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Schwarz Lemma at the Boundary for Holomorphic Mappings Between Non-equidimensional Unit Polydiscs*

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Abstract In this paper, the authors prove a Schwarz lemma at the boundary for holomorphic mappings between non-equidimensional unit polydiscs and consider an inequality for holomorphic self-mappings of the unit polydisc.

Keywords Schwarz lemma at the boundary, Unit polydisc, Holomorphic mappings
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1 Introduction

Let D be the unit disk in the complex plane $\mathbb{C}$. We denote the n-dimensional complex linear space by $\mathbb{C}^n$, whose inner product is defined as $\langle z, w\rangle = \sum\limits_{j=1}^{n} z_j\overline{w_j}$, where $z = (z_1, \cdots, z_n)^{\mathrm{T}}, w = (w_1, \cdots, w_n)^{\mathrm{T}} \in \mathbb{C}^n$. If $\|z\| = \langle z, z\rangle^{\frac{1}{2}}$ and $\|z\|_\infty = \max\limits_{1\le j\le n} |z_j|$ are defined on $\mathbb{C}^n$, then $\|\cdot\|$ and $\|\cdot\|_\infty$ are both norms on $\mathbb{C}^n$ and topologically equivalent. Let $B^n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ be the unit ball in $\mathbb{C}^n$ and its boundary is $\partial B^n = \{z \in \mathbb{C}^n : \|z\| = 1\}$. Denote the unit polydisc in $\mathbb{C}^n$ as $D^n = \{z \in \mathbb{C}^n : \|z\|_\infty < 1\}$. Let $\partial D^n = \{z \in \mathbb{C}^n : \|z\|_\infty = 1\}$ and $T^n = \{z \in \mathbb{C}^n : |z_j| = 1, 1 \le j \le n\}$ denote the topological boundary and characteristic boundary of D^n, respectively.

In this paper, the symbol $H(\Omega_1, \Omega_2)$ is used to represent the set of holomorphic mappings from $\Omega_1 \subset \mathbb{C}^n$ into $\Omega_2 \subset \mathbb{C}^m$. For any $f = (f_1, \cdots, f_m)^{\mathrm{T}} \in H(\Omega_1, \Omega_2)$, the complex Jacobian matrix of f at point $a \in \Omega_1$ is defined as $J_f(a) = \big(\frac{\partial f_j}{\partial z_k}(a)\big)_{m\times n}$.

For a bounded domain V in $\mathbb{C}^n$, $C^\alpha(V)$ denotes the set of all functions f defined on V such that for $0 < \alpha < 1$, the function f satisfies

$$\sup\Big\{\frac{|f(z) - f(z')|}{|z - z'|^\alpha} : z, z' \in \overline{V}, z \neq z'\Big\} < \infty.$$

$C^{k+\alpha}(V)$ is the set of all functions f on V whose k-th order partial derivatives exist and belong to $C^\alpha(V)$ for an integer $k \ge 0$.

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The boundary Schwarz lemma in complex analysis is a hot topic. It provides a powerful tool for the study of geometric function theory. Garnett first proposed the following classical boundary Schwarz lemma.

Theorem 1.1 (cf. [1]) *Let $f : D \to D$ be a holomorphic function. If f is holomorphic at $z = 1$ with $f(0) = 0$ and $f(1) = 1$, then $f'(1) \geq 1$. Moreover, $f'(1) = 1$ if and only if $f(z) \equiv z$.*

In recent years, since Garnett's work, based on different domains and mappings, many authors have studied different versions of the boundary Schwarz lemma and extended the above theorem to higher dimensions. For further details, please refer to references [2–5, 8–9, 13, 15–16, 19] and the citations contained therein.

More recently, Tang et al. proved the boundary Schwarz lemma of holomorphic self-mappings of the unit polydisc and considered the properties of the Jacobian matrix at boundary points in [18]. This result is stated as follows.

Theorem 1.2 (cf. [7]) *Suppose that $p = (p_1, \cdots, p_\beta, p_{\beta+1}, \cdots, p_n)^{\mathrm{T}} \in \partial D^n$ with $|p_1| = \cdots = |p_\beta| = 1$ and $|p_{\beta+1}| < 1, \cdots, |p_n| < 1$ and $w = (w_1, \cdots, w_\alpha, w_{\alpha+1}, \cdots, w_n)^{\mathrm{T}} \in \partial D^n$ with $|w_1| = \cdots = |w_\alpha| = 1$ and $|w_{\alpha+1}| < 1, \cdots, |w_n| < 1$. If $f : D^n \to D^n$ is a holomorphic mapping and f is holomorphic at p with $f(p) = w$, then the following four statements hold:*

(i) $\lambda_i = \sum\limits_{j=1}^{n} \overline{w_i} \frac{\partial f_i}{\partial z_j}(p) p_j \geq \frac{|1-\overline{w_i} f_i(0)|^2}{1-|\overline{w_i} f_i(0)|^2} > 0$ *for* $i = 1, \cdots, \alpha$.

(ii) *If $1 \leq i \leq \alpha$, then $\frac{\partial f_i}{\partial z_j}(p) = |\frac{\partial f_i}{\partial z_j}(p)| w_i \overline{p_j}$ for $j = 1, \cdots, \beta$; $\frac{\partial f_i}{\partial z_j}(p) = 0$ for $j = \beta+1, \cdots, n$; $\sum\limits_{j=1}^{\beta} |\frac{\partial f_i}{\partial z_j}(p)| = \lambda_i$.*

(iii) *If $\alpha + 1 \leq i \leq n$, then*

$$\frac{1}{1-|w_i|^2} \sum_{j=\beta+1}^{n} \Big|\frac{\partial f_i}{\partial z_j}(p)\Big| \leq \max_{\beta+1 \leq j \leq n} \frac{1}{1-|p_j|^2}.$$

(iv) *Suppose that $\mu_{\beta+1}, \cdots, \mu_n$ are all the eigenvalues of*

$$\Big(\frac{\partial f_i}{\partial z_j}(p)\Big)_{i,j=\beta+1,\cdots,n} \quad and \quad (\xi_{\beta+1}, \cdots, \xi_n)^{\mathrm{T}}$$

is a nonzero eigenvector of $\big(\frac{\partial f_i}{\partial z_j}(p)\big)_{i,j=\beta+1,\cdots,n}$ with respect to $\mu_i, i = \beta+1, \cdots, n$. If $\alpha \leq \beta$, then

$$|\mu_i| \max_{\beta+1 \leq j \leq n} \frac{|\xi_j|}{1-|w_j|^2} \leq \max_{\beta+1 \leq j \leq n} \frac{|\xi_j|}{1-|p_j|^2};$$

if $\alpha > \beta$, then

$$|\mu_i| \max_{\alpha+1 \leq j \leq n} \frac{|\xi_j|}{1-|w_j|^2} \leq \max_{\alpha+1 \leq j \leq n} \frac{|\xi_j|}{1-|p_j|^2}$$

and there are at least $(\alpha - \beta)$ zeros in $\mu_{\beta+1}, \cdots, \mu_n$.

In [14], Liu et al. proved the boundary Schwarz lemma for holomorphic mappings from the unit polydisc to the unit ball, which is the following result.

Theorem 1.3 (cf. [8]) *Let $f \in H(D^n, B^N)$ for any $n, N \geq 1$. Given $z_0 \in \partial D^n$, assume $z_0 \in E_r$ with the first r components at the boundary of D for some $1 \leq r \leq n$. If f is $C^{1+\alpha}$ at*

z_0 and $f(z_0) = w_0 \in \partial B^N$, then there exists a sequence of nonnegative real numbers $\gamma_1, \cdots, \gamma_r$ satisfying $\sum\limits_{j=1}^{r} \gamma_j \geq 1$ and $\lambda \in \mathbb{R}$ such that

$$\overline{J_f(z_0)}^{\mathrm{T}} w_0 = \lambda \mathrm{diag}(\gamma_1, \cdots, \gamma_r, 0, \cdots, 0) z_0,$$

where $\lambda = \frac{|1-\overline{a}^{\mathrm{T}} w_0|^2}{1-\|a\|^2} > 0$, $a = f(0)$ and diag *represents the diagonal matrix.*

As a generalization of Theorem 1.1, Liu et al. in [12] obtained a boundary rigidity theorem for holomorphic self-mappings of the unit polydisc.

Theorem 1.4 (cf. [14]) *Let $f : D^n \to D^n$ be a holomorphic mapping with $f(0) = 0$. Assume f is holomorphic at $\alpha_k \in T^n$ and $f(\alpha_k) \in T^n$ for $k = 1, \cdots, n$. Then*

$$\overline{f(\alpha_k)}^{\mathrm{T}} J_f(\alpha_k) \alpha_k \geq n, \quad k = 1, \cdots, n.$$

Moreover, if $\alpha_1, \cdots, \alpha_n$ are linearly independent and $f(\alpha_1), \cdots, f(\alpha_n)$ are linearly independent as well, then the n equalities above hold if and only if $f(z) \equiv J_f(0)z$, where $J_f(0)$ has exactly one complex unit in each row and each column and zeros elsewhere.

In this paper, inspired by [14, 20], we first prove a boundary Schwarz lemma for holomorphic mappings between non-equidimensional unit polydiscs, which is a meaningful addition to Theorem 1.2. Then we consider an inequality for holomorphic self-mappings of the unit polydisc. As an application, we establish another new version of the boundary Schwarz lemma on the unit polydisc, which is a generalization of [17].

2 Some Lemmas

In this section, we introduce two lemmas, which will be used in the following proof.

Lemma 2.1 (cf. [15]) *Let f be a holomorphic self-mapping of D. If f is holomorphic at $z = 1$ with $f(1) = 1$, then*

$$f'(1) \geq \frac{2|1 - f(0)|^2}{1 - |f(0)|^2 + |f'(0)|}.$$

The Kobayashi metric has many very nice properties. For example, the Kobayashi metric has the property of maintaining stability in complex geometric structures (cf. [7]), which provides a fundamental understanding for the proof of the boundary Schwarz lemma in Lemma 2.2. Many mathematicians have studied the boundary Schwarz lemma using the properties of the Kobayashi metric (cf. [2, 6–7, 10–11, 13]). In this paper, the non-increasing property of the Kobayashi distance defined on the basis of the Kobayashi metric is proved.

Lemma 2.2 (cf. [15]) *Let f be a holomorphic self-mapping of B^n. If f is holomorphic at $z = \alpha \in \partial B^n$ with $f(\alpha) = \beta \in \partial B^n$, then*

$$\overline{\beta}^{\mathrm{T}} J_f(\alpha) \alpha \geq \frac{2|1 - \overline{\beta}^{\mathrm{T}} f(0)|^2}{1 - |\overline{\beta}^{\mathrm{T}} f(0)|^2 + \|J_f(0)\|},$$

where $\|J_f(0)\|$ is the norm of $J_f(0)$ and the norm $\|\cdot\|$ is defined in such a way that, for an $n \times n$ complex matrix A, the operator norm $\|A\| = \sup\limits_{z \neq 0} \frac{\|Az\|}{\|z\|} = \max\{\|A\theta\| : \theta \in \partial B^n\}$.

3 Main Results

We now establish the Schwarz lemma at the boundary for holomorphic mappings of the unit polydisc.

Theorem 3.1 *Let $f \in H(D^n, D^N)$ with $f(0) = 0$ for any $n, N \geq 1$. Given $z_0 \in \partial D^n$, assume $z_0 \in E_{r_1}$, where E_{r_1} indicates that for $1 \leq r_1 \leq n$, the first r_1 components of z_0 are at the boundary of D. Similarly, E_{r_2} means that for $1 \leq r_2 \leq N$, the first r_2 components of w_0 are at the boundary of D. Then there exists a sequence of nonnegative real numbers $\lambda_1, \cdots, \lambda_{r_1}$ such that the following statements hold*:

(i) $\overline{J_f(z_0)}^{\mathrm{T}} w_0 = \mathrm{diag}(\lambda_1, \cdots, \lambda_{r_1}, 0, \cdots, 0) z_0$, *where* diag *represents the $n \times n$ diagonal matrix.*

(ii) $\mathrm{tr}(\mathrm{diag}(\lambda_1, \cdots, \lambda_{r_1}, 0, \cdots, 0)) \geq r_2$.

Proof We claim that for any $n, N \geq 1$, if $f \in H(D^n, D^N)$ with $f(0) = 0$, then $\|f(w)\|_\infty \leq \|w\|_\infty$, where $w \in D^n$. Due to the fact that for any $z, w \in D^n$, the Kobayashi distance of the unit polydisc is $K_{D^n}(z, w) = \frac{1}{2}\log\frac{1+\|\phi_z(w)\|_\infty}{1-\|\phi_z(w)\|_\infty}$, where $\phi_z(w)$ is an automorphism of D^n. Taking into account the property of the Kobayashi distance contraction under holomorphic mappings, this implies that $K_{D^N}(0, f(w)) \leq K_{D^n}(0, w)$. Hence

$$\frac{1}{2}\log\frac{1 + \|f(w)\|_\infty}{1 - \|f(w)\|_\infty} \leq \frac{1}{2}\log\frac{1 + \|w\|_\infty}{1 - \|w\|_\infty}.$$

It is obvious that $t \to \frac{1}{2}\log\frac{1+t}{1-t}$ is increasing for $t \in [0, 1)$. Therefore, $\|f(w)\|_\infty \leq \|w\|_\infty$.

Now, we prove the Schwarz lemma at the boundary for the unit polydisc in any dimension. Let e_i^n and e_s^N represent the i-th column of the identity matrix I_n and the s-th column of the identity matrix I_N, respectively.

Step 1 Without loss of generality, we assume that $z_0 = \sum\limits_{i=1}^{r_1} e_i^n \in E_{r_1}$ and the holomorphic mapping f in the neighborhood V of z_0 is $C^{1+\alpha}$ with $f(z_0) = w_0 = \sum\limits_{s=1}^{r_2} e_s^N \in E_{r_2}$.

Let $p = z_0, q_l = -\sum\limits_{i=1}^{r_1} e_i^n + \mathrm{i}ke_l^n$ for $1 \leq l \leq r_1$ and $k \in \mathbb{R}$, where i represents the imaginary unit. Then $p + tq_l = (1 - t)z_0 + \mathrm{i}kte_l^n$ for $t \in \mathbb{R}$. Note that $\|p + tq_l\|_\infty < 1 \Leftrightarrow |1 - t + \mathrm{i}kt| < 1$ and $|1 - t| < 1 \Leftrightarrow 0 < t < \frac{2}{1+k^2}$, as $t \to 0^+$, for any given $k \in \mathbb{R}$, $p + tq_l \in D^n \cap V$. For t that satisfies the above conditions, the Taylor expansion of $f(p + tq_l)$ at $t = 0$ is

$$f(p + tq_l) = w_0 + J_f(z_0)q_l t + O(t^{1+\alpha}).$$

Then

$$\|f(p + tq_l)\|_\infty = \|w_0 + J_f(z_0)q_l t + O(t^{1+\alpha})\|_\infty \leq \|p + tq_l\|_\infty.$$

This implies

$$\max_{1\leq j\leq N} |f_j(p + tq_l)|^2 \leq (\|p + tq_l\|_\infty)^2.$$

Through calculation, it can be concluded that

$$(\|p + tq_l\|_\infty)^2 = |1 - t + \mathrm{i}kt|^2 = 1 - 2t + O(t^2)$$

or

$$(\|p + tq_l\|_\infty)^2 = |1 - t|^2 = 1 - 2t + O(t^2).$$

Then

$$\max_{1\le j\le N}|(w_0)_j + J_{f_j}(z_0)q_l t + O(t^{1+\alpha})|^2 \le 1-2t+O(t^2).$$

Substituting $w_0 = \sum_{s=1}^{r_2} e_s^N$, we have

$$\max_{1\le j\le r_2}\{1+2\mathrm{Re}J_{f_j}(z_0)q_l t + O(t^{1+\alpha})\} \le 1-2t+O(t^2).$$

Let $t\to 0^+$, then

$$\max_{1\le j\le r_2}\{\mathrm{Re}J_{f_j}(z_0)q_l\} \le -1. \tag{3.1}$$

Substituting $q_l = -\sum_{i=1}^{r_1} e_i^n + \mathrm{i}ke_l^n$ into (3.1), we get

$$\max_{1\le j\le r_2}\Big\{\mathrm{Re}J_{f_j}(z_0)\Big(-\sum_{i=1}^{r_1} e_i^n + \mathrm{i}ke_l^n\Big)\Big\} \le -1.$$

That is

$$\max_{1\le j\le r_2}\Big\{\mathrm{Re}\Big(-\sum_{i=1}^{r_1}\frac{\partial f_j(z_0)}{\partial z_i} + \mathrm{i}k\frac{\partial f_j(z_0)}{\partial z_l}\Big)\Big\} \le -1. \tag{3.2}$$

Since $\frac{\partial f_j(z_0)}{\partial z_l} = \mathrm{Re}\frac{\partial f_j(z_0)}{\partial z_l} + \mathrm{i}\mathrm{Im}\frac{\partial f_j(z_0)}{\partial z_l}$, (3.2) is equivalent to

$$\max_{1\le j\le r_2}\Big\{\mathrm{Re}\Big(-\sum_{i=1}^{r_1}\frac{\partial f_j(z_0)}{\partial z_i}\Big) - k\mathrm{Im}\frac{\partial f_j(z_0)}{\partial z_l}\Big\} \le -1.$$

That is

$$\min_{1\le j\le r_2}\Big\{\mathrm{Re}\sum_{i=1}^{r_1}\frac{\partial f_j(z_0)}{\partial z_i} + k\mathrm{Im}\frac{\partial f_j(z_0)}{\partial z_l}\Big\} \ge 1. \tag{3.3}$$

As (3.3) is valid for any $k\in\mathbb{R}$, it holds that $\mathrm{Im}\frac{\partial f_j(z_0)}{\partial z_l} = 0, 1\le l\le r_1$.

Therefore, $\frac{\partial f_j(z_0)}{\partial z_l} = \mathrm{Re}\frac{\partial f_j(z_0)}{\partial z_l}, 1\le l\le r_1$ and $\min_{1\le j\le r_2}\big\{\mathrm{Re}\sum_{i=1}^{r_1}\frac{\partial f_j(z_0)}{\partial z_i}\big\} \ge 1$.

Step 2 Let $p = z_0, q_l = -\sum_{i=1}^{r_1} e_i^n + ke_l^n$ for $1\le l\le r_1$ and $k\le 0$. Then $p+tq_l = (1-t)z_0 + kte_l^n$ for $t\in\mathbb{R}$. Note that $\|p+tq_l\|_\infty < 1 \Leftrightarrow |1-t+kt|<1$ and $|1-t|<1 \Leftrightarrow 0<t<\frac{2}{1-k}$, as $t\to 0^+$, for any given $k\le 0$, $p+tq_l\in D^n\cap V$. For t that satisfies the above conditions, the Taylor expansion of $f(p+tq_l)$ at $t=0$ is

$$f(p+tq_l) = w_0 + J_f(z_0)q_l t + O(t^{1+\alpha}).$$

Then

$$\max_{1\le j\le N}|(w_0)_j + J_{f_j}(z_0)q_l t + O(t^{1+\alpha})|^2 \le 1-2t+O(t^2).$$

Using $w_0 = \sum_{s=1}^{r_2} e_s^N$, we have

$$\max_{1\le j\le r_2}\{1+2\mathrm{Re}J_{f_j}(z_0)q_l t + O(t^{1+\alpha})\} \le 1-2t+O(t^2).$$

Let $t \to 0^+$. Then

$$\max_{1\le j\le r_2}\{\mathrm{Re}J_{f_j}(z_0)q_l\} \le -1. \tag{3.4}$$

Substituting $q_l = -\sum\limits_{i=1}^{r_1} e_i^n + ke_l^n$ into (3.4), we obtain

$$\max_{1\le j\le r_2}\Big\{\mathrm{Re}J_{f_j}(z_0)\Big(-\sum_{i=1}^{r_1} e_i^n + ke_l^n\Big)\Big\} \le -1.$$

That is

$$\max_{1\le j\le r_2}\Big\{\mathrm{Re}\Big(-\sum_{i=1}^{r_1}\frac{\partial f_j(z_0)}{\partial z_i} + k\frac{\partial f_j(z_0)}{\partial z_l}\Big)\Big\} \le -1. \tag{3.5}$$

We know from Step 1 that $\frac{\partial f_j(z_0)}{\partial z_l} = \mathrm{Re}\frac{\partial f_j(z_0)}{\partial z_l}$ for $1 \le l \le r_1$, so (3.5) is equivalent to

$$\min_{1\le j\le r_2}\Big\{\mathrm{Re}\sum_{i=1}^{r_1}\frac{\partial f_j(z_0)}{\partial z_i} - k\frac{\partial f_j(z_0)}{\partial z_l}\Big\} \ge 1.$$

We also know from Step 1 that $\min\limits_{1\le j\le r_2}\big\{\mathrm{Re}\sum\limits_{i=1}^{r_1}\frac{\partial f_j(z_0)}{\partial z_i}\big\} \ge 1$ for $k \in \mathbb{R}$. Then we can conclude $\min\limits_{1\le j\le r_2}\big\{-k\frac{\partial f_j(z_0)}{\partial z_l}\big\} \ge 0$. Since $k \le 0$ is arbitrary, $\frac{\partial f_j(z_0)}{\partial z_l} \ge 0, 1 \le l \le r_1$.

Step 3 Let $p = z_0, q_l = -\sum\limits_{i=1}^{r_1} e_i^n + \mathrm{i}ke_l^n$ for $r_1 + 1 \le l \le n$ and $0 \ne k \in \mathbb{R}$. Then $p + tq_l = (1-t)z_0 + \mathrm{i}kte_l^n$ for $t \in \mathbb{R}$. It is easy to prove that $\|p + tq_l\|_\infty < 1 \Leftrightarrow |1-t| < 1$ and $|\mathrm{i}kt| < 1 \Leftrightarrow 0 < t < \min\big\{\frac{1}{|k|}, 2\big\}$. Therefore, for any given $0 \ne k \in \mathbb{R}$, there exists $t \to 0^+$ such that $p + tq_l \in D^n \cap V$. For t that satisfies the above conditions, based on the Taylor expansion of $f(p + tq_l)$ at $t = 0$, we can conclude that

$$\max_{1\le j\le N}|(w_0)_j + J_{f_j}(z_0)q_lt + O(t^{1+\alpha})|^2 \le 1 - 2t + O(t^2).$$

Substituting $w_0 = \sum\limits_{s=1}^{r_2} e_s^N$, we have

$$\max_{1\le j\le r_2}\{1 + 2\mathrm{Re}J_{f_j}(z_0)q_lt + O(t^{1+\alpha})\} \le 1 - 2t + O(t^2).$$

By letting $t \to 0^+$, we have

$$\max_{1\le j\le r_2}\{\mathrm{Re}J_{f_j}(z_0)q_l\} \le -1. \tag{3.6}$$

Substituting $q_l = -\sum\limits_{i=1}^{r_1} e_i^n + \mathrm{i}ke_l^n$ into (3.6), we obtain

$$\max_{1\le j\le r_2}\Big\{\mathrm{Re}\Big(-\sum_{i=1}^{r_1}\frac{\partial f_j(z_0)}{\partial z_i} + \mathrm{i}k\frac{\partial f_j(z_0)}{\partial z_l}\Big)\Big\} \le -1.$$

Similarly, write $\frac{\partial f_j(z_0)}{\partial z_l} = \mathrm{Re}\frac{\partial f_j(z_0)}{\partial z_l} + \mathrm{i}\mathrm{Im}\frac{\partial f_j(z_0)}{\partial z_l}$. Then

$$\max_{1\le j\le r_2}\Big\{\mathrm{Re}\Big(-\sum_{i=1}^{r_1}\frac{\partial f_j(z_0)}{\partial z_i} - k\mathrm{Im}\frac{\partial f_j(z_0)}{\partial z_l}\Big)\Big\} \le -1. \tag{3.7}$$

Since (3.7) holds for any $0 \neq k \in \mathbb{R}$, we have $\mathrm{Im}\frac{\partial f_j(z_0)}{\partial z_l} = 0, r_1+1 \leq l \leq n$. Therefore

$$\frac{\partial f_j(z_0)}{\partial z_l} = \mathrm{Re}\frac{\partial f_j(z_0)}{\partial z_l}, \quad r_1+1 \leq l \leq n. \tag{3.8}$$

Step 4 Let $p = z_0, q_l = -\sum\limits_{i=1}^{r_1} e_i^n + ke_l^n$ for $r_1+1 \leq l \leq n$ and $0 \neq k \in \mathbb{R}$. Then $p + tq_l = (1-t)z_0 + kte_l^n$ for $t \in \mathbb{R}$. It is easy to check that $\|p+tq_l\|_\infty < 1 \Leftrightarrow |1-t| < 1$ and $|kt| < 1 \Leftrightarrow 0 < t < \min\left\{\frac{1}{|k|}, 2\right\}$. Therefore, given any $k \neq 0 \in \mathbb{R}$, there exists $t \to 0^+$ such that $p + tq_l \in D^n \cap V$.

By the same analysis as in Step 2, we can conclude that

$$\max_{1\leq j\leq r_2} \{\mathrm{Re}J_{f_j}(z_0)q_l\} \leq -1. \tag{3.9}$$

Substituting $q_l = -\sum\limits_{i=1}^{r_1} e_i^n + ke_l^n$ into (3.9), we obtain

$$\max_{1\leq j\leq r_2} \Big\{\mathrm{Re}\Big(-\sum_{i=1}^{r_1}\frac{\partial f_j(z_0)}{\partial z_i} + k\frac{\partial f_j(z_0)}{\partial z_l}\Big)\Big\} \leq -1.$$

It follows from (3.8) that

$$\max_{1\leq j\leq r_2} \Big\{\mathrm{Re}\Big(-\sum_{i=1}^{r_1}\frac{\partial f_j(z_0)}{\partial z_i}\Big) + k\mathrm{Re}\frac{\partial f_j(z_0)}{\partial z_l}\Big\} \leq -1. \tag{3.10}$$

Since (3.10) holds for any $k \neq 0 \in \mathbb{R}$, we know that $\mathrm{Re}\frac{\partial f_j(z_0)}{\partial z_l} = 0$ for $r_1+1 \leq l \leq n$. It follows that $\frac{\partial f_j(z_0)}{\partial z_l} = 0, r_1+1 \leq l \leq n$. From the above discussion, we have

$$\overline{J_f(z_0)}^{\mathrm{T}} w_0 = \mathrm{diag}(\lambda_1, \cdots, \lambda_{r_1}, 0, \cdots, 0)z_0,$$

where $\lambda_i = \sum\limits_{j=1}^{r_2} \frac{\partial f_j(z_0)}{\partial z_i} \geq 0$ $(1 \leq i \leq r_1)$ and $\sum\limits_{i=1}^{r_1}\lambda_i = \sum\limits_{i=1}^{r_1}\sum\limits_{j=1}^{r_2}\frac{\partial f_j(z_0)}{\partial z_i} \geq \sum\limits_{j=1}^{r_2} 1 = r_2$.

Step 5 Let z_0 be any given point in ∂D^n. And the first r_1 components are at the boundary of D. Namely, z_0 is not necessarily $\sum\limits_{i=1}^{r_1} e_i^n$. According to [14], there exists a special kind of diagonal unitary matrix U_{z_0} such that $U_{z_0} \cdot z_0 = \sum\limits_{i=1}^{r_1} e_i^n$. Let $f(z_0) = w_0 \in E_{r_2}$. But w_0 may not necessarily be $\sum\limits_{s=1}^{r_2} e_s^N$. Similarly, there is a unitary matrix U_{w_0} such that $U_{w_0} \cdot w_0 = \sum\limits_{s=1}^{r_2} e_s^N$.

Set $g = U_{w_0} \circ f \circ \overline{U_{z_0}}^{\mathrm{T}}$. Then $g : D^n \to D^N$ is a holomorphic mapping and g is $C^{1+\alpha}$ at z_0 with $g(0) = 0$ and $g(\sum\limits_{i=1}^{r_1} e_i^n) = \sum\limits_{s=1}^{r_2} e_s^N$. From the analysis in Step 4, there exists a sequence of nonnegative real numbers $\lambda_1, \cdots, \lambda_{r_1}$ such that

$$\overline{J_g\Big(\sum_{i=1}^{r_1} e_i^n\Big)}^{\mathrm{T}} \sum_{s=1}^{r_2} e_s^N = \mathrm{diag}(\lambda_1, \cdots, \lambda_{r_1}, 0, \cdots, 0)\sum_{i=1}^{r_1} e_i^n, \tag{3.11}$$

where $\lambda_i = \sum\limits_{j=1}^{r_2} \frac{\partial g_j(z_0)}{\partial z_i} \geq 0$ $(1 \leq i \leq r_1)$ and $\sum\limits_{i=1}^{r_1}\lambda_i \geq r_2$.

By direct computation, we know $J_g(z) = U_{w_0} J_f(\overline{U_{z_0}}^{\mathrm{T}} z)\overline{U_{z_0}}^{\mathrm{T}}$. This implies that

$$U_{z_0}\overline{J_f(z_0)}^{\mathrm{T}}\overline{U_{w_0}}^{\mathrm{T}}\sum_{s=1}^{r_2} e_s^N = \mathrm{diag}(\lambda_1, \cdots, \lambda_{r_1}, 0, \cdots, 0)\sum_{i=1}^{r_1} e_i^n. \tag{3.12}$$

Multiplying both sides by $\overline{U_{z_0}}^{\mathrm{T}}$ of (3.12), we obtain

$$\overline{J_f(z_0)}^{\mathrm{T}}\overline{U_{w_0}}^{\mathrm{T}}\sum_{s=1}^{r_2} e_s^N = \overline{U_{z_0}}^{\mathrm{T}}\mathrm{diag}(\lambda_1, \cdots, \lambda_{r_1}, 0, \cdots, 0)\sum_{i=1}^{r_1} e_i^n.$$

Since $\overline{U_{z_0}}^{\mathrm{T}}$ is a diagonal matrix, we can verify that

$$\overline{U_{z_0}}^{\mathrm{T}}\mathrm{diag}(\lambda_1, \cdots, \lambda_{r_1}, 0, \cdots, 0) = \mathrm{diag}(\lambda_1, \cdots, \lambda_{r_1}, 0, \cdots, 0)\overline{U_{z_0}}^{\mathrm{T}}.$$

Hence, we have

$$\overline{J_f(z_0)}^{\mathrm{T}} w_0 = \mathrm{diag}(\lambda_1, \cdots, \lambda_{r_1}, 0, \cdots, 0)\overline{U_{z_0}}^{\mathrm{T}}\sum_{i=1}^{r_1} e_i^n = \mathrm{diag}(\lambda_1, \cdots, \lambda_{r_1}, 0, \cdots, 0)z_0.$$

Thus, the theorem is proved.

Similar to the above proof, we obtain the following theorem.

Theorem 3.2 *Let $f \in H(B^n, D^N)$ with $f(0) = 0$ for any $n, N \geq 1$. If f is $C^{1+\alpha}$ at $z_0 \in \partial B^n$ and $f(z_0) = w_0 \in E_r$, where E_r means that for $1 \leq r \leq N$, the first r components of w_0 are at the boundary of D, then there exists a nonnegative real number λ satisfying $\lambda \geq r$ such that $\overline{J_f(z_0)}^{\mathrm{T}} w_0 = \lambda z_0$.*

Next, we use the method of Lemma 2.2 to generalize Lemma 2.1 to the unit polydisc and establish an inequality relationship for the holomorphic self-mappings of the unit polydisc.

Theorem 3.3 *Let $f \in H(D^n, D^n)$ for any $n \geq 1$. If f is holomorphic at $z = z_0 \in T^n$ with $f(z_0) = w_0 \in T^n$, then*

$$\overline{w_0}^{\mathrm{T}} J_f(z_0)z_0 \geq \frac{2n(n - |\overline{w_0}^{\mathrm{T}} f(0)|)^2}{n^2 - |\overline{w_0}^{\mathrm{T}} f(0)|^2 + n^2\|J_f(0)z_0\|_\infty} \geq \frac{2n(1 - \|f(0)\|_\infty)^2}{1 - (\|f(0)\|_\infty)^2 + \|J_f(0)z_0\|_\infty}.$$

Proof Set $g(\xi) = \frac{1}{n}\overline{w_0}^{\mathrm{T}} f(\xi z_0), \xi \in D$. Then $g : D \to D$ is a holomorphic mapping with $g(1) = \frac{1}{n}\overline{w_0}^{\mathrm{T}} f(z_0) = 1$. According to Lemma 2.1, we get

$$g'(1) \geq \frac{2|1 - g(0)|^2}{1 - |g(0)|^2 + |g'(0)|}.$$

Note that $g(0) = \frac{1}{n}\overline{w_0}^{\mathrm{T}} f(0)$ and $g'(0) = \frac{1}{n}\overline{w_0}^{\mathrm{T}} J_f(0)z_0$. Then

$$g'(1) = \frac{1}{n}\overline{w_0}^{\mathrm{T}} J_f(z_0)z_0 \geq \frac{2\left|1 - \frac{1}{n}\overline{w_0}^{\mathrm{T}} f(0)\right|^2}{1 - \left|\frac{1}{n}\overline{w_0}^{\mathrm{T}} f(0)\right|^2 + \left|\frac{1}{n}\overline{w_0}^{\mathrm{T}} J_f(0)z_0\right|}.$$

That is

$$\overline{w_0}^{\mathrm{T}} J_f(z_0)z_0 \geq \frac{2n\left|1 - \frac{1}{n}\overline{w_0}^{\mathrm{T}} f(0)\right|^2}{1 - \left|\frac{1}{n}\overline{w_0}^{\mathrm{T}} f(0)\right|^2 + \left|\frac{1}{n}\overline{w_0}^{\mathrm{T}} J_f(0)z_0\right|}$$

$$\geq \frac{2n|n-\overline{w_0}^{\mathrm{T}}f(0)|^2}{n^2-|\overline{w_0}^{\mathrm{T}}f(0)|^2+n^2\|\overline{w_0}^{\mathrm{T}}\|_\infty\|J_f(0)z_0\|_\infty}$$
$$\geq \frac{2n(n-|\overline{w_0}^{\mathrm{T}}f(0)|)^2}{n^2-|\overline{w_0}^{\mathrm{T}}f(0)|^2+n^2\|J_f(0)z_0\|_\infty}. \tag{3.13}$$

Let $a=n^2+n^2\|J_f(0)z_0\|_\infty\geq n^2$ and $\sigma(x)=\frac{2(n-x)^2}{a-x^2}$, where $x=|\overline{w_0}^{\mathrm{T}}f(0)|\in(0,n)$. It is obvious that $\sigma(x)$ is decreasing for $x\in(0,n)$. Thus

$$\overline{w_0}^{\mathrm{T}}J_f(z_0)z_0\geq\frac{2n(n-n\|f(0)\|_\infty)^2}{n^2-(n\|f(0)\|_\infty)^2+n^2\|J_f(0)z_0\|_\infty}=\frac{2n(1-\|f(0)\|_\infty)^2}{1-(\|f(0)\|_\infty)^2+\|J_f(0)z_0\|_\infty}.$$

Corollary 3.1 *Let $f\in H(D^n,D^n)$ for any $n\geq1$. If f is holomorphic at $z=z_0\in T^n$ with $f(z_0)=w_0\in T^n$, then*

$$\|J_f(z_0)z_0\|_\infty\geq\frac{2(n-|\overline{w_0}^{\mathrm{T}}f(0)|)^2}{n^2-|\overline{w_0}^{\mathrm{T}}f(0)|^2+n^2\|J_f(0)z_0\|_\infty}\geq\frac{2(1-\|f(0)\|_\infty)^2}{1-(\|f(0)\|_\infty)^2+\|J_f(0)z_0\|_\infty}.$$

Especially, if $f(0)=0$, we have $\|J_f(z_0)z_0\|_\infty\geq\frac{2}{1+\|J_f(0)z_0\|_\infty}$.

Proof Since $|\overline{w_0}^{\mathrm{T}}J_f(z_0)z_0|\leq n\|J_f(z_0)z_0\|_\infty$, the result follows from Theorem 3.3.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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