Chin. Ann. Math. Ser. B
47(5), 2026, 803–812
DOI: 10.1007/s11401-026-0006-y

Chinese Annals of Mathematics, Series B

Synchronization Analysis of High Order Layered Complex Networks*

Tianping CHEN[1] Yujuan HAN[2] Wenlian LU[3]

Abstract In this paper, high-order layered complex networks are proposed. The synchronization is discussed in detail. The relations of synchronization, individual coupling matrices and the intrinsic function of the uncoupled system are given. As special cases, the synchronization of monolayer networks and multiplex networks discussed in the literature can be obtained.

Keywords High-order layered network, Linearly coupled systems, Synchronization
2020 MR Subject Classification 93D20, 34D06

1 Introduction

In recent years, synchronization of linearly coupled networks has been an important topic in the research of complex networks. Many models and mathematical concepts have been proposed (see [1–10]). For example, in [3–4], the following model was discussed:

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = f(x_i(t)) + c\sum_{j=1}^{N} l_{ij}x_j(t), \quad i = 1, \cdots, N, \tag{1.1}$$

where $x_i(t) \in \mathbb{R}^n$ is the state variable of the ith node, $f : \mathbb{R}^n \to \mathbb{R}^n$, $L = (l_{ij}) \in \mathbb{R}^{N\times N}$ is the coupling matrix with zero-sum rows and $l_{ij} \geq 0$ for $i \neq j$. The roles played by the intrinsic property of $f(x)$ and the coupling matrix L in synchronization were discussed in [3].

In practice, many networks are interconnected, forming a network of networks, which can be viewed as multi-layer networks (see [2, 10]). A multiplex network is one kind of multi-layer networks that consists of several layers, where different layers have the same number of nodes and possibly different interaction topologies. Each layer can describe a specific type of interaction among nodes. Layers interact with each other only via counterpart nodes across

Manuscript received November 13, 2023. Revised January 17, 2025.
[1]School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: tchen@fudan.edu.cn
[2]Corresponding author. College of Information Engineering, Shanghai Maritime University, Shanghai 201306, China. E-mail: yjhan@shmtu.edu.cn
[3]School of Mathematical Sciences, Fudan University, Shanghai 200433, China; Shanghai Center for Mathematical Sciences, Fudan University, Shanghai 200438, China; Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Shanghai 200433, China.
E-mail: wenlian@fudan.edu.cn
*This work was supported by the China Brain Project (No. 2021ZD0200407) and the National Natural Science Foundation of China (No. 62072111) .

different layers. In [7, 11–17], synchronization of multiplex networks was analyzed. In [11–12, 17], the following model was studied:

$$\frac{\mathrm{d}x_i^k(t)}{\mathrm{d}t} = f(x_i^k(t)) + c\sum_{j=1}^{N} a_{ij}x_j^k(t) + d\sum_{l=1}^{K} b_{kl}x_i^l(t), \quad i = 1, \cdots, N, \ k = 1, \cdots, K, \tag{1.2}$$

where $x_i^k(t) \in \mathbb{R}^n$ is the state of the ith node in the kth layer, $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ is the intra-layer coupling matrix within layers with zero-sum rows and $a_{ij} \geq 0$ for $i \neq j$, $B = (b_{kl}) \in \mathbb{R}^{K\times K}$ represents the inter-layer coupling matrix with zero-sum rows and $b_{kl} \geq 0$ for $k \neq l$, and c and d are coupling strengths.

For multiplex networks, intra-layer and inter-layer synchronization are respectively investigated (see [7, 13–14]). Intra-layer synchronization requires all nodes within the same layer to reach synchronization, while inter-layer synchronization requires that each node evolves synchronously with all its counterparts in other layers.

In this paper, we propose more general multi-layer networks, called high-order layered networks, and study the synchronization of these networks. More precisely, we investigate the following high-order layered coupled networks:

$$\begin{aligned}\frac{\mathrm{d}x_{i_1,\cdots,i_m}(t)}{\mathrm{d}t} =& f(x_{i_1,\cdots,i_m}(t)) + c_1\sum_{j=1}^{I_1} l_{i_1 j}^{(1)} x_{j,i_2,\cdots,i_m}(t)\\ &+\cdots + c_k\sum_{j=1}^{I_k} l_{i_k j}^{(k)} x_{i_1,\cdots,i_{k-1},j,i_{k+1},\cdots,i_m}(t)\\ &+\cdots + c_m\sum_{j=1}^{I_m} l_{i_m j}^{(m)} x_{i_1,\cdots,j}(t).\end{aligned} \tag{1.3}$$

It is clear that models (1.1)–(1.2) are special cases of the general model (1.3). Therefore, the results on model (1.3) can be applied to these special cases.

In this paper, we study the synchronization of high-order layered networks (1.3) and explore how synchronization depends on the eigenvalues of the coupling matrices $L^{(k)}$ and the coupling strengths c_k, $k = 1, \cdots, m$.

This paper is organized as follows. Section 2 introduces the models and gives some basic notations, definitions, and assumptions. Section 3 investigates the synchronization of high-order layered networks in detail. Numerical examples are given in Section 4 to illustrate the effectiveness of the theoretical results. Finally, Section 5 concludes the paper.

2 Model Description and Background

Let us consider a setup in which the dynamics of nodes in high-order layered networks is linearly coupled. The coupling among nodes are assumed to be undirected.

We call complex network (1.1) the 1st order layered network and multiplex network (1.2) the 2nd order layered network.

Notice that each layer of multiplex networks is a 1st order layered network. If we take each layer as a "node" whose own dynamics follows model (1.1) and b_{kl} as the coupling strength between the kth and lth "nodes", then these linearly coupled "nodes" will generate the multiplex network (1.2). This implies that the 2nd order layered network is in fact a network composed of identical 1st order layered networks. For 2nd order layered networks, we call the network topology among 1st order layered networks the 2nd order subnetwork and the network topology of each 1st order layered network the 1st order subnetwork.

Then we can obtain a 3rd order layered network by regarding the 2nd order layered network as a "node" and linearly coupling these "nodes". The network topology among these identical 2nd order layered networks is called the 3rd order subnetwork (see Figure 1).

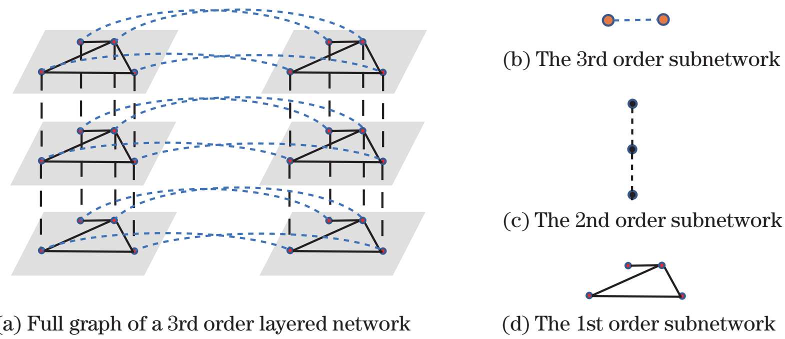

Figure 1 Example of a 3rd order layered network. (a) The full graph of the whole network. (b) The 3rd order subnetwork. (c) The 2nd order subnetwork. (d) The 1st order subnetwork. Nodes in the same size belong to the same order subnetwork and follow the same uncoupled dynamics.

Inductively, we can obtain an mth order layered network. And the kth order subnetwork of the mth order layered network can be defined in sequence, $k = m, \cdots, 1$.

In fact, the mth order layered network defined above is undirected and satisfies the symmetric property that every kth order subnetwork can be regarded as the 1st order subnetwork, $k = 2, \cdots, m$. For instance, if we regard the subnetwork in Figure 1(b) as the 1st order subnetwork, and the subnetwork in Figure 1(d) as the 3rd order subnetwork, then the graph topology of the newly generated 3th order layered network is equivalent to the original graph in Figure 1(a).

Interactions between subnetworks occur because the same nodes participate in multiple subnetworks. These shared nodes allow the propagation of influence, information, or behaviors between subnetworks. Suppose that the mth order layered network has I_k nodes in its kth order subnetwork, $k = 1, \cdots, m$. Denote by $L^{(k)} \in \mathbb{R}^{I_k \times I_k}$ the coupling matrix of the kth order subnetwork with zero-sum rows, i.e., $[L^{(k)}]_{ij} \geq 0$ for $i \neq j$ and $[L^{(k)}]_{ii} = -\sum_{j=1, j\neq i}^{I_k} [L^{(k)}]_{ij}$, and c_k the coupling strength among nodes in the kth order subnetwork.

Next, we will introduce the model of high-order layered networks inductively. Firstly, we consider the 1st order layered network. Denote by $x_{i_1}^{(1)}(t) \in \mathbb{R}^n$ the state of the i_1th node in the 1st order layered network. Then the dynamics of the 1st order layered network satisfies

$$\frac{\mathrm{d}x_{i_1}^{(1)}(t)}{\mathrm{d}t} = f(x_{i_1}^{(1)}(t)) + c_1 \sum_{j=1}^{I_1} l_{i_1,j}^{(1)} x_j^{(1)}(t), \quad i_1 = 1, \cdots, I_1. \tag{2.1}$$

Denote $x^{(1)}(t)=[x_1^{(1)}(t)^{\mathrm{T}},\cdots,x_{I_1}^{(1)}(t)^{\mathrm{T}}]^{\mathrm{T}}$, $f(x^{(1)}(t))=[f(x_1^{(1)}(t))^{\mathrm{T}},\cdots,f(x_{I_1}^{(1)}(t))^{\mathrm{T}}]^{\mathrm{T}}$. Then $x^{(1)}(t)\in\mathbb{R}^{nI_1}$. System (2.1) can be written in the matrix form

$$\frac{\mathrm{d}x^{(1)}(t)}{\mathrm{d}t}=f(x^{(1)}(t))+c_1(L^{(1)}\otimes E_n)x^{(1)}(t), \tag{2.2}$$

where $\otimes$ is the Kronecker product and E_n is the $n\times n$ identity matrix. Consider that the 2nd order layered network is obtained by linearly coupling I_2 identical 1st order layered networks. Denote by $x_{i_2}^{(2)}(t)\in\mathbb{R}^{nI_1}$ the state of the i_2th 1st order layered network, whose uncoupled dynamics follows model (2.2). Hence, the dynamics of the 2nd order layered network satisfies

$$\frac{\mathrm{d}x_{i_2}^{(2)}(t)}{\mathrm{d}t}=f(x_{i_2}^{(2)}(t))+c_1(L^{(1)}\otimes E_n)x_{i_2}^{(2)}(t)+c_2\sum_{j=1}^{I_2}l_{i_2j}^{(2)}x_j^{(2)}(t),\quad i_2=1,2,\cdots,I_2.$$

Denote $x^{(2)}(t)=[x_1^{(2)}(t)^{\mathrm{T}},\cdots,x_{I_2}^{(2)}(t)^{\mathrm{T}}]^{\mathrm{T}}$, $f(x^{(2)}(t))=[f(x_1^{(2)}(t))^{\mathrm{T}},\cdots,f(x_{I_2}^{(2)}(t))^{\mathrm{T}}]^{\mathrm{T}}$. Apparently, $x^{(2)}(t)\in\mathbb{R}^{nI_1I_2}$. Then the dynamics of 2nd order layered networks can be written in the matrix form:

$$\frac{\mathrm{d}x^{(2)}(t)}{\mathrm{d}t}=f(x^{(2)}(t))+c_1(E_{I_2}\otimes L^{(1)}\otimes E_n)x^{(2)}(t)+c_2(L^{(2)}\otimes E_{I_1}\otimes E_n)x^{(2)}(t). \tag{2.3}$$

Inductively, we obtain the dynamics of the mth order layered network as follows:

$$\begin{aligned}\frac{\mathrm{d}x^{(m)}(t)}{\mathrm{d}t}=f(x^{(m)}(t))+\sum_{k=1}^{m}c_k(&E_{I_m}\otimes\cdots\otimes E_{I_{k+1}}\otimes L^{(k)}\\&\otimes E_{I_{k-1}}\cdots\otimes E_{I_1}\otimes E_n)x^{(m)}(t),\end{aligned} \tag{2.4}$$

where $x^{(m)}(t)=[x_1^{(m)}(t)^{\mathrm{T}},\cdots,x_{I_m}^{(m)}(t)^{\mathrm{T}}]^{\mathrm{T}}\in\mathbb{R}^{nI_1\cdots I_m}$.

Notice that the mth order layered network has a total number of $(I_m\cdots I_2I_1)$ nodes in the network (see Figure 1(a)). Let us consider the dynamics of these nodes, which are denoted by $v_{i_1,i_2,\cdots,i_m}$, with i_k being the node index in the kth order subnetwork, $k=1,\cdots,m$. The state of each node $v_{i_1,i_2,\cdots,i_m}$ is represented as $x_{i_1,i_2,\cdots,i_m}\in\mathbb{R}^n$. Apparently, $x_{i_1,i_2,\cdots,i_m}=[\cdots[[x_{i_m}^{(m)}]_{i_{m-1}}]_{i_{m-2}}\cdots]_{i_1}$. Then from (2.4), we get that the dynamics of node $v_{i_1,i_2,\cdots,i_m}$ in the mth order layered network satisfies model (1.3).

Similar to synchronization in multiplex networks, different forms of synchronization phenomenon can emerge in high-order layered networks.

In this paper, synchronization of high-order layered networks is defined as follows.

Definition 2.1 *Consider the* mth *order layered network* (1.3).

(1) *For any* $1\le k\le m$, *network* (1.3) *is said to reach* kth *order synchronization if for any* $1\le p,q\le I_k$,

$$\lim_{t\to\infty}\|x_{i_1,\cdots,i_{k-1},p,i_{k+1},\cdots,i_m}(t)-x_{i_1,\cdots,i_{k-1},q,i_{k+1},\cdots,i_m}(t)\|=0$$

holds for any fixed $i_s=1,\cdots,I_s$, $s=1,\cdots,k-1,k+1,\cdots,m$.

(2) *Network* (1.3) *is said to realize complete synchronization if*

$$\lim_{t\to\infty}\|x_{i_1,i_2,\cdots,i_m}(t)-x_{j_1,j_2,\cdots,j_m}(t)\|=0 \tag{2.5}$$

holds for any $1\le i_k,j_k\le I_k$, $k=1,\cdots,m$.

In other words, the high-order layered network can reach complete synchronization if and only if it can reach every kth order synchronization, $k = 1, \cdots, m$.

Throughout this paper, we suppose f satisfies the following QUAD condition.

Assumption 2.1 The continuous function $f(x,t) : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ is said to satisfy QUAD condition if there exists a constant α such that

$$(x-y)^{\mathrm{T}}[f(x) - f(y)] \le \alpha (x-y)^{\mathrm{T}}(x-y) \tag{2.6}$$

holds for all $x, y \in \mathbb{R}^n$.

The role of the QUAD condition in synchronization of complex networks was investigated in [1, 4]. Many well-known systems, such as the Lorenz system, the Chuas circuit, the Chen system and some cellular neural networks, satisfy the QUAD condition.

Assumption 2.2 All coupling matrices $L^{(k)} = [l_{i,j}^{(k)}] \in \mathbb{R}^{I_k \times I_k}$, $k = 1, \cdots, m$ are symmetric. For any k, the eigenvalues of $L^{(k)}$ are denoted as $0 = \lambda_1^{(k)} \ge \lambda_2^{(k)} \ge \cdots \ge \lambda_{I_k}^{(k)}$.

3 Main Results

In this section, we provide a unified approach to analyse synchronization of the network (1.3). The following theorem reveals how synchronization depends on the eigenvalues of the coupling matrices $L^{(k)}$ and the coupling strengths c_k, $k = 1, \cdots, m$.

Theorem 3.1 *Suppose Assumptions* 2.1–2.2 *are satisfied. For any $k = 1, \cdots, m$, the m*th *order layered network* (1.3) *can reach k*th *order synchronization if $c_k \lambda_2^{(k)} + \alpha < 0$, where α is the constant defined in Assumption* 2.1.

Proof Here we mainly consider the case $m = 3$. The proof for case $m > 3$ can be given in a similar argument. For any fixed $1 \le i_2 \le I_2$ and $1 \le i_3 \le I_3$, we define

$$\overline{x}_{\cdot,i_2,i_3}^{(1)}(t) = \frac{1}{I_1} \sum_{j=1}^{I_1} x_{j,i_2,i_3}(t). \tag{3.1}$$

Then for any function $J(t)$ independents of the index i_1, we have

$$\sum_{i_1,i_2,i_3} [x_{i_1,i_2,i_3}(t) - \overline{x}_{\cdot,i_2,i_3}^{(1)}(t)]^{\mathrm{T}} J(t) = 0.$$

Note that $\frac{\mathrm{d}\overline{x}_{\cdot,i_2,i_3}^{(1)}(t)}{\mathrm{d}t}$ and $f(\overline{x}_{\cdot,i_2,i_3}^{(1)}(t))$ are independent of the index i_1, we have

$$\sum_{i_1,i_2,i_3} [x_{i_1,i_2,i_3}(t) - \overline{x}_{\cdot,i_2,i_3}^{(1)}(t)]^{\mathrm{T}} \frac{\mathrm{d}\overline{x}_{\cdot,i_2,i_3}^{(1)}(t)}{\mathrm{d}t} = 0 \tag{3.2}$$

and

$$\sum_{i_1,i_2,i_3} [x_{i_1,i_2,i_3}(t) - \overline{x}_{\cdot,i_2,i_3}^{(1)}(t)]^{\mathrm{T}} f(\overline{x}_{\cdot,i_2,i_3}^{(1)}(t)) = 0. \tag{3.3}$$

Define a Lyapunov function

$$V(X(t)) = \frac{1}{2} \sum_{i_1,i_2,i_3} \|x_{i_1,i_2,i_3}(t) - \overline{x}^{(1)}_{\cdot,i_2,i_3}(t)\|_2^2. \tag{3.4}$$

Differentiating $V(X(t))$ and by (3.2), we have

$$\begin{aligned}\frac{\mathrm{d}V(X(t))}{\mathrm{d}t} &= \sum_{i_1,i_2,i_3} [x_{i_1,i_2,i_3}(t) - \overline{x}^{(1)}_{\cdot,i_2,i_3}(t)]^{\mathrm{T}} f(x_{i_1,i_2,i_3}(t)) \\ &\quad + c_1 \sum_{i_1,i_2,i_3} \sum_{j=1}^{I_1} [x_{i_1,i_2,i_3}(t) - \overline{x}^{(1)}_{\cdot,i_2,i_3}(t)]^{\mathrm{T}} l^{(1)}_{i_1 j} x_{j,i_2,i_3}(t) \\ &\quad + c_2 \sum_{i_1,i_2,i_3} \sum_{j=1}^{I_2} [x_{i_1,i_2,i_3}(t) - \overline{x}^{(1)}_{\cdot,i_2,i_3}(t)]^{\mathrm{T}} l^{(2)}_{i_2 j} x_{i_1,j,i_3}(t) \\ &\quad + c_3 \sum_{i_1,i_2,i_3} \sum_{j=1}^{I_3} [x_{i_1,i_2,i_3}(t) - \overline{x}^{(1)}_{\cdot,i_2,i_3}(t)]^{\mathrm{T}} l^{(3)}_{i_3 j} x_{i_1,i_2,j}(t) \\ &= V_1(t) + V_2(t) + V_3(t) + V_4(t). \end{aligned} \tag{3.5}$$

From (3.3) and Assumption 2.1, we have

$$V_1(t) \le \alpha \sum_{i_1,i_2,i_3} \|x_{i_1,i_2,i_3}(t) - \overline{x}^{(1)}_{\cdot,i_2,i_3}(t)\|_2^2. \tag{3.6}$$

By Assumption 2.2, $L^{(1)}$ is symmetric and has zero-sum rows, then we have

$$\begin{aligned} V_2(t) &= c_1 \sum_{i_1,i_2,i_3} \sum_{j=1}^{I_1} l^{(1)}_{i_1 j} [x_{i_1,i_2,i_3}(t) - \overline{x}^{(1)}_{\cdot,i_2,i_3}(t)]^{\mathrm{T}} \cdot [x_{j,i_2,i_3}(t) - \overline{x}^{(1)}_{\cdot,i_2,i_3}(t)] \\ &\le c_1 \lambda_2^{(1)} \sum_{i_1,i_2,i_3} \|x_{i_1,i_2,i_3}(t) - \overline{x}^{(1)}_{\cdot,i_2,i_3}(t)\|_2^2. \end{aligned} \tag{3.7}$$

Because $l^{(2)}_{i_2 j} \overline{x}^{(1)}_{\cdot,j,i_3}(t)$ is independent of i_1 and $\sum_{i_1} [x_{i_1,i_2,i_3}(t) - \overline{x}^{(1)}_{\cdot,i_2,i_3}(t)] = 0$, we have

$$\sum_{i_1,i_2,i_3} \sum_{j=1}^{I_2} [x_{i_1,i_2,i_3}(t) - \overline{x}^{(1)}_{\cdot,i_2,i_3}(t)]^{\mathrm{T}} l^{(2)}_{i_2 j} \overline{x}^{(1)}_{\cdot,j,i_3}(t) = 0. \tag{3.8}$$

Let $\delta x^{(1)}_{i_1,i_2,i_3}(t) = x_{i_1,i_2,i_3}(t) - \overline{x}^{(1)}_{\cdot,i_2,i_3}(t)$. Then by (3.8) and Assumption 2.2, $L^{(2)}$ is symmetric and semi-negative, and we have

$$\begin{aligned} V_3(t) &= c_2 \sum_{i_1,i_2,i_3} \sum_{j=1}^{I_2} \delta x^{(1)}_{i_1,i_2,i_3}(t)^{\mathrm{T}} l^{(2)}_{i_2 j} [x_{i_1,j,i_3}(t) - \overline{x}^{(1)}_{\cdot,j,i_3}(t)] \\ &= c_2 \sum_{i_1,i_3} \sum_{i_2,j} \delta x^{(1)}_{i_1,i_2,i_3}(t)^{\mathrm{T}} l^{(2)}_{i_2 j} \delta x^{(1)}_{i_1,j,i_3}(t) \le 0. \end{aligned} \tag{3.9}$$

Similarly, we can derive $V_4(t) \le 0$. Therefore,

$$\frac{\mathrm{d}V(X(t))}{\mathrm{d}t} \le 2(c_1 \lambda_2^{(1)} + \alpha) V(t), \tag{3.10}$$

which implies for any fixed $1 \le i_2 \le I_2$ and $1 \le i_3 \le I_3$,

$$\|x_{p,i_2,i_3}(t) - x_{q,i_2,i_3}(t)\| = O(\mathrm{e}^{2(c_1\lambda_2^{(1)}+\alpha)t}), \quad 1 \le p, q \le I_1, \tag{3.11}$$

i.e., the 1st order synchronization can be realized if $c_1\lambda_2^{(1)} + \alpha < 0$. Similarly, we can derive that for any fixed $1 \le i_1 \le I_1$ and $1 \le i_3 \le I_3$,

$$\|x_{i_1,p,i_3}(t) - x_{i_1,q,i_3}(t)\| = O(\mathrm{e}^{2(c_2\lambda_2^{(2)}+\alpha)t}), \quad 1 \le p, q \le I_2 \tag{3.12}$$

and for any fixed $1 \le i_1 \le I_1$ and $1 \le i_2 \le I_2$,

$$\|x_{i_1,i_2,p}(t) - x_{i_1,i_2,q}(t)\| = O(\mathrm{e}^{2(c_3\lambda_2^{(3)}+\alpha)t}), \quad 1 \le p, q \le I_3. \tag{3.13}$$

From (3.11)–(3.13), we have that the kth order synchronization can be realized if $c_k\lambda_2^{(k)}+\alpha < 0$, $k = 1, 2, 3$. This completes the proof.

Combining (3.11)–(3.13), we have that for any $1 \le i_k \le I_k$ and $1 \le j_k \le I_k$, $k = 1, 2, 3$,

$$\|x_{i_1,i_2,i_3}(t) - x_{j_1,j_2,j_3}(t)\| = O(\mathrm{e}^{\max\limits_k 2(c_k\lambda_2^{(k)}+\alpha)t}). \tag{3.14}$$

This implies that complete synchronization of the 3rd order layered network can be realized if $\max\limits_{k=1,2,3}(c_k\lambda_2^{(k)} + \alpha) < 0$, which leads to the following corollary.

Corollary 3.1 *Suppose Assumptions* 2.1–2.2 *are satisfied. Then the* mth *order layered network* (1.3) *will realize complete synchronization if* $\max\limits_{k=1,\cdots,m}(c_k\lambda_2^{(k)} + \alpha) < 0$.

In fact, if we consider the case $m = 2$, model (1.3) turns to be the multiplex network model (1.2) with $L^{(1)} = A$ and $L^{(2)} = B$. Then, as a special case of Theorem 3.1, we obtain the results on multiplex networks with identical layers given in [17].

Corollary 3.2 (see [17, Theorem 1]) *Consider the multiplex network* (1.2)*. Suppose Assumption* 2.1 *is satisfied and coupling matrices A and B are symmetric. The eigenvalues of A are $0 = \mu_1 \ge \mu_2 \ge \cdots \ge \mu_N$. The eigenvalues of B are $0 = \nu_1 \ge \nu_2 \ge \cdots \ge \nu_K$. Then the multiplex network will realize* (i) *intra-layer synchronization if $c\mu_2 + \alpha < 0$,* (ii) *inter-layer synchronization if $d\nu_2 + \alpha < 0$, and* (iii) *complete synchronization if* $\max\{c\mu_2, d\nu_2\} + \alpha < 0$.

4 Numerical Simulations

In this section, we give some numerical examples to verify the effectiveness of theoretical results given in previous sections.

In these examples, the network is a 3rd order layered network whose full graph is given in Figure 1(a).

Denote by $x_{i_1,i_2,i_3}(t)$ the state of node v_{i_1,i_2,i_3} at time t, where $1 \le i_1 \le 4$, $1 \le i_2 \le 3$, $1 \le i_3 \le 2$. The intrinsic dynamics of each node is the Lorenz system described by

$$\dot{s}(t) = f(s(t)) = \begin{pmatrix} 10(s^2(t) - s^1(t)) \\ 28s^1(t) - s^2(t) - s^1(t)s^3(t) \\ s^1(t)s^2(t) - \dfrac{8s^3(t)}{3} \end{pmatrix}. \tag{4.1}$$

Define qualities $E_1(t)$, $E_2(t)$ and $E_3(t)$ for the 1st order, 2nd order and 3rd order synchronization errors, respectively, where

$$\begin{aligned} E_1(t) &= \sum_{p<q}\sum_{i_2=1}^{3}\sum_{i_3=1}^{2} \|x_{p,i_2,i_3}(t) - x_{q,i_2,i_3}(t)\|_2^2, \\ E_2(t) &= \sum_{i_1=1}^{4}\sum_{p<q}\sum_{i_3=1}^{2} \|x_{i_1,p,i_3}(t) - x_{i_1,q,i_3}(t)\|_2^2, \\ E_3(t) &= \sum_{i_1=1}^{4}\sum_{i_2=1}^{3} \|x_{i_1,i_2,1}(t) - x_{i_1,i_2,2}(t)\|_2^2. \end{aligned} \tag{4.2}$$

As for the coupling matrices, we pick

$$\begin{aligned} L^{(1)} &= \begin{pmatrix} -3 & 3 & 0 & 0 \\ 3 & -6 & 2 & 1 \\ 0 & 2 & -4 & 2 \\ 0 & 1 & 2 & -3 \end{pmatrix}, \\ L^{(2)} &= \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \\ L^{(3)} &= \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}. \end{aligned} \tag{4.3}$$

The largest nonzero eigenvalues of $L^{(1)}, L^{(2)}$ and $L^{(3)}$ are $\lambda_2^{(1)} = -2.0905$, $\lambda_2^{(2)} = -1$, $\lambda_2^{(3)} = -4$.

Example 4.1 In this example, the initial state of each node in the network is randomly chosen in $[0, 1]$. The coupling coefficients are $c_1 = 1$, $c_2 = 1$ and $c_3 = 1$.

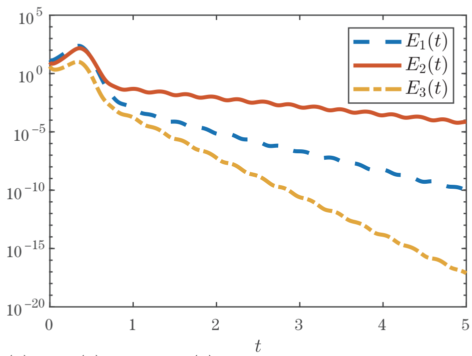

Figure 2 Dynamics of $E_1(t)$, $E_2(t)$ and $E_3(t)$ with respect to time t for the linearly coupled high order layered network (1.3).

In Figure 2, we plot the dynamics of $E_1(t)$, $E_2(t)$ and $E_3(t)$ with respect to time, which shows that $E_3(t)$ converges to zero fastest and $E_2(t)$ converges slowest. This numerical result coincides with the fact that $c_3\lambda_2^{(3)} \le c_1\lambda_2^{(1)} \le c_2\lambda_2^{(2)} < 0$.

From Figure 3(a), it can be seen that the larger c_1 is, the faster $E_1(t)$ converges to zero. Figure 3(b) shows that the convergence rate of $E_1(t)$ scales linearly with respect to c_1, which coincides with the estimation (3.11). That is, the 1st order synchronization depends on the coupling coefficient c_1.

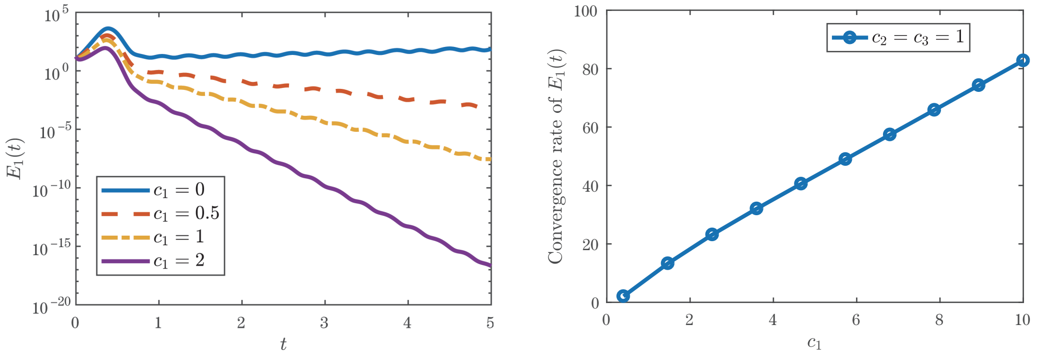

Figure 3 (a) Dynamics of $E_1(t)$ under different values of c_1. (b) The convergence rate of $E_1(t)$ with respect to c_1.

Example 4.2 In this example, we want to reveal the relevance between synchronization and coupling coefficients. For brevity, we only present the results for coupling coefficient c_1 since the results for the other two coefficients are similar. The parameters that are kept fixed are $c_2 = 1$, $c_3 = 1$.

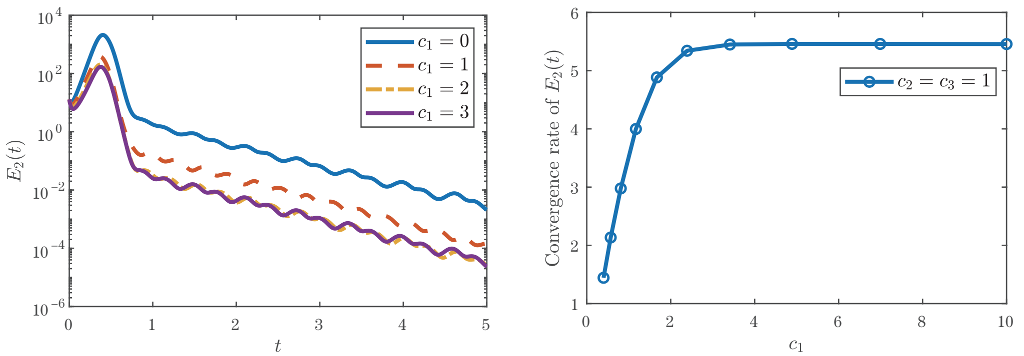

Figure 4 (a) Dynamics of $E_2(t)$ under different values of c_1. (b) The convergence rate of $E_2(t)$ with respect to c_1.

In comparison with $E_1(t)$, dynamics of $E_2(t)$ and $E_3(t)$ give different results. Here we only present the results on $E_2(t)$ since dynamics of $E_2(t)$ and $E_3(t)$ give the similar results. From Figure 4, it can be seen that as c_1 increases, the convergence rate of $E_2(t)$ increases distinctly at first, then rarely changes for $c_1 > 2$. This is because strengthening interactions among nodes in the 1st order subnetwork enhances the 2nd order synchronization. But the enhancement is limited since the 2th order synchronization is mainly governed by the value of c_2 and $\lambda_2^{(2)}$.

5 Conclusions

In this paper, we investigate the synchronization of the high-order layered network (1.3) and explore the relations between synchronization and the eigenvalues of the coupling matrices. An effective theoretical analysis is provided. Numerical examples confirm our analytic results.

Declarations

Conflicts of interest The authors declare no conflicts of interest.

References

[1] Delellis, P., Bernardo, M. D. and Russo, G., On QUAD, Lipschitz, and contracting vector fields for consensus and synchronization of networks, *IEEE Trans. Circuits Syst. I Regul. Pap.*, **58**(3), 2011, 576–583.

[2] Gao, J., Buldyrev, S. V., Stanley, H. E. and Havlin, S., Networks formed from interdependent networks, *Nat. Phys.*, **8**, 2012, 40–48.

[3] Lu, W. and Chen, T., New approach to synchronization analysis of linearly coupled ordinary differential systems, *Physica D*, **213**(2), 2006, 214–230.

[4] Lu, W. and Chen, T., QUAD-condition, synchronization, consensus of multiagents, and anti-synchronization of complex networks, *IEEE Trans. Cybern.*, **51**(6), 2021, 3384–3388.

[5] Pecora, L., Carroll, T., Johnson, G., et al., Synchronization stability in coupled oscillator arrays: solution for arbitrary configuration, *Int. J. Bifurcat. Chaos*, **10**(2), 2000, 273–290.

[6] Pecora, L. M. and Carroll, T. L., Synchronization in chaotic systems, *Phys. Rev. Lett.*, **64**(8), 1990, 821–824.

[7] Sevilla-Escoboza, R., Sendiña-Nadal, I., Leyva, I., et al., Inter-layer synchronization in multiplex networks of identical layers, *Chaos*, **26**, 2016, 065304.

[8] Wu, C. W. and Chua, L. O., Synchronization in an array of linearly coupled dynamical systems, *IEEE Trans. Circuits Syst. I. Fundam. Theory Appl.*, **42**(8), 1995, 430–447.

[9] Wu, C. W., Synchronization in networks of nonlinear dynamical systems coupled via a directed graph, *Nonlinearity*, **18**(3), 2005, 1057–1064.

[10] Wu, D., Tang, M., Liu, Z. and Lai, Y., Impact of inter-layer hopping on epidemic spreading in a multilayer network, *Commun. Nonlinear Sci. Numer. Simul.*, **90**, 2020, 105403.

[11] Deng, Y., Jia, Z., Deng, G. and Zhang, Q., Eigenvalue spectrum and synchronizability of multiplex chain networks, *Physica A*, **537**, 2020, 122631.

[12] Khalaf, A. J. M., Alsaadi, F. E., Alsaadi, F. E., et al., Synchronization in a multiplex network of gene oscillators, *Phys. Lett. A*, **383**(31), 2019, 125919.

[13] Bera, B. K., Rakshit, S. and Ghosh, D., Intralayer synchronization in neuronal multiplex network, *Eur. Phys. J. Spec. Top.*, **228**, 2019, 2441–2454.

[14] Tang, L., Wu, X., Lü, J., et al., Master stability functions for complete, intra-layer and inter-layer synchronization in multiplex networks, *Phys. Rev. E*, **99**, 2019, 012304.

[15] Gómez, S., Díaz-Guilera, A., Gómez-Gardeñes, J., et al., Diffusion dynamics on multiplex networks, *Phys. Rev. Lett.*, **110**, 2013, 028701.

[16] Xu, M., Zhou, J., Lu, J. and Wu, X., Synchronizability of two-layer networks, *Eur. Phys. J. B*, **88**, 2015, 240.

[17] Chen, T., Synchronization of multi-cluster complex networks, *Neural Netw.*, **156**, 2022, 239–243.