with $\lambda^m=1$ such that either

$$\phi(A)=\lambda UAU^* \quad \text{for all } A\in\mathcal{B}(H),$$

where $U \in \mathcal{B}(H)$ is a unitary operator, or

$$\phi(A) = \lambda U A^* U^* \quad \text{for all } A \in \mathcal{B}(H),$$

where $U \in \mathcal{B}(H)$ is a conjugate unitary operator, and the second case occur only when $m = 2p-1$.

Now, by a similar discussion to that in the proof of Theorem 3.3, one can show that the second case also cannot occur if $(i_1, \cdots, i_m) \neq (i_m, \cdots, i_1)$. The proof is finished.

4 Maps Preserving the k-Dimensional Numerical Range of Generalized Jordan Products

In this section, we will discuss all maps preserving the k-numerical range of generalized Jordan products on $\mathcal{B}_s(H)$ and $\mathcal{B}(H)$, respectively.

Theorem 4.1 *Let H be a complex Hilbert space with $\dim H \geq 2$ and k be a positive integer with $k < \dim H$. Suppose that $\phi : \mathcal{B}_s(H) \to \mathcal{B}_s(H)$ is a surjective map. Then the following statements are equivalent*:

(1) *$W_k(A_1 * \cdots * A_l) = W_k(\phi(A_1) * \cdots * \phi(A_l))$ holds for all $A_1, \cdots, A_l \in \mathcal{B}_s(H)$, where $A_1 * \cdots * A_l$ is defined in Definition* 1.1.

(2) *$W_k(B^r A B^s + B^s A B^r) = W_k(\phi(B)^r \phi(A) \phi(B)^s + \phi(B)^s \phi(A) \phi(B)^r)$ holds for all $A, B \in \mathcal{B}_s(H)$, where $r + s + 1 = m$.*

(3) *There exists some $\lambda \in \mathbb{R}$ with $\lambda^m = 1$ and a unitary or conjugate unitary operator U on H such that $\phi(A) = \lambda U A U^*$ for all $A \in \mathcal{B}_s(H)$.*

Theorem 4.2 *Let H be a complex Hilbert space with $\dim H \geq 2$ and k be a positive integer with $k < \dim H$. Suppose that $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a surjective map. Then the following statements are equivalent*:

(1) *$W_k(A_1 * \cdots * A_l) = W_k(\phi(A_1) * \cdots * \phi(A_l))$ holds for all $A_1, \cdots, A_l \in \mathcal{B}(H)$, where $A_1 * \cdots * A_l$ is defined in Definition* 1.1.

(2) *$W_k(B^r A B^s + B^s A B^r) = W_k(\phi(B)^r \phi(A) \phi(B)^s + \phi(B)^s \phi(A) \phi(B)^r)$ holds for all $A, B \in \mathcal{B}(H)$, where $r + s + 1 = m$.*

(3) *There exists some $\lambda \in \mathbb{C}$ with $\lambda^m = 1$ such that either $\phi(A) = \lambda U A U^*$ for all $A \in \mathcal{B}(H)$, where $U \in \mathcal{B}(H)$ is a unitary operator, or $\phi(A) = \lambda U A^* U^*$ for all $A \in \mathcal{B}(H)$, where $U : H \to H$ is a conjugate unitary operator.*

Particularly, for the Jordan product, we have the following results.

Corollary 4.1 *Let H be a complex Hilbert space with $\dim H \geq 2$ and k be a positive integer with $k < \dim H$. Suppose that $\phi : \mathcal{B}_s(H) \to \mathcal{B}_s(H)$ is a surjective map. Then $W_k(AB + BA) = W_k(\phi(A)\phi(B) + \phi(B)\phi(A))$ holds for all $A, B \in \mathcal{B}_s(H)$ if and only if there exists some*

$\lambda \in \{-1,1\}$ and a unitary or conjugate unitary operator U on H such that $\phi(A)=\lambda UAU^*$ for all $A \in \mathcal{B}_s(H)$.

Corollary 4.2 *Let H be a complex Hilbert space with $\dim H \geq 2$ and k be a positive integer with $k < \dim H$. Suppose that $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a surjective map. Then $W_k(AB+BA) = W_k(\phi(A)\phi(B)+\phi(B)\phi(A))$ holds for all $A, B \in \mathcal{B}(H)$ if and only if there exists some $\lambda \in \{-1,1\}$ such that either $\phi(A)=\lambda UAU^*$ for all $A \in \mathcal{B}(H)$, where $U \in \mathcal{B}(H)$ is a unitary operator, or $\phi(A) = \lambda UA^*U^*$ for all $A \in \mathcal{B}(H)$, where $U : H \to H$ is a conjugate unitary operator.*

Now, we are in the position to give proofs of Theorems 4.1–4.2.

Proof of Theorem 4.1 (3)⇒(1)⇒(2) This is obvious.

(2)⇒(3) We divide the proof into several claims.

Claim 1 ϕ is injective.

Let $A, B \in \mathcal{B}_s(H)$ and suppose that $\phi(A)=\phi(B)$. For any unit vector $x \in H$, we have

$$\begin{aligned}
&W_k((x\otimes x)^r A(x\otimes x)^s + (x\otimes x)^s A(x\otimes x)^r)\\
&= W_k(\phi(x\otimes x)^r \phi(A)\phi(x\otimes x)^s + \phi(x\otimes x)^s \phi(A)\phi(x\otimes x)^r)\\
&= W_k(\phi(x\otimes x)^r \phi(B)\phi(x\otimes x)^s + \phi(x\otimes x)^s \phi(B)\phi(x\otimes x)^r)\\
&= W_k((x\otimes x)^r B(x\otimes x)^s + (x\otimes x)^s B(x\otimes x)^r).
\end{aligned}$$

If $rs > 0$, then the above equation gives $\langle Ax, x\rangle W_k(x\otimes x) = \langle Bx, x\rangle W_k(x\otimes x)$, which means $\langle Ax, x\rangle = \langle Bx, x\rangle$ for all unit vectors $x \in H$. If $rs = 0$, say $r = 0$, then the above equation reduces to $W_k(Ax\otimes x + x\otimes xA) = W_k(Bx\otimes x + x\otimes xB)$. By Proposition 2.1(7), one gets $\langle Ax, x\rangle = \langle Bx, x\rangle$ for all unit vectors $x \in H$. It follows that $A = B$.

Claim 2 $\phi(0) = 0$.

For any unit vector $x \in H$, by the surjectivity of ϕ, there exists some $S \in \mathcal{B}_s(H)$ such that $\phi(S) = x\otimes x$. By the assumption, we have

$$\{0\} = W_k(S^r 0 S^s + S^s 0 S^r) = W_k((x\otimes x)^r \phi(0)(x\otimes x)^s + (x\otimes x)^s \phi(0)(x\otimes x)^r).$$

By the same argument as in the proof of Claim 1, one can show $\phi(0) = 0$.

Claim 3 There exists some $\lambda \in \mathbb{R}$ with $\lambda^{r+s+1} = 1$ such that $\phi(I) = \lambda I$.

By the assumption, one has $\{2\} = W_k(2I^{r+s+1}) = W_k(2\phi(I)^{r+s+1})$, which, together with Proposition 2.1(2), implies $\phi(I)^{r+s+1} = I$. By the same discussion as that of Claim 3 in the proof of Theorem 3.1, if $r+s+1$ is odd, then $\phi(I) = I$; if $r+s+1$ is even, then one of r, s is odd and $\phi(I) = 2P - I$ for some projection $P \in \mathcal{B}(H)$. If $P \notin \{0, I\}$, we can take $x_0 \in [\operatorname{ran} P]$ and $y_0 \in [\operatorname{ran} P]^\perp$. Let $S \in \mathcal{B}_s(H)$ be such that $\phi(S) = x_0\otimes y_0 + y_0\otimes x_0$. A direct calculation gives $T = \phi(I)^r(x_0\otimes y_0 + y_0\otimes x_0)\phi(I)^s + \phi(I)^s(x_0\otimes y_0 + y_0\otimes x_0)\phi(I)^r = 0$. So $W_k(2S) = W_k(I^r S I^s + I^s S I^r) = W_k(T) = \{0\}$. This implies $S = 0$, a contradiction to Claim 2. Hence $P \in \{0, I\}$ and $\phi(I) = \pm I$.

Claim 4 There exists some $\lambda \in \mathbb{R}$ with $\lambda^{r+s+1} = 1$ such that $\phi(x \otimes x) = \lambda y_x \otimes y_x$ for all unit vectors $x \in H$, where $y_x \in H$ is a unit vector depending on x.

Take any unit vector $x \in H$. By Claim 3, one gets

$$\begin{aligned}
2\Big[0, \frac{1}{k}\Big] &= W_k(2(x \otimes x)^{r+s+1}) = W_k(2\phi(x \otimes x)^{r+s+1}),\\
2\Big[0, \frac{1}{k}\Big] &= W_k((x \otimes x)^r I (x \otimes x)^s + (x \otimes x)^s I (x \otimes x)^r)\\
&= \lambda W_k(\phi(x \otimes x)^r I \phi(x \otimes x)^s + \phi(x \otimes x)^s I \phi(x \otimes x)^r)\\
&= \frac{2}{\lambda^{r+s}} W_k(\phi(x \otimes x)^{r+s})
\end{aligned}$$

and

$$\begin{aligned}
2\Big[0, \frac{1}{k}\Big] &= W_k(I^r x \otimes x I^s + I^s x \otimes x I^r)\\
&= \lambda^{r+s} W_k(I^r \phi(x \otimes x) I^s + I^s \phi(x \otimes x) I^r)\\
&= \frac{2}{\lambda} W_k(\phi(x \otimes x)).
\end{aligned}$$

Comparing the above equalities gives

$$W_k\Big(\frac{1}{\lambda}\phi(x \otimes x)\Big) = W_k\Big(\Big(\frac{1}{\lambda}\phi(x \otimes x)\Big)^{r+s}\Big) = W_k\Big(\Big(\frac{1}{\lambda}\phi(x \otimes x)\Big)^{r+s+1}\Big) = \Big[0, \frac{1}{k}\Big]. \tag{4.1}$$

If $k \geq 2$, then (4.1) and Proposition 2.4 imply $\frac{1}{\lambda}\phi(x \otimes x) = y_x \otimes y_x$ for some unit vector $y_x \in H$, that is, $\phi(x \otimes x) = \lambda y_x \otimes y_x$.

If $k = 1$, take any $B \in \mathcal{B}_s(H)$. By Proposition 2.7, if

$$2W(B) = W((x \otimes x)^r B (x \otimes x)^s + (x \otimes x)^s B (x \otimes x)^r) = [0, 2], \tag{4.2}$$

then we have

$$\begin{aligned}
&\{C \in \mathcal{B}_s(H) : B^r C B^s + B^s C B^r = 0\}\\
\subseteq &\{C \in \mathcal{B}_s(H) : (x \otimes x)^r C (x \otimes x)^s + (x \otimes x)^s C (x \otimes x)^r = 0\}.
\end{aligned} \tag{4.3}$$

Note that

$$\begin{aligned}
&W\Big(\Big(\frac{1}{\lambda}\phi(x \otimes x)\Big)^r \lambda^{r+s} \phi(B) \Big(\frac{1}{\lambda}\phi(x \otimes x)\Big)^s + \Big(\frac{1}{\lambda}\phi(x \otimes x)\Big)^s \lambda^{r+s} \phi(B) \Big(\frac{1}{\lambda}\phi(x \otimes x)\Big)^r\Big)\\
= &W(\phi(x \otimes x)^r \phi(B) \phi(x \otimes x)^s + \phi(x \otimes x)^s \phi(B) \phi(x \otimes x)^r).
\end{aligned} \tag{4.4}$$

And by Claim 3,

$$2W(\lambda^{r+s}\phi(B)) = \lambda^{r+s} W(I^r \phi(B) I^s + I^s \phi(B) I^r) = W(I^r B I^s + I^s B I^r) = 2W(B). \tag{4.5}$$

Now, if

$$2W(\lambda^{r+s}\phi(B))$$

$$
\begin{aligned}
&= W\Big(\Big(\frac{1}{\lambda}\phi(x\otimes x)\Big)^r\lambda^{r+s}\phi(B)\Big(\frac{1}{\lambda}\phi(x\otimes x)\Big)^s + \Big(\frac{1}{\lambda}\phi(x\otimes x)\Big)^s\lambda^{r+s}\phi(B)\Big(\frac{1}{\lambda}\phi(x\otimes x)\Big)^r\Big)\\
&= [0,2],
\end{aligned}
$$

then by (4.4)–(4.5) and the assumption about ϕ, we see that (4.2) holds. So (4.3) also holds. Thus, by Proposition 2.1(2), we have

$$
\begin{aligned}
&\{\phi(C)\in\mathcal{B}_s(H):\phi(B)^r\phi(C)\phi(B)^s+\phi(B)^s\phi(C)\phi(B)^r=0\}\\
\subseteq&\{\phi(C)\in\mathcal{B}_s(H):\phi(x\otimes x)^r\phi(C)\phi(x\otimes x)^s+\phi(x\otimes x)^s\phi(C)\phi(x\otimes x)^r=0\},
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&\{\phi(C)\in\mathcal{B}_s(H):(\lambda^{r+s}\phi(B))^r\phi(C)(\lambda^{r+s}\phi(B))^s+(\lambda^{r+s}\phi(B))^s\phi(C)(\lambda^{r+s}\phi(B))^r=0\}\\
\subseteq&\Big\{\phi(C)\in\mathcal{B}_s(H):\Big(\frac{1}{\lambda}\phi(x\otimes x)\Big)^r\phi(C)\Big(\frac{1}{\lambda}\phi(x\otimes x)\Big)^s+\Big(\frac{1}{\lambda}\phi(x\otimes x)\Big)^s\phi(C)\Big(\frac{1}{\lambda}\phi(x\otimes x)\Big)^r=0\Big\}.
\end{aligned}
$$

It follows from Proposition 2.7 and the surjectivity of ϕ that there exists a unit vector $y_x\in H$ such that $\frac{1}{\lambda}\phi(x\otimes x)=y_x\otimes y_x$, that is, $\phi(x\otimes x)=\lambda y_x\otimes y_x$.

Claim 5 The statement (3) holds, that is, there exists some $\lambda\in\mathbb{R}$ with $\lambda^m=1$ and a unitary or conjugate unitary operator U on H such that $\phi(A)=\lambda UAU^*$ for all $A\in\mathcal{B}_s(H)$.

Firstly, take any unit vectors $x_1,x_2\in H$. By Claim 4, there exist some unit vectors $y_{x_1},y_{x_2}\in H$ such that

$$\phi(x_1\otimes x_1)=\lambda y_{x_1}\otimes y_{x_1}\quad\text{and}\quad\phi(x_2\otimes x_2)=\lambda y_{x_2}\otimes y_{x_2}.$$

This and the fact that $\lambda^{r+s+1}=1$ imply

$$
\begin{aligned}
&W_k((x_1\otimes x_1)^r(x_2\otimes x_2)(x_1\otimes x_1)^s+(x_1\otimes x_1)^s(x_2\otimes x_2)(x_1\otimes x_1)^r)\\
=&W_k((y_{x_1}\otimes y_{x_1})^r(y_{x_2}\otimes y_{x_2})(y_{x_1}\otimes y_{x_1})^s+(y_{x_1}\otimes y_{x_1})^s(y_{x_2}\otimes y_{x_2})(y_{x_1}\otimes y_{x_1})^r).
\end{aligned}
$$

If $rs\neq 0$, we arrive at $2|\langle x_1,x_2\rangle|^2W_k(x_1\otimes x_1)=2|\langle y_{x_1},y_{x_2}\rangle|^2W_k(y_{x_1}\otimes y_{x_1})$, which means $|\langle x_1,x_2\rangle|=|\langle y_{x_1},y_{x_2}\rangle|$. If $rs=0$, say $s=0$, we get

$$
\begin{aligned}
&W_k((x_1\otimes x_1)(x_2\otimes x_2)+(x_2\otimes x_2)(x_1\otimes x_1))\\
=&W_k((y_{x_1}\otimes y_{x_1})(y_{x_2}\otimes y_{x_2})+(y_{x_2}\otimes y_{x_2})(y_{x_1}\otimes y_{x_1})),
\end{aligned}
$$

which, together with Proposition 2.1(7), implies $\langle(x_1\otimes x_1)x_2,x_2\rangle=\langle(y_{x_1}\otimes y_{x_1})y_{x_2},y_{x_2}\rangle$, that is, $|\langle x_1,x_2\rangle|=|\langle y_{x_1},y_{x_2}\rangle|$.

Now, by a similar argument to that of Claim 4 in the proof of Theorem 3.1, one can show that the statement (3) holds.

The proof of the theorem is completed.

Proof of Theorem 4.2 (3)⇒(1)⇒(2) This is a direct calculation.

(2)⇒(3) We will prove it by a series of claims.

Claim 1 ϕ is injective and homogeneous.

The proof of injectivity is similar to that of Claim 1 in the proof of Theorem 4.1, and we omit it.

To show that ϕ is homogeneous, take any $A, C \in \mathcal{B}(H)$ and any $\lambda \in \mathbb{C}$. Then we have

$$\begin{aligned} & W_k(\phi(C)^r\phi(\lambda A)\phi(C)^s + \phi(C)^s\phi(\lambda A)\phi(C)^r) \\ = & W_k(C^r\lambda AC^s + C^s\lambda AC^r) \\ = & \lambda W_k(C^rAC^s + C^sAC^r) \\ = & \lambda W_k(\phi(C)^r\phi(A)\phi(C)^s + \phi(C)^s\phi(A)\phi(C)^r) \\ = & W_k(\phi(C)^r\lambda\phi(A)\phi(C)^s + \phi(C)^s\lambda\phi(A)\phi(C)^r). \end{aligned}$$

Since ϕ is surjective, for any unit vector $x \in H$, we can take $\phi(C) = x \otimes x$. Now by a similar argument to that of Claim 1 of Theorem 4.1, again, one can prove $\phi(\lambda A) = \lambda\phi(A)$.

Claim 2 If $rs > 0$, then there exists some $\lambda \in \mathbb{C}$ with $\lambda^{r+s+1} = 1$ such that $\phi(I) = \lambda I$ and ϕ is additive.

Assume that $rs > 0$. For any unit vector $x \in H$, by the surjectivity of ϕ, there exists some $Y_x \in \mathcal{B}(H)$ such that

$$\phi(Y_x) = x \otimes x.$$

Then for any $B \in \mathcal{B}(H)$, we have

$$W_k(Y_x^{r+s+1}) = W_k((x \otimes x)^{r+s+1}) = \left[0, \frac{1}{k}\right]$$

and

$$W_k(Y_x^rBY_x^s + Y_x^sBY_x^r) = W_k((x \otimes x)^r\phi(B)(x \otimes x)^s + (x \otimes x)^s\phi(B)(x \otimes x)^r).$$

Applying Proposition 2.8 to $x\otimes x$, one gets that $W_k((x\otimes x)^r\phi(B)(x\otimes x)^s+(x\otimes x)^s\phi(B)(x\otimes x)^r)$ is a closed line segment which has 0 as an endpoint, and so $W_k(Y_x^rBY_x^s+Y_x^sBY_x^r)$ is also a closed line segment with 0 as an endpoint. Applying Proposition 2.8 again to Y_x, there exists some unit vector $y_x \in H$ such that under the space decomposition $H = [y_x] \oplus [y_x]^\perp$, $Y_x = \lambda_x \oplus Z_x$ with $\lambda_x^{r+s+1} = 1$ and $Z_x^r = Z_x^s = 0$. This yields

$$\begin{aligned} 2\lambda_x^{r+s}\left[0, \frac{1}{k}\right] & = W_k((\lambda_x \oplus Z_x)^rI(\lambda_x \oplus Z_x)^s + (\lambda_x \oplus Z_x)^sI(\lambda_x \oplus Z_x)^r) \\ & = W_k((x \otimes x)^r\phi(I)(x \otimes x)^s + (x \otimes x)^s\phi(I)(x \otimes x)^r) \\ & = 2\langle\phi(I)x, x\rangle W_k(x \otimes x), \end{aligned}$$

which implies $|\langle\phi(I)x, x\rangle| = |\lambda_x^{r+s}| = 1$. Note that $W(\phi(I))$ is convex. It follows that there exists a scalar $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\phi(I) = \lambda I$. As $W_k(I^{r+s+1}) = W_k(\phi(I)^{r+s+1}) = \{1\}$, we get $\lambda^{r+s+1} = 1$.

In addition, one has

$$2W_k(\lambda_x \oplus Z_x) = W_k(I^r(\lambda_x \oplus Z_x)I^s + I^s(\lambda_x \oplus Z_x)I^r) = W_k(2\lambda^{r+s}x \otimes x) \subseteq \lambda^{r+s}\mathbb{R}.$$

This means $\lambda Z_x \in \mathcal{B}_s([y_x]^\perp)$. As $Z_x^r = 0$, we obtain $Z_x = 0$, and so

$$\phi(\lambda_x y_x \otimes y_x) = x \otimes x \quad \text{for all unit vectors } x \in H. \tag{4.6}$$

Now, for any $A \in \mathcal{B}(H)$, one has

$$\begin{aligned}
&2W_k(\lambda_x^{r+s}(y_x \otimes y_x)A(y_x \otimes y_x)) \\
&= W_k((\lambda_x y_x \otimes y_x)^r A(\lambda_x y_x \otimes y_x)^s + (\lambda_x y_x \otimes y_x)^s A(\lambda_x y_x \otimes y_x)^r) \\
&= W_k((x \otimes x)^r \phi(A)(x \otimes x)^s + (x \otimes x)^s \phi(A)(x \otimes x)^r) \\
&= 2W_k((x \otimes x)\phi(A)(x \otimes x)),
\end{aligned}$$

which, together with Proposition 2.5, implies

$$\mathrm{tr}(\lambda_x^{r+s} A y_x \otimes y_x) = \mathrm{tr}(\phi(A)x \otimes x).$$

So, for any $B \in \mathcal{B}(H)$, we have

$$\mathrm{tr}(\lambda_x^{r+s} B y_x \otimes y_x) = \mathrm{tr}(\phi(B)x \otimes x) \quad \text{and} \quad \mathrm{tr}(\lambda_x^{r+s}(A+B) y_x \otimes y_x) = \mathrm{tr}(\phi(A+B)x \otimes x).$$

Combining the above equations gives $\mathrm{tr}((\phi(A) + \phi(B))x \otimes x) = \mathrm{tr}(\phi(A+B)x \otimes x)$, that is, $\langle(\phi(A)+\phi(B))x, x\rangle = \langle\phi(A+B)x, x\rangle$ holds for all unit vectors $x \in H$. It follows that $\phi(A+B) = \phi(A) + \phi(B)$.

Claim 3 If $rs > 0$, then the statement (3) holds.

For any $A \in \mathcal{B}(H)$, by Claim 2, we have

$$2W_k(A) = W_k(I^r A I^s + I^s A I^r) = W_k(\phi(I)^r \phi(A)\phi(I)^s + \phi(I)^s \phi(A)\phi(I)^r) = 2\lambda^{r+s} W_k(\phi(A)).$$

So, by Claims 1–2, the map $\lambda^{r+s}\phi(\cdot)$ is a linear bijective map preserving the k-dimensional numerical range of operators. Based on the result in [21], and by using (4.6), either there exists a unitary operator $U \in \mathcal{B}(H)$ such that $\phi(A) = \lambda U A U^*$ for all $A \in \mathcal{B}(H)$, or there exists a conjugate unitary operator U such that $\phi(A) = \lambda U A^* U^*$ for all $A \in \mathcal{B}(H)$.

Claim 4 If $rs = 0$ and $\dim H \geq 3$, then the statement (3) holds.

Without loss of generality, let $s = 0$. Take any unit vector $x \in H$. We have

$$W_k((x \otimes x)^{r+1}) = W_k(\phi(x \otimes x)^{r+1}) = \left[0, \frac{1}{k}\right].$$

Note that

$$W_k(B^r(x \otimes x) + (x \otimes x)B^r) = W_k(\phi(B)^r \phi(x \otimes x) + \phi(x \otimes x)\phi(B)^r) \text{ for all } B \in \mathcal{B}(H).$$

By the surjectivity of ϕ and Proposition 2.9 twice, one gets

$$\phi(x \otimes x) = \lambda_x y_x \otimes y_x \quad \text{for some unit vector } y_x \in H \text{ and some } \lambda_x \in \mathbb{C} \text{ with } \lambda_x^{r+1} = 1.$$

Now, for any two unit vectors $x_1, x_2 \in H$, there are two unit vectors $y_{x_1}, y_{x_2} \in H$ and scalars $\lambda_{x_1}, \lambda_{x_2} \in \mathbb{C}$ with $\lambda_{x_1}^{r+1} = \lambda_{x_2}^{r+1} = 1$ such that

$$\phi(x_1 \otimes x_1) = \lambda_{x_1} y_{x_1} \otimes y_{x_1} \quad \text{and} \quad \phi(x_2 \otimes x_2) = \lambda_{x_2} y_{x_2} \otimes y_{x_2}.$$

Then

$$\begin{aligned} &W_k((x_1 \otimes x_1)^r (x_2 \otimes x_2) + (x_2 \otimes x_2)(x_1 \otimes x_1)^r) \\ &= \lambda_{x_1}^r \lambda_{x_2} W_k((y_{x_1} \otimes y_{x_1})^r (y_{x_2} \otimes y_{x_2}) + (y_{x_2} \otimes y_{x_2})(y_{x_1} \otimes y_{x_1})^r), \end{aligned}$$

which, together with Proposition 2.1(7), implies

$$|\langle x_1, x_2 \rangle|^2 = \lambda_{x_1}^r \lambda_{x_2} |\langle y_{x_1}, y_{x_2} \rangle|^2. \tag{4.7}$$

It follows that $\langle x_1, x_2 \rangle = 0$ if and only if $\langle y_{x_1}, y_{x_2} \rangle = 0$. Combining this, Claim 1 and Uhlhorn's Theorem ([24]), one obtains that there exists a unitary or conjugate unitary operator U on H and $\theta_x \in \mathbb{C}$ with $|\theta_x| = 1$ such that $y_x = \theta_x U x$. Hence

$$\phi(x \otimes x) = \lambda_x Ux \otimes Ux \quad \text{for all unit vectors } x \in H. \tag{4.8}$$

Furthermore, (4.7) becomes

$$|\langle x_1, x_2 \rangle|^2 = \lambda_{x_1}^r \lambda_{x_2} |\langle Ux_1, Ux_2 \rangle|^2 = \lambda_{x_1}^r \lambda_{x_2} |\langle x_1, x_2 \rangle|^2 \quad \text{for all unit vectors } x_1, x_2 \in H.$$

If $\langle x_1, x_2 \rangle \neq 0$, then $\lambda_{x_1}^r \lambda_{x_2} = 1$, which together with $\lambda_{x_1}^{r+1} = 1$ imply $\lambda_{x_1} = \lambda_{x_2}$. If $\langle x_1, x_2 \rangle = 0$, we can take $x_3 = \alpha x_1 + \beta x_2 \in H$ with $|\alpha|^2 + |\beta|^2 = 1$ and $\alpha\beta \neq 0$. By (4.8), one has $\phi(x_3 \otimes x_3) = \lambda_{x_3} Ux_3 \otimes Ux_3$ for some $\lambda_{x_3} \in \mathbb{C}$ with $|\lambda_{x_3}| = 1$. Note that $\langle x_1, x_3 \rangle \neq 0$ and $\langle x_2, x_3 \rangle \neq 0$. One achieves $\lambda_{x_1} = \lambda_{x_3} = \lambda_{x_2}$.

So far, we have proved that there exists a scalar $\lambda \in \mathbb{C}$ with $\lambda^{r+1} = 1$ and a unitary or conjugate unitary operator U on H such that

$$\phi(x \otimes x) = \lambda Ux \otimes Ux \quad \text{for all unit vectors } x \in H.$$

Then, for any $A \in \mathcal{B}(H)$ and any unit vector $x \in H$, we have

$$W_k((x \otimes x)^r A + A(x \otimes x)^r) = \lambda^r W_k((Ux \otimes Ux)^r \phi(A) + \phi(A)(Ux \otimes Ux)^r).$$

It follows from Proposition 2.1(7) that $\langle Ax, x \rangle = \lambda^r \langle \phi(A)Ux, Ux \rangle$ holds for all unit vectors $x \in H$. If U is unitary, then we have $\phi(A) = \lambda UAU^*$ for all $A \in \mathcal{B}(H)$; if U is conjugate unitary, then one gets $\langle x, A^*x \rangle = \lambda^r \langle x, U^*\phi(A)Ux \rangle$, that is, $\phi(A) = \lambda UA^*U^*$ for all $A \in \mathcal{B}(H)$.

Claim 5 If $rs = 0$ and $\dim H = 2$, then the statement (3) holds.

Still, we assume $s = 0$. In this case, it must be $k = 1$. Note that, for any $A \in \mathcal{B}(H)$ with $\dim H = 2$, $W(A)$ is an ellipse with foci the eigenvalues of A. So, one has

$$\sigma(B^r A + AB^r) = \sigma(\phi(B)^r\phi(A) + \phi(A)\phi(B)^r) \quad \text{for all } A, B \in \mathcal{B}(H).$$

Thus, by [17, Theorem 3.1], there is a scalar $\lambda \in \mathbb{C}$ with $\lambda^{r+1} = 1$ and an invertible operator $T \in \mathcal{B}(H)$ such that either

$$\phi(A) = \lambda TAT^{-1} \quad \text{for all } A \in \mathcal{B}(H), \tag{4.9}$$

or

$$\phi(A) = \lambda TA^*T^{-1} \quad \text{for all } A \in \mathcal{B}(H). \tag{4.10}$$

This means $\phi(I) = \lambda I$, and so

$$\begin{aligned} 2W(A) &= W(I^r A + AI^r) = \lambda^r W(I^r\phi(A) + \phi(A)I^r) \\ &= 2\lambda^r W(\phi(A)) \quad \text{for all } A \in \mathcal{B}(H). \end{aligned} \tag{4.11}$$

If (4.9) holds, then $\lambda^r\phi(\cdot)$ is a linear bijective map satisfying $W(A) = W(\lambda^r\phi(A))$. By [21], and keeping in mind that $\lambda^{r+1} = 1$, there exists either a unitary operator $U \in \mathcal{B}(H)$ such that $\phi(A) = \lambda UAU^*$ for all $A \in \mathcal{B}(H)$, or a conjugate unitary operator U on H such that $\phi(A) = \lambda UA^*U^*$ for all $A \in \mathcal{B}(H)$, as desired.

If (4.10) holds, then (4.11) becomes

$$W(A) = W(TA^*T^{-1}) \quad \text{for all } A \in \mathcal{B}(H), \tag{4.12}$$

which implies $W(x\otimes x) = W(Tx\otimes T^{-1*}x)$ for all unit vectors $x \in H$. Note that $x\otimes x \in \mathcal{B}_s(H)$. So Tx and $T^{-1*}x$ are linearly dependent, and hence $T = \mu T^{-1*}$ for some $\mu \in \mathbb{C}$. Obviously, $TT^* = \mu I$, and thus $\mu > 0$. Write $V = \frac{1}{\sqrt{\mu}}T$. It is clear that $V \in \mathcal{B}(H)$ is a unitary operator. Now, by (4.12), one has $W(A) = W(VA^*V^*) = W(A^*)$ for all $A \in \mathcal{B}(H)$, which is impossible. So (4.10) cannot occur.

The proof of the theorem is completed.

Remark 4.1 Finally, we remark that the surjectivity assumption of maps in Sections 3–4 cannot be removed. For example, let H be an infinite dimensional complex Hilbert space. Define a map $\phi : \mathcal{B}(H) \to \mathcal{B}(H\oplus H)$ by $\phi(A) = A\oplus A$ for every $A \in \mathcal{B}(H)$. It is clear that ϕ is not surjective. Note that $W(X\oplus Y) = \mathrm{conv}\{W(X), W(Y)\}$ for all $X, Y \in \mathcal{B}(H)$. It is obvious that $W(A_1 \circ \cdots \circ A_l) = W(\phi(A_1)\circ\cdots\circ\phi(A_l))$ and $W(A_1 * \cdots * A_l) = W(\phi(A_1) * \cdots * \phi(A_l))$ hold for all $A_1, \cdots, A_l \in \mathcal{B}(H)$. However, ϕ does not have the standard forms represented in theorems.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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