Chin. Ann. Math. Ser. B
47(5), 2026, 823–832
DOI: 10.1007/s11401-026-0039-2

**Chinese Annals of Mathematics, Series B**


# Monotonicity and Symmetry of Positive Solutions to Logarithmic Laplacian Equations and Systems with Gradient Terms*

Xianghui XU[1]  Tingzhi CHENG[2]

**Abstract** This paper is concerned with logarithmic Laplacian equations and systems with gradient terms in bounded convex domains. Under some assumptions on the nonlinearities, the authors prove the strict monotonicity and symmetry of positive solutions. In particular, the authors establish one new estimate of the logarithmic Laplacian operator that plays an important role in applying the direct method of moving planes. This paper seems to be the first one to deal with the monotonicity and symmetry of positive solutions to logarithmic Laplacian equations and systems with gradient terms.

**Keywords** Logarithmic Laplacian, Monotonicity, Positive solution, Convex domain
**2020 MR Subject Classification** 35J67, 35R11

## 1 Introduction

Due to the significant applications and profound boost in the perception of nonlocal phenomena, the interest in the research of nonlinear integro-differential equations has been fueled by the high frequency study on the qualitative analysis of solutions to integro-differential equations and systems. Among a large quantity of nonlocal operators, the fractional Laplacian has received tremendous attention from mathematicians. The prototype of the fractional Laplacian can be defined as follows:

$$(-\Delta)^s u(x) = c_{n,s} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}} \mathrm{d}y,$$

where $0 < s < 1$ and $c_{n,s} = 2^{2s}\pi^{-\frac{n}{2}} s \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)}$. To overcome the difficulty caused by the nonlocal operator $(-\Delta)^s$, there have been several methods such as the extension method, the integral equations method, the direct method of moving planes and the sliding method (see [1, 3–5, 9] and the references therein).

In 2019, Chen and Weth [2] first introduced the logarithmic Laplacian operator $(-\Delta)^L$

Manuscript received October 9, 2023. Revised May 16, 2024.
[1]School of Mathematics and Statistics, Ludong University, Yantai 264025, Shandong, China.
E-mail: xvxianghui@163.com
[2]Corresponding author. School of Mathematics and Statistics, Ludong University, Yantai 264025, Shandong, China. E-mail: chengtingzhi1989@163.com
*This work was supported by the National Natural Science Foundation of China (Nos. 12501263, 12401259), the Shandong Provincial Natural Science Foundation (No. ZR2021QA070) and the Shanghai Frontier Research Center of Modern Analysis.

which is given by

$$\begin{aligned}(-\Delta)^L u(x) &= C_n \int_{\mathbb{R}^n} \frac{u(x)1_{B_1(x)}(y) - u(y)}{|x-y|^n}\mathrm{d}y + \varrho_n u(x) \\ &= C_n \int_{B_1(x)} \frac{u(x) - u(y)}{|x-y|^n}\mathrm{d}y - C_n \int_{\mathbb{R}^n \setminus B_1(x)} \frac{u(y)}{|x-y|^n}\mathrm{d}y + \varrho_n u(x).\end{aligned}$$

Here $C_n = \pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)$, $\varrho_n = 2\log 2 + \psi\left(\frac{n}{2}\right) - \gamma$, $\psi = \Gamma'/\Gamma$ is the Digamma function and $\Gamma$ is the Gamma function. Moreover, $\gamma = -\Gamma'(1)$ is the Euler Mascheroni constant and $B_1(x)$ denotes the unit ball with center $x$. They pointed out that the logarithmic Laplacian operator can be used to describe the asymptotic behavior of principal Dirichlet eigenvalues and eigenfunctions of $(-\Delta)^s$ as $s \to 0$.

Recently, the qualitative analysis of solutions to various logarithmic Laplacian equations has attracted more and more attention (see [6–7, 10] and the references therein). For instance, Liu [7] generalized the direct method of moving planes (see [4]) to prove the symmetry and monotonicity of solutions for the following Dirichlet problem of a nonlinear logarithmic Laplacian equation

$$\begin{cases} (-\Delta)^L u(x) = f(u) & \text{in } \Omega, \\ u(x) > 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \Omega^c, \end{cases} \tag{1.1}$$

where $\Omega = B_R(0)$. In 2021, Zhang and Nie [10] also considered problem (1.1) in a unit ball and a nonlinear Schrödinger equation with logarithmic Laplacian in $\mathbb{R}^n$. With the establishment of various maximum principles under a restricted condition $h_\Omega(x) + \varrho_n \geq 0, x \in \Omega$, they applied the method of moving planes successfully to the monotonicity and symmetry analysis of positive solutions, where

$$h_\Omega(x) = C_n \int_{B_1(x)\setminus\Omega} \frac{1}{|x-y|^n}\mathrm{d}y - C_n \int_{\Omega\setminus B_1(x)} \frac{1}{|x-y|^n}\mathrm{d}y.$$

For more details about the monotonicity and symmetry properties of solutions to nonlinear integro-differential equations and systems, we refer to [3–4, 8–9] and the references therein. Although there have already emerged some works related to the monotonicity and symmetry research on nonlinear integro-differential equations and systems, we should point out that the nonlinear terms involved in the available research lack gradient terms. That is to say, compared to the tremendous results on nonlinear term $f(u)$, the nonlinear term containing gradient ingredient such as $f(x, u, \nabla u)$ has fewer occurrences in the monotonicity and symmetry research.

Inspired by the above observations, we focus on the monotonicity and symmetry analysis of the integro-differential equations and systems with both logarithmic Laplacian and gradient terms. To our best knowledge, our paper seems to be the first one to deal with logarithmic Laplacian equations and systems with gradient terms. More precisely, we consider the following Dirichlet problems of a logarithmic Laplacian equation:

$$\begin{cases} (-\Delta)^L u(x) = f(x, u, \nabla u) & \text{in } \Omega, \\ u(x) > 0 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \Omega^c, \end{cases} \tag{1.2}$$

and system

$$\begin{cases} (-\Delta)^L u_i(x) = f_i(x, u_1, u_2, \cdots, u_m, \nabla u_i) & \text{in } \Omega, \\ u_i(x) > 0 & \text{in } \Omega, \\ u_i(x) = 0 & \text{on } \Omega^c, \quad i = 1, 2, \cdots, m, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $\Omega$ is convex in $x_1$-direction (i.e., for any $(x_1, x'), (\overline{x}_1, x') \in \Omega$ and $0 < t < 1$, $(tx_1 + (1-t)\overline{x}_1, x') \in \Omega$). It is clear that $(-\Delta)^L u_i$ is well defined if $u_i \in C^{1,1}_{\rm loc}(\mathbb{R}^n) \cap L_0$ for $i = 1, 2, \cdots, m$. Here

$$L_0 = \Big\{ u_i : \mathbb{R}^n \to \mathbb{R} \mid \int_{\mathbb{R}^n} \frac{|u_i(x)|}{1 + |x|^n} \mathrm{d}x < \infty \Big\}.$$

For the sake of narrative convenience, we introduce some notations which will be frequently used in the following parts. Set

$$\begin{aligned} & x = (x_1, x') \in \mathbb{R}^n, \quad x^\lambda = (2\lambda - x_1, x'), \quad T_\lambda = \{(x_1, x') \in \mathbb{R}^n \mid x_1 = \lambda\}, \\ & \Sigma_\lambda = \{(x_1, x') \in \mathbb{R}^n \mid x_1 < \lambda\}, \quad \widetilde{\Sigma}_\lambda = \{(x_1, x') \in \mathbb{R}^n \mid x_1 > \lambda\}, \\ & R_{x_1}(A) = \{x^\lambda \in \mathbb{R}^n \mid x \in A\}, \quad u_i^\lambda(x) = u_i(x^\lambda), \quad w_i^\lambda(x) = u_i^\lambda(x) - u_i(x). \end{aligned}$$

We let $\overrightarrow{p} = (p_1, p_2, \cdots, p_n)$ denote $\nabla u$ and $\nabla u_i$ in the following Theorem 1.1 and Theorem 1.2, respectively.

With the aid of the direct method of moving planes for the logarithmic Laplacian, we will show the strict monotonicity and symmetry of solutions to (1.2) and (1.3), which can generalize the results in [7, 10]. In particular, this paper has two major advantages of innovation. Firstly, the bounded domain $\Omega$ in [7, 10] is a ball which is a strictly convex domain. The reflection point of the boundary $R_{x_1}(\Sigma_\lambda \cap \partial\Omega)$ lies in the interior part of $\Omega$ entirely (see Figure 1(a) below), which means that $w_i^\lambda(x) > 0$ for $x \in \Sigma_\lambda \cap \partial\Omega$. However, this fact cannot hold for a general domain which is not strictly convex (see Figure 1(b) below). Due to the different geometry between the strictly and non-strictly convex domains, the arguments used in [7, 10] fail here. Thus, we will establish a new estimate of the logarithmic Laplacian operator to overcome this setback (see Lemma 2.1 in Section 2 for details). Secondly, comparing with various maximum principles in [10], we will not need the condition $h_\Omega(x) + \varrho_n \geq 0, x \in \Omega$ any longer. In addition, we also need some new ideas to deal with the appearance of gradient terms.

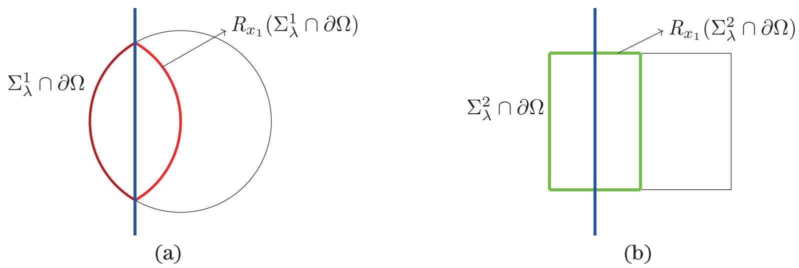


Figure 1 Reflection part of $\Sigma_\lambda \cap \partial\Omega$ in strictly and non-strictly convex domains.

Our main results can be exactly illustrated as follows.

**Theorem 1.1** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ which is convex in $x_1$-direction and symmetric about $x_1 = 0$. Suppose that $u \in C^{1,1}_{\rm loc}(\Omega) \cap C(\mathbb{R}^n)$ solves (1.2) and $f(x, u, \overrightarrow{p})$ is Lipschitz continuous with respect to $u$. If $f(x, u, \overrightarrow{p})$ satisfies*

(F) $f(x_1, x', u, p_1, p_2, \cdots, p_n) \le f(\overline{x}_1, x', u, -p_1, p_2, \cdots, p_n)$ *for any* $x_1, -p_1 \le 0$, $x_1 \le \overline{x}_1 \le -x_1$.

*Then $u(x_1, x')$ is strictly increasing in the left half of $\Omega$ in $x_1$-direction and*

$$u(x_1, x') \le u(-x_1, x'), \quad \forall x_1 < 0,\ (x_1, x') \in \Omega.$$

*In particular, if $f(x_1, x', u, \overrightarrow{p}) = f(-x_1, x', u, \overrightarrow{p})$, then*

$$u(x_1, x') = u(-x_1, x').$$

**Theorem 1.2** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ which is convex in $x_1$-direction and symmetric about $x_1 = 0$. Suppose that $u_i \in C^{1,1}_{\rm loc}(\Omega) \cap C(\mathbb{R}^n)$ $(i = 1, 2, \cdots, m)$ solve (1.3) and $f_i(x, \mathrm{u}, \overrightarrow{p})$ $(i = 1, 2, \cdots, m)$ are Lipschitz continuous with respect to $\mathrm{u} = (u_1, u_2, \cdots, u_m)$. If $f_i(x, \mathrm{u}, \overrightarrow{p})$ $(i = 1, 2, \cdots, m)$ satisfy*

($F_1$) $\frac{\partial f_i}{\partial u_k} \ge 0$ *for* $k \ne i$, $1 \le i, k \le m$;

($F_2$) $f_i(x_1, x', \mathrm{u}, p_1, p_2, \cdots, p_n) \le f_i(\overline{x}_1, x', \mathrm{u}, -p_1, p_2, \cdots, p_n)$ *for any* $x_1, -p_1 \le 0$, $x_1 \le \overline{x}_1 \le -x_1$.

*Then $u_i(x_1, x')$ $(i = 1, 2, \cdots, m)$ are strictly increasing in the left half of $\Omega$ in $x_1$-direction and*

$$u_i(x_1, x') \le u_i(-x_1, x'), \quad \forall\ x_1 < 0, \quad (x_1, x') \in \Omega.$$

*In particular, if $f_i(x_1, x', \mathrm{u}, \overrightarrow{p}) = f_i(-x_1, x', \mathrm{u}, \overrightarrow{p})$, then*

$$u_i(x_1, x') = u_i(-x_1, x').$$

The rest of this paper is organized as follows. In Section 2, one new crucial estimate for the logarithmic Laplacian operator is derived. Then we present the detailed proofs of main results in Section 3.

## 2 Preliminaries

In this section, we establish a new estimate lemma that will play a vital role in the application of the method of moving planes.

**Lemma 2.1** *Let $w \in L_0$ satisfy $w(x_1, x_2, \cdots, x_n) = -w(2\lambda - x_1, x_2, \cdots, x_n)$, $x \in \Sigma_\lambda$ and $w(x) \ge 0$, $x \in \Sigma_\lambda \backslash \Omega$. Assume that there exists one point $x \in \Sigma_\lambda \cap \Omega$ such that*

$$w(x) = \inf_{\Sigma_\lambda \cap \Omega} w(y) < 0.$$

*If $w$ is $C^{1,1}$ at $x$, then we have*

$$(-\Delta)^L w(x) \le \Big(C_n C' \ln \frac{1}{2\delta} - C_n C + \varrho_n\Big) w(x) - n\delta C_n \int_{\Sigma_\lambda \cap \Omega} \frac{(w(y) - w(x))(\lambda - y_1)}{|x - y^\lambda|^{n+2}} \mathrm{d}y,$$

*where $\delta = d(x, T_\lambda) = |x_1 - \lambda| \in \big(0, \frac{1}{2}\big)$, $C'$ and $C$ both denote some positive constants.*

**Proof** It follows from the definition of logarithmic Laplacian operator that

$$\begin{aligned}(-\Delta)^L w(x) &= C_n \int_{B_1(x)} \frac{w(x)-w(y)}{|x-y|^n}\mathrm{d}y - C_n \int_{\mathbb{R}^n\setminus B_1(x)} \frac{w(y)}{|x-y|^n}\mathrm{d}y + \varrho_n w(x)\\ &= C_n(\mathrm{I}+\mathrm{J}) + \varrho_n w(x).\end{aligned}$$

Set $A = B_1(x^\lambda)\cap\Sigma_\lambda$, $B = B_1(x)\cap(\Sigma_\lambda\setminus A)$. Then $R_{x_1}(A) = B_1(x)\setminus\Sigma_\lambda$, $B_1(x) = A\cup B\cup R_{x_1}(A)$ and

$$\begin{aligned}\mathrm{I} &= \int_A \frac{w(x)-w(y)}{|x-y|^n}\mathrm{d}y + \int_B \frac{w(x)-w(y)}{|x-y|^n}\mathrm{d}y + \int_{R_{x_1}(A)} \frac{w(x)-w(y)}{|x-y|^n}\mathrm{d}y\\ &= \int_A \frac{w(x)-w(y)}{|x-y|^n}\mathrm{d}y + \int_B \frac{w(x)-w(y)}{|x-y|^n}\mathrm{d}y + \int_A \frac{w(x)+w(y)}{|x-y^\lambda|^n}\mathrm{d}y\\ &\le -\int_A (w(y)-w(x))\Big(\frac{1}{|x-y|^n} - \frac{1}{|x-y^\lambda|^n}\Big)\mathrm{d}y\\ &\quad + \int_A \frac{2w(x)}{|x-y^\lambda|^n}\mathrm{d}y + \int_B \frac{w(x)-w(y)}{|x-y|^n}\mathrm{d}y. \end{aligned}\tag{2.1}$$

Let $C = \Sigma_\lambda\setminus B_1(x)$, $R_{x_1}(B), R_{x_1}(C)$ denote the reflection of $B$, $C$ with respect to $T_\lambda$. Then we have

$$\begin{aligned}\mathrm{J} &= -\int_C \frac{w(y)}{|x-y|^n}\mathrm{d}y - \int_{R_{x_1}(B)} \frac{w(y)}{|x-y|^n}\mathrm{d}y - \int_{R_{x_1}(C)} \frac{w(y)}{|x-y|^n}\mathrm{d}y\\ &= -\int_C \frac{w(y)}{|x-y|^n}\mathrm{d}y + \int_B \frac{w(y)}{|x-y^\lambda|^n}\mathrm{d}y + \int_C \frac{w(y)}{|x-y^\lambda|^n}\mathrm{d}y\\ &= -\int_C w(y)\Big(\frac{1}{|x-y|^n} - \frac{1}{|x-y^\lambda|^n}\Big)\mathrm{d}y + \int_B \frac{w(y)}{|x-y^\lambda|^n}\mathrm{d}y\\ &\le -\int_{C\cap\Omega} w(y)\Big(\frac{1}{|x-y|^n} - \frac{1}{|x-y^\lambda|^n}\Big)\mathrm{d}y + \int_B \frac{w(y)}{|x-y^\lambda|^n}\mathrm{d}y. \end{aligned}\tag{2.2}$$

Combining (2.1) and (2.2), we see

$$\begin{aligned}\mathrm{I}+\mathrm{J} &\le -\int_{A\cup B\cup(C\cap\Omega)} (w(y)-w(x))\Big(\frac{1}{|x-y|^n} - \frac{1}{|x-y^\lambda|^n}\Big)\mathrm{d}y + \int_A \frac{2w(x)}{|x-y^\lambda|^n}\mathrm{d}y\\ &\quad + \int_B \frac{w(x)}{|x-y^\lambda|^n}\mathrm{d}y - w(x)\int_{C\cap\Omega}\Big(\frac{1}{|x-y|^n} - \frac{1}{|x-y^\lambda|^n}\Big)\mathrm{d}y\\ &\le -\int_{\Sigma_\lambda\cap\Omega} (w(y)-w(x))\Big(\frac{1}{|x-y|^n} - \frac{1}{|x-y^\lambda|^n}\Big)\mathrm{d}y\\ &\quad + \int_{A\cup B} \frac{w(x)}{|x-y^\lambda|^n}\mathrm{d}y - \int_{C\cap\Omega} \frac{w(x)}{|x-y|^n}\mathrm{d}y. \end{aligned}\tag{2.3}$$

Applying the mean value theorem, we can compute that

$$\begin{aligned}&\int_{\Sigma_\lambda\cap\Omega} (w(y)-w(x))\Big(\frac{1}{|x-y|^n} - \frac{1}{|x-y^\lambda|^n}\Big)\mathrm{d}y\\ &= \int_{\Sigma_\lambda\cap\Omega} (w(y)-w(x))\frac{n}{\theta^{n+1}}(|x-y^\lambda| - |x-y|)\mathrm{d}y\\ &\ge n\int_{\Sigma_\lambda\cap\Omega} (w(y)-w(x))\frac{(\lambda-x_1)(\lambda-y_1)}{|x-y^\lambda|^{n+2}}\mathrm{d}y\end{aligned}$$

$$= n\delta \int_{\Sigma_\lambda \cap \Omega} \frac{(w(y)-w(x))(\lambda - y_1)}{|x-y^\lambda|^{n+2}} \mathrm{d}y, \tag{2.4}$$

where $\theta \in (|x-y|, |x-y^\lambda|)$. While, let $D = \{y \mid x_1+\delta < y_1 < x_1+\frac{1}{2}, |y'-x'| < \frac{1}{4}\}$, and we can claim that $D \subset R_{x_1}(A \cup B) \subset \widetilde{\Sigma}_\lambda$. In fact, from the information displayed in Figure 2, we can rewrite the coordinate of $M$, $R_{x_1}M$ as $M = (x_1-1, x')$ and $R_{x_1}M = (2\lambda - x_1 + 1, x')$. Moreover, we can also set the coordinate of $Q$ as $Q = (\lambda, Q')$, where $(\lambda - x_1)^2 + |x'-Q'|^2 = 1$. Hence, for $y = (y_1, y') \in R_{x_1}(A \cup B)$, we can deduce from Figure 2 that $\lambda < y_1 \leq 2\lambda - x_1 + 1$ and $|y'-x'|^2 \leq |Q'-x'|^2 = 1-(\lambda - x_1)^2$. Furthermore, from the definition of $\delta$, we can also know that $\delta = \lambda - x_1 < y_1 - x_1 \leq 2\lambda - 2x_1 + 1 = 2\delta + 1$ and $|y'-x'|^2 \leq 1-\delta^2$. In a word, for $y = (y_1, y') \in R_{x_1}(A\cup B)$, we can achieve that $y_1 \in (x_1+\delta, x_1+1+2\delta]$ and $|y'-x'| \leq \sqrt{1-\delta^2}$. Thanks to the fact that $0 < \delta < \frac{1}{2}$, we can obtain the conclusion that

$$\begin{aligned} D &= \Big\{y \mid x_1+\delta < y_1 < x_1+\frac{1}{2}, |y'-x'| < \frac{1}{4}\Big\} \\ &\subset E = \{y \mid x_1+\delta < y_1 < x_1+1+2\delta, |y'-x'| \leq \sqrt{1-\delta^2}\} \\ &\subset R_{x_1}(A \cup B). \end{aligned}$$

Then, for $\omega_{n-2} = |\partial B_1(0)|$ in $\mathbb{R}^{n-1}$, we can also get

$$\begin{aligned} &\int_{A\cup B} \frac{1}{|x-y^\lambda|^n} \mathrm{d}y \\ &\geq \int_D \frac{1}{|x-y|^n}\mathrm{d}y = \int_\delta^{\frac{1}{2}} \int_0^{\frac{1}{4}} \frac{\omega_{n-2}\tau^{n-2}}{(\rho^2+\tau^2)^{\frac{n}{2}}} \mathrm{d}\tau \mathrm{d}\rho = \int_\delta^{\frac{1}{2}} \int_0^{\frac{1}{4\rho}} \frac{\omega_{n-2}\rho^{n-2}t^{n-2}}{(\rho^2+\rho^2t^2)^{\frac{n}{2}}} \rho \mathrm{d}t\mathrm{d}\rho \\ &\geq \int_\delta^{\frac{1}{2}} \int_0^{\frac{1}{2}} \frac{\omega_{n-2}t^{n-2}}{(1+t^2)^{\frac{n}{2}}} \mathrm{d}t \frac{1}{\rho}\mathrm{d}\rho \geq C' \ln\frac{1}{2\delta}. \end{aligned} \tag{2.5}$$

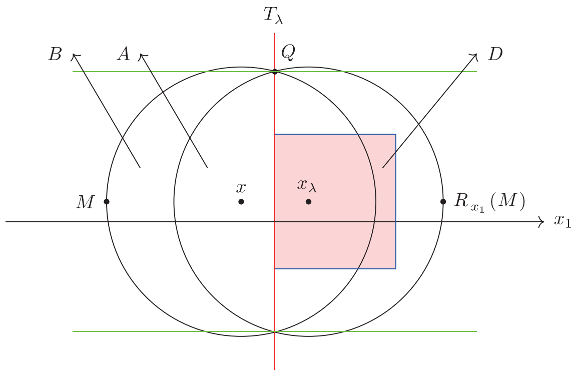


Figure 2 Location of domain $D$.

Moreover, since $|x-y| > 1$ for all $y \in C$, we have

$$\int_{C\cap\Omega} \frac{1}{|x-y|^n}\mathrm{d}y < |C \cap \Omega| \leq |\Omega| < C. \tag{2.6}$$

Thus, by (2.3)–(2.6), we see that

$$(-\Delta)^L w(x) \leq \Big(C_nC' \ln\frac{1}{2\delta} - C_nC + \varrho_n\Big) w(x) - n\delta C_n \int_{\Sigma_\lambda\cap\Omega} \frac{(w(y)-w(x))(\lambda - y_1)}{|x-y^\lambda|^{n+2}}\mathrm{d}y.$$

The proof of Lemma 2.1 is completed.

## 3 Proofs of Main Results

In this section, we mainly present the detailed proof of Theorem 1.2. The proof of Theorem 1.1 can be done by the similar arguments to those of Theorem 1.2. For simplicity, we leave it to the readers.

**Proof of Theorem 1.2** For the sake of discussion, we can suppose that $\Omega \subset \{|x_1| \leq 1\}$ and $\partial\Omega \cap \{x_1 = -1\} \neq \emptyset$. We will apply the direct method of moving planes to complete the proof of this theorem. Exactly, two steps can be presented as follows.

**Step 1** Show that there exists $\iota > 0$ small enough such that

$$w_i^\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda,\ \forall \lambda \in [-1, -1+\iota],\ i = 1, 2, \cdots, m.$$

Otherwise, by the conditions of $u_i$, there exist $i_0 \in \{1, 2, \cdots, m\}$, $\lambda_0 > -1$ and $x_0 \in \Sigma_{\lambda_0} \cap \Omega$ such that

$$w_{i_0}^{\lambda_0}(x_0) = \inf\{w_i^\lambda(x) \mid \forall x \in \overline{\Sigma_\lambda},\ \forall \lambda \in [-1, -1+\iota],\ i = 1, 2, \cdots, m\} < 0$$

and

$$\frac{\partial w_{i_0}^\lambda(x_0)}{\partial \lambda}\Big|_{\lambda=\lambda_0} \leq 0, \qquad \nabla w_{i_0}^{\lambda_0}(x)|_{x=x_0} = 0.$$

Thus in fact

$$\frac{\partial u_{i_0}^\lambda(x_0)}{\partial \lambda}\Big|_{\lambda=\lambda_0} = -\frac{\partial u_{i_0}^{\lambda_0}(x)}{\partial x_1}\Big|_{x=x_0} \leq 0, \quad \nabla u_{i_0}^{\lambda_0}(x_0) = \nabla u_{i_0}(x_0). \tag{3.1}$$

Inspired by (3.1) and condition ($\mathrm{F}_2$), we can compute that as follows:

$$\begin{aligned}
&(-\Delta)^L w_{i_0}^{\lambda_0}(x_0)\\
&= f_{i_0}\Big(x_0^{\lambda_0}, u_1^{\lambda_0}(x_0), u_2^{\lambda_0}(x_0), \cdots, u_m^{\lambda_0}(x_0), -\frac{\partial u_{i_0}^{\lambda_0}(x_0)}{\partial x_1}, \frac{\partial u_{i_0}^{\lambda_0}(x_0)}{\partial x_2}, \cdots, \frac{\partial u_{i_0}^{\lambda_0}(x_0)}{\partial x_n}\Big)\\
&\quad - f_{i_0}\Big(x_0, u_1(x_0), u_2(x_0), \cdots, u_m(x_0), \frac{\partial u_{i_0}(x_0)}{\partial x_1}, \frac{\partial u_{i_0}(x_0)}{\partial x_2}, \cdots, \frac{\partial u_{i_0}(x_0)}{\partial x_n}\Big)\\
&\geq f_{i_0}(x_0, u_1^{\lambda_0}(x_0), u_2^{\lambda_0}(x_0), \cdots, u_m^{\lambda_0}(x_0), \nabla u_{i_0}^{\lambda_0}(x_0))\\
&\quad - f_{i_0}(x_0, u_1(x_0), u_2(x_0), \cdots, u_m(x_0), \nabla u_{i_0}(x_0))\\
&= f_{i_0}(x_0, u_1^{\lambda_0}(x_0), u_2^{\lambda_0}(x_0), \cdots, u_m^{\lambda_0}(x_0), \nabla u_{i_0}(x_0))\\
&\quad - f_{i_0}(x_0, u_1(x_0), u_2^{\lambda_0}(x_0), \cdots, u_m^{\lambda_0}(x_0), \nabla u_{i_0}(x_0)) + \cdots\\
&\quad + f_{i_0}(x_0, u_1(x_0), u_2(x_0), \cdots, u_k^{\lambda_0}(x_0), u_{k+1}^{\lambda_0}(x_0), \cdots, u_m^{\lambda_0}(x_0), \nabla u_{i_0}(x_0))\\
&\quad - f_{i_0}(x_0, u_1(x_0), u_2(x_0), \cdots, u_k(x_0), u_{k+1}^{\lambda_0}(x_0), \cdots, u_m^{\lambda_0}(x_0), \nabla u_{i_0}(x_0)) + \cdots\\
&\quad + f_{i_0}(x_0, u_1(x_0), u_2(x_0), \cdots, u_m^{\lambda_0}(x_0), \nabla u_{i_0}(x_0))\\
&\quad - f_{i_0}(x_0, u_1(x_0), u_2(x_0), \cdots, u_m(x_0), \nabla u_{i_0}(x_0))\\
&\geq c_1(x_0)w_1^{\lambda_0}(x_0) + c_2(x_0)w_2^{\lambda_0}(x_0) + \cdots + c_k(x_0)w_k^{\lambda_0}(x_0) + \cdots + c_m(x_0)w_m^{\lambda_0}(x_0),
\end{aligned} \tag{3.2}$$

where

$$c_k(x_0) = \frac{f_{i_0}(x_0, \cdots, u_k^{\lambda_0}(x_0), u_{k+1}^{\lambda_0}(x_0), \cdots, \nabla u_{i_0}(x_0))}{u_k^{\lambda_0}(x_0) - u_k(x_0)}$$

$$-\frac{f_{i_0}(x_0,\cdots,u_k(x_0),u_{k+1}^{\lambda_0}(x_0),\cdots,\nabla u_{i_0}(x_0))}{u_k^{\lambda_0}(x_0)-u_k(x_0)}.$$

It can be easily deduced from the conditions of $u_i$ and $f_{i_0}$ that $c_k(x_0)$ are uniformly bounded.

Furthermore, by $(\mathrm{F}_1)$, we see that $c_k(x_0)\geq 0$ for $k\neq i_0$ and

$$\begin{aligned}&\sum_{k=1}^m c_k(x_0)w_k^{\lambda_0}(x_0)\\&=c_1(x_0)w_1^{\lambda_0}(x_0)+c_2(x_0)w_2^{\lambda_0}(x_0)+\cdots+c_{i_0}(x_0)w_{i_0}^{\lambda_0}(x_0)+\cdots+c_m(x_0)w_m^{\lambda_0}(x_0)\\&\geq c_1(x_0)w_{i_0}^{\lambda_0}(x_0)+c_2(x_0)w_{i_0}^{\lambda_0}(x_0)+\cdots+c_{i_0}(x_0)w_{i_0}^{\lambda_0}(x_0)+\cdots+c_m(x_0)w_{i_0}^{\lambda_0}(x_0)\\&=\Big[\sum_{k=1}^m c_k(x_0)\Big]w_{i_0}^{\lambda_0}(x_0)\\&=\widetilde{c}(x_0)w_{i_0}^{\lambda_0}(x_0).\end{aligned}$$

Thus, we have

$$(-\Delta)^L w_{i_0}^{\lambda_0}(x_0)-\widetilde{c}(x_0)w_{i_0}^{\lambda_0}(x_0)\geq 0. \tag{3.3}$$

On the other hand, by Lemma 2.1, it turns out that for sufficiently small $\iota>0$

$$(-\Delta)^L w_{i_0}^{\lambda_0}(x_0)-\widetilde{c}(x_0)w_{i_0}^{\lambda_0}(x_0)\leq\Big(C_nC'\ln\frac{1}{2\iota}-C_nC+\varrho_n-\widetilde{c}(x_0)\Big)w_{i_0}^{\lambda_0}(x_0)<0,$$

which contradicts (3.3).

**Step 2** Show that

$$\lambda_*=\sup\Big\{\lambda\in[-1,0]\mid\min_{1\leq i\leq m}w_i^{\mu}(x)\geq 0,\ \forall x\in\Sigma_\mu,\ \forall\mu\leq\lambda\Big\}=0.$$

Combining the definition of $\lambda_*$ and the continuity of $u_i(x)$, we know that $w_i^{\lambda_*}(x)\geq 0$ for all $x\in\Sigma_{\lambda_*}$. Suppose that $\lambda_*<0$. Then we can get the following claim.

**Claim 1** If $\lambda_*<0$, then $w_i^{\lambda_*}(x)>0$ for any $x\in\Sigma_{\lambda_*}\cap\Omega$, $i=1,2,\cdots,m$.

If this claim is not true, then there must exist one $i_*\in\{1,2,\cdots,m\}$ and $x_*\in\Sigma_{\lambda_*}\cap\Omega$ such that $w_{i_*}^{\lambda_*}(x_*)=0$. Applying the similar arguments as in Step 1, we can obtain that

$$(-\Delta)^L w_{i_*}^{\lambda_*}(x_*)\geq\sum_{k=1}^m c_k(x_*)w_k^{\lambda_*}(x_*)\geq c_{i_*}(x_*)w_{i_*}^{\lambda_*}(x_*)=0. \tag{3.4}$$

On the other hand, by the definition of $(-\Delta)^L$, we can compute that

$$\begin{aligned}(-\Delta)^L w_{i_*}^{\lambda_*}(x_*)&=C_n\int_{\mathbb{R}^n}\frac{-w_{i_*}^{\lambda_*}(y)}{|x_*-y|^n}\mathrm{d}y\\&=C_n\Big(\int_{\Sigma_{\lambda_*}}\frac{-w_{i_*}^{\lambda_*}(y)}{|x_*-y|^n}\mathrm{d}y+\int_{\widetilde{\Sigma}_{\lambda_*}}\frac{-w_{i_*}^{\lambda_*}(y)}{|x_*-y|^n}\mathrm{d}y\Big)\\&=C_n\Big(\int_{\Sigma_{\lambda_*}}\frac{-w_{i_*}^{\lambda_*}(y)}{|x_*-y|^n}\mathrm{d}y+\int_{\Sigma_{\lambda_*}}\frac{w_{i_*}^{\lambda_*}(y)}{|x_*-y^{\lambda_*}|^n}\mathrm{d}y\Big)\\&=C_n\int_{\Sigma_{\lambda_*}}w_{i_*}^{\lambda_*}(y)\Big(\frac{1}{|x_*-y^{\lambda_*}|^n}-\frac{1}{|x_*-y|^n}\Big)\mathrm{d}y<0,\end{aligned}$$

which contradicts (3.4). As a consequence of Claim 1, we can also get the following claim.

**Claim 2** There exists a sufficiently small $\epsilon > 0$ such that if $\lambda_* < 0$, then

$$w_i^\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda,\ \forall \lambda \leq \lambda_* + \epsilon,\ i = 1, 2, \cdots, m.$$

Otherwise, there exists a sequence $\lambda_k \searrow \lambda_*$ as $k \to \infty$ and $j_* \in \{1, 2, \cdots, m\}$, $\mu_k \in (\lambda_*, \lambda_k]$, $x^k \in \Sigma_{\mu_k} \cap \Omega$ such that

$$w_{j_*}^{\mu_k}(x^k) = \inf\{w_i^\lambda(x) \mid \forall x \in \overline{\Sigma_\lambda},\ \lambda \in [\lambda_*,\ \lambda_k],\ i = 1, 2, \cdots, m\} < 0. \tag{3.5}$$

For simplicity, we denote $d_k = \operatorname{dist}(x^k, T_{\mu_k})$. Applying the similar arguments as in Step 1, we can see that

$$(-\Delta)^L w_{j_*}^{\mu_k}(x^k) - \widetilde{c}(x^k) w_{j_*}^{\mu_k}(x^k) \geq 0 \tag{3.6}$$

and

$$\begin{aligned} &(-\Delta)^L w_{j_*}^{\mu_k}(x^k) - \widetilde{c}(x^k) w_{j_*}^{\mu_k}(x^k) \\ \leq &\Big(C_n C' \ln \frac{1}{2d_k} - C_n C + \varrho_n - \widetilde{c}(x^k)\Big) w_{j_*}^{\mu_k}(x^k) \\ &- n d_k C_n \int_{\Sigma_{\mu_k} \cap \Omega} \frac{(w_{j_*}^{\mu_k}(y) - w_{j_*}^{\mu_k}(x^k))(\mu_k - y_1)}{|x^k - y^{\mu_k}|^{n+2}} \mathrm{d}y. \end{aligned} \tag{3.7}$$

It can be easily deduced from (3.6) and (3.7) that

$$C_n C' \ln \frac{1}{2d_k} - C_n C + \varrho_n - \widetilde{c}(x^k) \leq 0.$$

Hence we have

$$\ln \frac{1}{2d_k} \leq \frac{C_n C + |\varrho_n| + |\widetilde{c}(x^k)|}{C_n C'} \leq \overline{C},$$

i.e., $d_k \geq \frac{1}{2\mathrm{e}^{\overline{C}}} > 0$. By the boundedness of $\Omega$, we can choose a convergent subsequence. For simplicity, we assume that $x^k \to x^* \in \overline{\Sigma_{\lambda_*} \cap \Omega}$ and $d(x^*, T_{\lambda_*}) \geq \frac{1}{2\mathrm{e}^{\overline{C}}} > 0$.

On the other hand, from the definition of $\lambda_*$ and (3.5), we can easily check that $\lim\limits_{k \to \infty} w_{j_*}^{\mu_k}(x^k) = w_{j_*}^{\lambda_*}(x^*) = 0$. Thus, by the Lebesgue dominated convergence theorem, we obtain

$$0 \leq \int_{\Sigma_{\lambda_*} \cap \Omega} \frac{w_{j_*}^{\lambda_*}(y)(\lambda_* - y_1)}{|x^* - y^{\lambda_*}|^{n+2}} \mathrm{d}y \leq 0.$$

Obviously, we obtain that $w_{j_*}^{\lambda_*}(x) \equiv 0$ for $x \in \Sigma_{\lambda_*} \cap \Omega$, which contradicts Claim 1. So the proof of Claim 2 is done, i.e.,

$$u_i(x_1, x') \leq u_i(-x_1, x'), \quad \forall x_1 < 0,\ (x_1, x') \in \Omega. \tag{3.8}$$

Based upon the above proof, we see that $w_i^\lambda(x) > 0$ for any $x \in \Sigma_\lambda \cap \Omega$, $\lambda < 0$. By taking $\lambda = \frac{x_1 + \overline{x}_1}{2}$ and $(x_1, x'), (\overline{x}_1, x') \in \Omega$ with $\overline{x}_1 < x_1 < 0$, we have $u_i(\overline{x}_1, x') < u_i(x_1, x')$ for $i = 1, 2, \cdots, m$. Moreover, deducing from the assumption $f(x_1, x', u, \overrightarrow{p}) = f(-x_1, x', u, \overrightarrow{p})$, we can get that $\widetilde{u}_i(x_1, x') = u_i(-x_1, x')\,(i = 1, 2, \cdots, m)$ can also serve as solutions of system (1.3). Applying the above procedure to $\widetilde{u}_i$ once more, we obtain that

$$\widetilde{u}_i(x_1, x') \le \widetilde{u}_i(-x_1, x'), \quad \forall x_1 < 0,\ (x_1, x') \in \Omega, \tag{3.9}$$

i.e.,

$$u_i(x_1, x') \ge u_i(-x_1, x'), \quad \forall x_1 < 0,\ (x_1, x') \in \Omega. \tag{3.10}$$

Combining (3.8) and (3.10), we see that

$$u_i(x_1, x') = u_i(-x_1, x'), \quad \forall x_1 < 0,\ (x_1, x') \in \Omega.$$

The proof of Theorem 1.2 is completed.

**Acknowledgement** The authors would like to deeply thank all the reviewers for their insightful and constructive comments.

## Declarations

**Conflicts of interest** The authors declare no conflicts of interest.

## References

[1] Caffarelli, L. and Silvestre, L., An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations*, **32**(8), 2007, 1245–1260.

[2] Chen, H. and Weth, T., The Dirichlet problem for the logarithmic Laplacian, *Comm. Partial Differential Equations*, **44**(11), 2019, 1100–1139.

[3] Chen, W. and Li, C., Maximum principles for the fractional $p$-Laplacian and symmetry of solutions, *Adv. Math.*, **335**, 2018, 735–758.

[4] Chen, W., Li, C. and Li, Y., A direct method of moving planes for fractional Laplacian, *Adv. Math.*, **308**, 2017, 404–437.

[5] Chen, W., Li, C. and Ou, B., Classification of solutions for an integral equation, *Comm. Pure Appl. Math.*, **59**(3), 2006, 330–343.

[6] Jarohs, S., Saldaña, A. and Weth, T., A new look at the fractional Poisson problem via the logarithmic Laplacian, *J. Funct. Anal.*, **279**(11), 2020, 108732, 50 pp.

[7] Liu, B., Direct method of moving planes for logarithmic Laplacian system in bounded domains, *Discrete Contin. Dyn. Syst.*, **38**(10), 2018, 5339–5349.

[8] Wang, G. and Ren, X., Radial symmetry of standing waves for nonlinear fractional Laplacian Hardy-Schrödinger systems, *Appl. Math. Lett.*, **110**, 2020, 106560, 8 pp.

[9] Wu, L. and Chen, W., The sliding method for the fractional $p$-Laplacian, *Adv. Math.*, **361**, 2020, 106933, 26 pp.

[10] Zhang, L. and Nie, X., A direct method of moving planes for the logarithmic Laplacian, *Appl. Math. Lett.*, **118**, 2021, 107141, 7 pp.