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Resolution of Singularities of Quasitoric Orbifolds and Equivariant Cobordism of Certain T-manifolds*

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Abstract Quasitoric orbifolds and locally 1-standard T-manifolds are the topological generalizations of simplicial projective toric varieties and good contact toric manifolds respectively. In this paper, the authors show that there is a resolution of singularities of a quasitoric orbifold. Then, they give an explicit construction of a $(2n+1)$-dimensional smooth orientable manifold with T^{n+1}-action whose boundary is a given locally 1-standard T-manifold. Moreover, they conclude that good contact toric manifolds and generalized lens spaces are equivariantly cobordant to zero.

Keywords Resolution of singularity, Quasitoric orbifold, Torus action, Locally 1-standard T-manifold, Equivariant cobordism
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1 Introduction and Main Results

This paper has two objectives. The first goal is to resolve the singularities of quasitoric orbifolds. On the other hand, we discuss the equivariant oriented cobordism class of contact toric manifolds.

Satake [29] introduced orbifolds with the name "V-manifolds" by generalizing manifolds that include singularities. V-manifolds were called orbifolds by Thurston in [31]. Orbifolds appear naturally in many areas of mathematics and physics. An orbifold X is a topological space locally modeled on the quotients of $U \subseteq \mathbb{R}^n$ by an effective action of a finite group G where the G-invariant map ϕ: $U \to X$ induces a homeomorphism from U/G to an open subset of X. The orbifold X is defined by a family of charts $\{(U, G, \phi)\}$ with compatibility of charts similar to that of manifolds which respects the action of G. Let $x \in X$ such that $x = \phi(y) \in \phi(U)$ and $G_y \subseteq G$ denote the isotropy subgroup of the point y. Note that this group is uniquely determined up to a conjugation. Then x is called a singular point in X if G_y is not trivial. The group G_y measures the order of singularity at $x \in \phi^{-1}(y)$. The readers are referred to [2] for more details on orbifolds and singularities on them.

Gromov posed the problem of resolving symplectic singularities in his book [14] and there are several developments (see [6, 19]). The resolution of singularities of orbifolds has been studied using algebraic methods (see [1, 8]) and blowup method (see [16]). Given an orbifold X with

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singular locus X_{sing}, a resolution of singularities is a proper morphism $\zeta\colon Y \to X$ which induces an isomorphism on the smooth locus $Y \setminus \zeta^{-1}(X_{\text{sing}}) \cong X \setminus X_{\text{sing}}$ where Y is a manifold (see [7, Definition 10.1.7]). Resolution of a singularity has been used to study several properties such as McKay correspondence, Hodge structure and hard Lefschetz property of certain orbifolds (see [11, 13, 32]).

If X is a singular toric variety, then there is a resolution of singularities of X (see [10, Chapter 2]). Generalizing smooth projective toric varieties and simplicial projective toric varieties, Davis and Januszkiewicz introduced the notion of (quasi)toric manifolds and (quasi)toric orbifolds respectively in the pioneering paper [9]. Briefly, a $2n$-dimensional effective orbifold with a "locally standard T^n-action" is called a quasitoric orbifold if the orbit space has the structure of an n-dimensional simple polytope. The resolution of singularities of four-dimensional quasitoric orbifolds was studied in [12]. We note that there are infinitely many quasitoric orbifolds which are not toric varieties. For example, an equivariant connected sum of two weighted projective spaces is a quasitoric orbifold but not a toric variety. In the first part of this article, we discuss the resolution of singularities of any quasitoric orbifold. Our technique is different from the proof of the resolution of singularities of a singular toric variety.

On the other hand, Pontryagin introduced the notion of cobordism in [26]. Cobordism theory discusses an equivalence relation on same-dimensional compact manifolds using the concept of the boundary of a manifold. One can get a ring structure on these equivalence classes, considering the disjoint union as addition and the Cartesian product as multiplication. We have complete information about oriented, non-oriented and complex cobordism rings (see [30]). The notion of equivariant cobordism was introduced in the early 1970s (see [17–18, 21]). Briefly, two ℓ-dimensional oriented G-manifolds M_1 and M_2 are said to be equivariantly cobordant if there exists an $(\ell+1)$-dimensional oriented G-manifold W with boundary ∂W such that ∂W is G-equivariantly diffeomorphic to $M_1 \sqcup (-M_2)$ under an orientation preserving diffeomorphism. Note that by $-M_2$, we denote the manifold M_2 with the opposite orientation. This gives an equivalence relation on the set of oriented G-manifolds, and one can define the equivariant oriented cobordism ring on the equivalence classes. The equivariant oriented cobordism rings for some nontrivial groups can be found in [15]. However, they are not known for arbitrary nontrivial group G. One of the key reasons is that in the equivariant category, the Thom transversality theorem does not hold. Thus the equivariant oriented cobordism theory cannot be reduced to the equivariant homotopy theory. In the second part of this article, we study the equivariant oriented cobordism of locally 1-standard T-manifolds, and our approach is constructive using some methods from toric topology.

We note that contact toric manifolds are odd-dimensional analogues of symplectic toric manifolds with Hamiltonian torus actions (see [3]). Lerman provided a complete classification of compact connected contact toric manifolds in [20, Theorem 2.18]. Motivated by the work of Davis and Januszkiewicz [9], Luo discussed the combinatorial construction of good contact toric manifolds and studied some of their topological properties in [22]. Recently, the construction of toric manifolds in [9] has been extended to introduce the notion of locally k-standard T-manifolds, where T is a compact abelian torus in [27]. The paper [27] considers the invariant subset $\mathbb{C}^n \times (S^1)^k$ of $\mathbb{C}^{n+k}$ with respect to the standard T^{n+k}-action, and calls the restricted T^{n+k}-action on $\mathbb{C}^n \times (S^1)^k$ the local model to define a locally k-standard T-manifold. In particular, when $k = 1$ we call this manifold a locally 1-standard T-manifold. Briefly, a $(2n+1)$-dimensional smooth compact T^{n+1}-manifold N is called a locally 1-standard T-manifold if each point of N has an invariant neighbourhood which is diffeomorphic to an invariant open subset of $\mathbb{C}^n \times S^1$ such that the orbit space N/T^{n+1} has the structure of an n-dimensional simple polytope. The article [27] shows that the category of locally 1-standard T-manifolds contains all good contact toric manifolds. We use the term "locally 1-standard T-manifold" as the topological counterpart of "good contact toric manifold". One can conclude from [28] that the generalized lens spaces are examples of locally 1-standard T-manifolds.

The paper is organized as follows. In Section 2, we recall the notion of a quasitoric manifold and a quasitoric orbifold following [9]. Then we recall the notion of blowup of a simple polytope as well as blowup of a quasitoric orbifold following [4]. We discuss the orbifold singularity on any face in the orbit space of a quasitoric orbifold. Then we apply the techniques of blowup of a quasitoric orbifold for resolution of singularity and prove the following result.

Theorem A (Theorem 2.1) *There exists a resolution of singularities of a quasitoric orbifold.*

In Section 3, we recall the concept of locally 1-standard T-manifolds and discuss some of their properties. The idea of the construction of these spaces is similar to the construction of toric manifolds introduced by Davis and Januszkiewicz [9]. Generalized lens spaces (see [28]) and good contact toric manifolds are some well known examples of locally 1-standard T-manifolds, see Examples 3.1–3.2, respectively. Then we construct a class of manifolds with boundaries which proves the following theorem.

Theorem B (Theorem 3.1) *Let N be a $(2n+1)$-dimensional locally 1-standard T-manifold. Then there is a smooth oriented T^{n+1}-manifold with boundary M such that $\partial M = N$.*

As a consequence, one can obtain that good contact toric manifolds and generalized lens spaces are equivariantly cobordant to zero, and hence, all the Stiefel-Whitney numbers of these manifolds are zero.

2 Resolution of Singularities of Quasitoric Orbifolds

In this section, we discuss the constructive definition of quasitoric manifolds and quasitoric orbifolds following [9]. These spaces are even dimensional orbifolds and equipped with half-dimensional locally standard torus actions where the orbit spaces are simple polytopes. We also recall the notion of blowup of a quasitoric orbifold along certain invariant suborbifolds. We discuss when an open convex bounded subset of $\mathbb{R}^k$ contains an integral point. We prove that there is a resolution of singularities of a quasitoric orbifold.

A convex hull of finitely many points in $\mathbb{R}^n$ for some $n \in \mathbb{Z}_{\geq 1}$ is called a convex polytope. If each vertex (zero-dimensional face) in an n-dimensional convex polytope is the intersection of n facets (codimension one faces), then the polytope is called a simple polytope. One can find the basic properties of simple polytopes in [5, 33]. For a simple polytope P we denote the vertex set of P by $V(P) := \{b_1, \cdots, b_m\}$ and the facet set by $\mathcal{F}(P) := \{F_1, \cdots, F_r\}$ throughout this paper.

Definition 2.1 *Let P be an n-dimensional simple polytope and $\lambda\colon \mathcal{F}(P) \to \mathbb{Z}^n$ be a function* *such that* $\lambda(F_i)$ *is primitive for* $i \in \{1, \cdots, r\}$ *and*

$$\{\lambda(F_{i_1}), \cdots, \lambda(F_{i_k})\} \textit{ is linearly independent if } \bigcap_{j=1}^{k} F_{i_j} \neq \emptyset. \tag{2.1}$$

Then λ *is called an* $\mathcal{R}$*-characteristic function on* P*, and the pair* (P, λ) *is called an* $\mathcal{R}$*-characteristic pair.*

If the set $\{\lambda(F_{i_1}), \cdots, \lambda(F_{i_k})\}$ *in* (2.1) *spans a* k*-dimensional unimodular submodule of* $\mathbb{Z}^n$*, then* λ *is called a characteristic function. In this case, the pair* (P, λ) *is called a characteristic pair.*

Observe that a characteristic function is an $\mathcal{R}$-characteristic function. We denote $\lambda(F_i)$ by λ_i and call it the $\mathcal{R}$-characteristic vector or characteristic vector assigned to the facet F_i according to the situation.

Next, we recall the construction of a quasitoric orbifold from an $\mathcal{R}$-characteristic pair (P, λ). We denote the standard n-dimensional torus by T^n. Note that $\mathbb{Z}^n$ is the standard n-dimensional lattice in the Lie algebra of T^n. Also, each $\lambda_i \in \mathbb{Z}^n$ determines a line in $\mathbb{R}^n\big(=\mathbb{Z}^n \bigotimes_{\mathbb{Z}} \mathbb{R}\big)$, whose image under the exponential map $\exp\colon \mathbb{R}^n \to T^n$ is a circle subgroup, denoted by T_i.

Let F be a codimension-k face of P where $0 < k \leq n$ and $\mathring{F}$ be the relative interior of F. Then $F = \bigcap_{j=1}^{k} F_{i_j}$ for some unique facets $F_{i_1}, \cdots, F_{i_k}$ of P. Let

$$T_F := \langle T_{i_1}, \cdots, T_{i_k} \rangle$$

be the k-dimensional subtorus of T^n generated by $T_{i_1}, \cdots, T_{i_k}$. We define $T_P := \{1\} \in T^n$.

We consider the identification space $M(P, \lambda) := (T^n \times P)/\sim$, where the equivalence relation $\sim$ is defined by

$$(t, x) \sim (s, y) \text{ if and only if } x = y \in \mathring{F} \text{ and } t^{-1}s \in T_F.$$

The identification space $M(P, \lambda)$ has a $2n$-dimensional orbifold structure with a natural T^n-action induced by the group operation on the first factor of $T^n \times P$. The projection onto the second factor gives the orbit map

$$\pi\colon M(P, \lambda) \to P \;\text{ defined by } [t, x]_\sim \mapsto x,$$

where $[t, x]_\sim$ is the equivalence class of (t, x). An explicit construction of the orbifold structure on $M(P, \lambda)$ is discussed in [12, 25]. The authors gave the axiomatic definition of quasitoric orbifolds and showed that the constructive and axiomatic definitions of quasitoric orbifolds are equivalent, see [25, Proposition 2.9].

We note that if λ is a characteristic function, then $M(P, \lambda)$ is a quasitoric manifold (see [9]). The equivalence of the axiomatic and constructive definitions of quasitoric manifolds is discussed in [9].

Now, we review the notion of blowup of a simple polytope P as well as blowup of the quasitoric orbifold $M(P, \lambda)$ along an invariant suborbifold.

Definition 2.2 *Let* P *be an* n*-dimensional simple polytope in* $\mathbb{R}^n$ *and* F *be a face of* P*. Take an* $(n-1)$*-dimensional hyperplane* H *in* $\mathbb{R}^n$ *such that* $V(F)$ *is a subset of the open half* *space $H_{>0}$ and $V(P) \setminus V(F)$ is a subset of the other open half space $H_{<0}$. Then $\overline{P} := P \cap H_{\leq 0}$ is called a blowup of P along F.*

Note that $\overline{P} \subset P$ and $\overline{P}$ is an n-dimensional simple polytope. If $\mathcal{F}(P) = \{F_1, \cdots, F_r\}$ and $\dim(F) < (n-1)$, then the facets of $\overline{P}$ are $\mathcal{F}(\overline{P}) := \{\overline{F}_1, \cdots, \overline{F}_r, \overline{F}_{r+1}\}$ where

$$\overline{F}_i := \begin{cases} F_i \cap \overline{P} & \text{for } i = 1, \cdots, r, \\ H \cap P & \text{for } i = r+1. \end{cases} \tag{2.2}$$

In this paper, a blowup $\overline{P}$ of P might also be denoted by $P^{(1)}$ and a blowup of $P^{(\ell)}$ is denoted by $P^{(\ell+1)}$, for $\ell \geq 1$.

Definition 2.3 *Let $\overline{P}$ be a blowup of a simple polytope P along a face F. Let (P, λ) and $(\overline{P}, \bar{\lambda})$ be two $\mathcal{R}$-characteristic pairs such that*

$$\overline{\lambda}(\overline{F}_i) = \lambda(F_i), \quad \textit{if } \overline{F}_i = F_i \cap \overline{P} \textit{ for } F_i \in \mathcal{F}(P).$$

Then the quasitoric orbifold $M(\overline{P}, \overline{\lambda})$ is called a blowup of the quasitoric orbifold $M(P, \lambda)$ along the suborbifold $\pi^{-1}(F)$.

Example 2.1 (see [4, Example 5.2]) Let (P, λ) be an $\mathcal{R}$-characteristic pair and $\overline{P}$ be a blowup of P along a codimension k face F where $0 < k \leq n$. Then $F = \bigcap\limits_{j=1}^{k} F_{i_j}$ for some unique facets $F_{i_1}, \cdots, F_{i_k}$ of P. Let

$$\mathbb{Q}(F) := \Big\{(c_1, \cdots, c_k) \in \mathbb{Q}^k \mid c_j \in \mathbb{Q} \setminus \{0\} \text{ and } \mathbf{0} \neq \sum_{j=1}^{k} c_j \lambda_{i_j} \in \mathbb{Z}^n\Big\}.$$

We define $\overline{\lambda}\colon \mathcal{F}(\overline{P}) \to \mathbb{Z}^n$ by

$$\overline{\lambda}(\overline{F_i}) := \begin{cases} \lambda_i, & \text{if } \overline{F}_i = F_i \cap \overline{P} \text{ for } i = 1, 2, \cdots, r, \\ \operatorname{prim}\Big(\sum\limits_{j=1}^{k} c_j \lambda_{i_j}\Big), & \text{if } \overline{F}_i = \overline{F}_{r+1} \text{ and } (c_1, \cdots, c_k) \in \mathbb{Q}(F), \end{cases} \tag{2.3}$$

where $\operatorname{prim}(\alpha)$ indicates the primitive vector of $\alpha \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Then $\overline{\lambda}$ is an $\mathcal{R}$-characteristic function on $\overline{P}$. The pair (P, λ) and $(\overline{P}, \overline{\lambda})$ satisfy Definition 2.3. Thus $M(\overline{P}, \overline{\lambda})$ is a blowup of the quasitoric orbifold $M(P, \lambda)$ along the suborbifold $\pi^{-1}(F)$.

Let $\overline{P}$ be a blowup of P along the face F such that F is not a facet of P. If $\overline{F}$ is the new facet $H \cap P$ in $\overline{P}$ arising in the place of F then there is a face preserving map $\zeta\colon \overline{P} \to P$ such that $\zeta(\overline{F}) = F$. This induces a map

$$\overline{\zeta}\colon M(\overline{P}, \bar{\lambda}) \to M(P, \lambda) \tag{2.4}$$

(see [13, Section 4]). Then, the restriction of $\overline{\zeta}$ gives an isomorphism of orbifolds between $M(\overline{P}, \overline{\lambda}) \setminus \pi^{-1}(\overline{F})$ and $M(P, \lambda) \setminus \pi^{-1}(F)$ (see [13, Lemma 4.1]).

Let (P, λ) be an $\mathcal{R}$-characteristic pair and F be a face as in Example 2.1. Let $\lambda_{i_j} = \lambda(F_{i_j})$ for all $j = 1, 2, \cdots, k$. Then the group

$$G_F(P, \lambda) := \frac{\Big(\langle \lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k} \rangle \bigotimes\limits_{\mathbb{Z}} \mathbb{R}\Big) \cap \mathbb{Z}^n}{\langle \lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k} \rangle}$$

is finite and abelian, where $\langle \lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k} \rangle$ is the submodule generated by the set $\{\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k}\}$. This group measures the order of singularity of points in $\pi^{-1}(\mathring{F}) \subseteq M(P, \lambda)$. For simplicity, we may denote $G_F(P, \lambda)$ by G_F, whenever the context is clear. The order of G_F may be called the order of singularity over a point in $\mathring{F}$. Note that if F is a face of F', then $|G_{F'}|$ divides $|G_F|$. Also, the group G_F is trivial if $M(P, \lambda)$ is a quasitoric manifold.

We recall the definition of the volume of a parallelepiped in an inner product space. Let $\{u_1, u_2, \cdots, u_k\}$ be an orthonormal basis in the k-dimensional real inner product space V. Let

$$C_V := \Big\{ \sum_{i=1}^{k} r_i u_i \in V \mid 0 \leq r_i \leq \alpha_i, r_i \in \mathbb{R} \text{ and } 0 < \alpha_i \in \mathbb{R} \Big\}.$$

Then C_V is a k-dimensional parallelepiped and

$$\mathrm{vol}(C_V) = \alpha_1 \alpha_2 \cdots \alpha_k.$$

For a face $F = \bigcap_{j=1}^{k} F_{i_j}$, we consider the vector space $V_F = \big(\langle \lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k} \rangle \bigotimes_{\mathbb{Z}} \mathbb{R}\big)$. Let $\{v_1, \cdots, v_k\}$ be a basis of the lattice

$$L_F := \Big(\langle \lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k} \rangle \bigotimes_{\mathbb{Z}} \mathbb{R} \Big) \cap \mathbb{Z}^n$$

in V_F. So $L_F = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_k$. Define

$$C := \Big\{ \sum_{i=1}^{k} r_i v_i \mid 0 \leq r_i < 1,\ r_i \in \mathbb{R} \Big\}.$$

This C is called a fundamental parallelepiped for the lattice L_F. It is well known that $\mathrm{vol}(C)$ is independent of the bases of L_F. Let

$$C_F = \Big\{ \sum_{j=1}^{k} r_j \lambda_{i_j} \mid 0 \leq r_j < 1,\ r_j \in \mathbb{R} \Big\}. \tag{2.5}$$

Then $|G_F|$ measures the volume of the k-dimensional parallelepiped $C_F \subset V_F$ made by the k vectors $\{\lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k}\}$, where $|G_F|$ is the order of the group G_F. Mathematically, one can write

$$\mathrm{vol}(C_F) = |G_F| \times \mathrm{vol}(C).$$

The following Minkowski theorem says when an open convex subset of $\mathbb{R}^n$ contains a non-zero lattice point.

Proposition 2.1 (see [24, Theorem 5.2]) *Let X be an open convex subset of a k-dimensional inner product space $V \subset \mathbb{R}^n$ and C be the fundamental parallelepiped for the lattice $V \cap \mathbb{Z}^n$. If X is symmetric about origin and has volume greater than $2^k \mathrm{vol}(C)$, then X contains a point $\mathbf{b} = (b_1, b_2, \cdots, b_n) \in V \cap \mathbb{Z}^n$ other than the origin.*

Lemma 2.1 *Let λ be an $\mathcal{R}$-characteristic function on an n-dimensional simple polytope P and $F = \bigcap_{j=1}^{k} F_{i_j}$ be a face in P. If $|G_F| > 1$, then there exists a non-zero $\lambda_F \in \mathbb{Z}^n$ such that*

$\lambda_F = \sum_{j=1}^{k} c_j \lambda_{i_j}$ *with* $|c_j| < 1$ *and* $c_j \in \mathbb{Q}$. *Moreover, if* $|G_{F'}| = 1$ *for every face* F' *properly containing* F *in* P, *then there exists a non-zero* $\lambda_F \in \mathbb{Z}^n$ *such that* $\lambda_F = \sum_{j=1}^{k} c_j \lambda_{i_j}$, *where* $0 < |c_j| < 1$ *and* $c_j \in \mathbb{Q}$ *for* $j = 1, 2, \cdots, k$.

Proof The parallelepiped C_F constructed in (2.5) is convex but not symmetric about the origin. Consider the union of 2^k many parallelepipeds defined by

$$\widetilde{C}_F = \Big\{ \sum_{j=1}^{k} r_j \lambda_{i_j} \mid -1 < r_j < 1,\ r_j \in \mathbb{R} \Big\}.$$

This is an open parallelepiped which is convex and symmetric about the origin such that

$$\mathrm{vol}(\widetilde{C}_F) = 2^k \mathrm{vol}(C_F) = 2^k |G_F| \mathrm{vol}(C) > 2^k \mathrm{vol}(C).$$

Then by Proposition 2.1, there exists a non-zero point $\lambda_F \in \mathbb{Z}^n \cap \widetilde{C}_F$. Then there exist $c_j \in \mathbb{Q}$ with $|c_j| < 1$ for all $j \in \{1, 2, \cdots, k\}$ such that $\lambda_F = \sum_{j=1}^{k} c_j \lambda_{i_j}$.

Now for the second part, without loss of generality, we may assume that $c_1 = 0$. Consider the face $F' := \bigcap_{j=2}^{k} F_{i_j}$. Then F' contains the face F properly. By our assumption, we have $|G_{F'}| = 1$. Then $\{\lambda_{i_2}, \lambda_{i_3}, \cdots, \lambda_{i_k}\}$ is a $\mathbb{Z}$-basis of the lattice $\big(\langle \lambda_{i_2}, \lambda_{i_3}, \cdots, \lambda_{i_k} \rangle \bigotimes_{\mathbb{Z}} \mathbb{R}\big) \cap \mathbb{Z}^n$. Observe that this contradicts the existence of a non-zero lattice point $\lambda_F = \sum_{j=2}^{k} c_j \lambda_{i_j}$ such that $|c_j| < 1$ for $j \in \{2, 3, \cdots, k\}$. Therefore $0 < |c_1| < 1$. Similarly, we can show $0 < |c_j| < 1$ for $j \in \{2, 3, \cdots, k\}$.

Remark 2.1 Let $F = \bigcap_{j=1}^{k} F_{i_j}$ be a face of P and λ be an $\mathcal{R}$-characteristic function on P.

(1) If $|G_F| = 1$, then

$$\Big(\langle \lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k} \rangle \bigotimes_{\mathbb{Z}} \mathbb{R}\Big) \cap \mathbb{Z}^n = \mathbb{Z}\lambda_{i_1} + \mathbb{Z}\lambda_{i_2} + \cdots + \mathbb{Z}\lambda_{i_k}$$

and there does not exist a point of $\big(\langle \lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k} \rangle \bigotimes_{\mathbb{Z}} \mathbb{R}\big) \cap \mathbb{Z}^n$ in the parallelepiped C_F except the origin.

(2) If $|G_F| > 1$, then the non-zero lattice points of $\big(\langle \lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_k} \rangle \bigotimes_{\mathbb{Z}} \mathbb{R}\big) \cap \mathbb{Z}^n$ in the parallelepiped C_F have a one-one correspondence to non-identity elements of the group G_F. So the total number of lattice points in the parallelepiped C_F is $|G_F|$. Thus λ_F obtained in Lemma 2.1 is not unique if $|G_F| > 2$.

Theorem 2.1 *For a quasitoric orbifold* $M(P, \lambda)$, *there exists a chain of blowups*

$$M(P^{(d)}, \lambda^{(d)}) \to \cdots \to M(P^{(1)}, \lambda^{(1)}) \to M(P, \lambda)$$

such that $M(P^{(d)}, \lambda^{(d)})$ *is a quasitoric manifold, the quasitoric orbifold* $M(P^{(i)}, \lambda^{(i)})$ *is a blowup of* $M(P^{(i-1)}, \lambda^{(i-1)})$ *and the map* $M(P^{(i)}, \lambda^{(i)}) \to M(P^{(i-1)}, \lambda^{(i-1)})$ *is defined as in* (2.4) *for* $i = 1, 2, \cdots, d$.

Proof Let

$$\mathcal{L} := \{F \mid F \text{ is a face of } P \text{ and } |G_F(P,\lambda)| \neq 1\}.$$

Define a partial order "$\leq$" on $\mathcal{L}$ by $F \leq F'$ if $F \subseteq F'$. Without loss of generality, let F be a maximal element in the set $\mathcal{L}$ with respect to the partial order "$\leq$".

Now consider the blowup of the polytope P along the face F and denote this blowup by $P^{(1)}$. If $\mathcal{F}(P) = \{F_1, \cdots, F_r\}$, then we have

$$\mathcal{F}(P^{(1)}) = \{\overline{F}_1, \cdots, \overline{F}_r, \overline{F}_{r+1}\}$$

defined as in (2.2). Let $F = \bigcap\limits_{j=1}^{k} F_{i_j}$ where $F_{i_j} \in \mathcal{F}(P)$ and $\lambda_{i_j} = \lambda(F_{i_j})$ for $j \in \{1, 2, \cdots, k\}$. We define a function $\lambda^{(1)} : \mathcal{F}(P^{(1)}) \to \mathbb{Z}^n$ by

$$\lambda^{(1)}(\overline{F_i}) := \begin{cases} \lambda_i, & \text{if } \overline{F}_i = F_i \cap \overline{P} \text{ for } i = 1, 2, \cdots, r, \\ \operatorname{prim}(\lambda_F) = \operatorname{prim}\Big(\sum\limits_{j=1}^{k} c_j \lambda_{i_j}\Big), & \text{if } \overline{F}_i = \overline{F}_{r+1}, \end{cases} \tag{2.6}$$

where λ_F is obtained by Lemma 2.1. Then $(P^{(1)}, \lambda^{(1)})$ is an $\mathcal{R}$-characteristic pair.

There does not exist a face F' containing F such that $|G_{F'}| \neq 1$, by the definition of $\mathcal{L}$. Thus we can assume $0 < |c_j| < 1$ and $c_j \in \mathbb{Q}$ for every $j \in \{1, 2, \cdots, k\}$ by Lemma 2.1. Let

$$V(F) := \{b_1, b_2, \cdots, b_\ell\} \quad \text{and} \quad V(P) = \{b_1, b_2, \cdots, b_\ell, b_{\ell+1}, \cdots, b_m\}.$$

Note that there is a homeomorphism from $\overline{F}_{r+1}$ to $F \times \Delta^{k-1}$ preserving the face structure. Let $V(\Delta^{k-1}) = \{u_1, u_2, \cdots, u_k\}$. We have $P^{(1)} \subset P$ from Definition 2.2. Then

$$V\big(P^{(1)}\big) := \{(b_p, u_s) \mid 1 \leq p \leq \ell; 1 \leq s \leq k\} \cup \{b_{\ell+1}, \cdots, b_m\}.$$

We can assume (b_p, u_s) is not a vertex of $\overline{F}_{i_s}$ for $1 \leq s \leq k$. For a vertex $b_p \in V(F) \subset P$, let $b_p = \big(\bigcap\limits_{j=k+1}^{n} F_{i_j}\big) \cap F = \bigcap\limits_{j=1}^{n} F_{i_j}$. Then for all $1 \leq s \leq k$ and $1 \leq p \leq \ell$,

$$(b_p, u_s) = \bigcap_{j=1\ j\neq s}^{n} \overline{F}_{i_j} \cap \overline{F}_{r+1}.$$

Now we calculate the order of singularity over each vertex in $P^{(1)}$. Note that

$$\begin{aligned} |G_{(b_p,u_s)}(P^{(1)}, \lambda^{(1)})| &= \Big|\det\Big[\lambda_{i_1}, \cdots, \widehat{\lambda_{i_s}}, \cdots, \lambda_{i_n}, \operatorname{prim}\Big(\sum_{j=1}^{k} c_j \lambda_{i_j}\Big)\Big]\Big| \\ &= \frac{|c_s|}{d_F} |G_{b_p}(P,\lambda)| < |G_{b_p}(P,\lambda)| \end{aligned}$$

for $1 \leq p \leq \ell$ and $1 \leq s \leq k$ and d_F is a positive integer satisfying $\lambda_F = d_F \operatorname{prim}(\lambda_F)$, and

$$|G_{b_{\ell+i}}(P^{(1)}, \lambda^{(1)})| = |G_{b_{\ell+i}}(P,\lambda)| \quad \text{for } 1 \leq i \leq m - \ell.$$

Therefore, in the above process, if a vertex of P is not in the face F, then the corresponding singularity remains same. Also corresponding to every vertex b in F we get exactly k many new vertices in $\overline{F}_{r+1}$ such that at each of them the singularity is strictly less than the singularity on $b \in F \subset P$ in the given orbifold.

If $|G_F| = 1$ for every face of $P^{(1)}$, then $M(P^{(1)}, \lambda^{(1)})$ is the desired resolution of singularities of $M(P, \lambda)$. Otherwise we repeat the above process on $M(P^{(1)}, \lambda^{(1)})$ to obtain an $\mathcal{R}$-characteristic pair $(P^{(2)}, \lambda^{(2)})$ where $P^{(2)}$ is a blowup of $P^{(1)}$, and $\lambda^{(2)}$ is defined similarly as in (2.6) from $\lambda^{(1)}$. If $|G_F| = 1$ for every face of $P^{(2)}$ corresponding to the pair $(P^{(2)}, \lambda^{(2)})$ then we are done. Otherwise, we repeat the process. Since the order of G_F is finite for any face F of P, this inductive process ends after finite steps.

Example 2.2 Consider the $\mathcal{R}$-characteristic function and the edge F of the triangular prism P_1 in Figure 1(a).

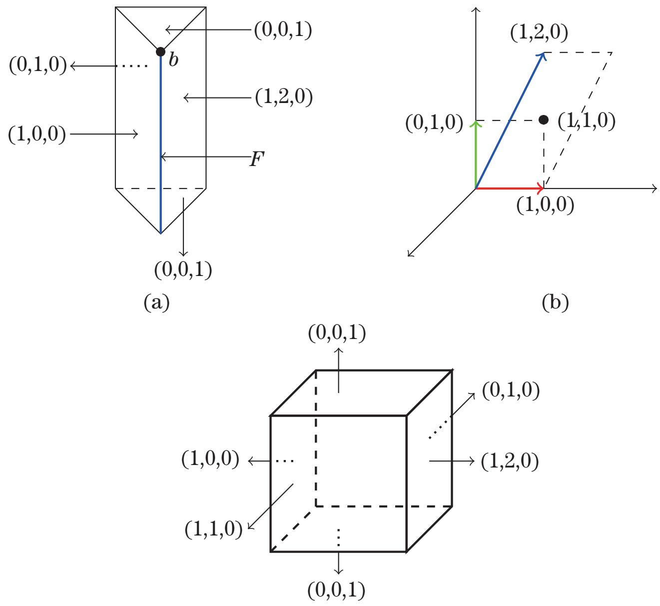

Figure 1 Some characteristic functions before and after blow up.

We have the $\mathbb{Z}$-module generated by $\{(1, 2, 0), (1, 0, 0)\}$. Then

$$\Big(\langle (1,2,0), (1,0,0)\rangle \bigotimes_{\mathbb{Z}} \mathbb{R}\Big) \cap \mathbb{Z}^3 = \{(x, y, 0) \mid x, y \in \mathbb{Z}\} \cong \mathbb{Z}^2,$$

which has a basis $\{(1, 0, 0), (0, 1, 0)\}$. In this case, $\mathrm{vol}(C_F) = 2$. Thus for this edge F, we get $|G_F| = |\mathbb{Z}^2/\langle (1, 2, 0), (1, 0, 0)\rangle| = 2$. Since the faces containing F are facets of P_1, by Lemma 2.1 there exists a non-zero $\lambda_F \in \mathring{C}_F \cap \mathbb{Z}^3$. Here, the only non-zero lattice point in the parallelepiped C_F is $\lambda_F = (1, 1, 0)$ which can be represented as (Figure 1(b))

$$(1, 1, 0) = \frac{1}{2}(1, 0, 0) + \frac{1}{2}(1, 2, 0).$$

Also, similarly, one can calculate $|G_b| = 2$ for $b \in V(F)$ and $|G_b| = 1$ for $b \in V(P_1) - V(F)$. So, the maximal element in $\mathcal{L}$ for this case is F. Thus we blow up P_1 along the face F, and get the cube $P_1^{(1)}$ as in Figure 1(c). One can define $\lambda^{(1)}$ on $P_1^{(1)}$ as in (2.6). Then in the quasitoric orbifold $M(P_1^{(1)}, \lambda^{(1)})$, we have $|G_E| = 1$ for every face E of $P_1^{(1)}$. Thus $M(P_1^{(1)}, \lambda^{(1)})$ is a quasitoric manifold, and it is a resolution of singularities of $M(P_1, \lambda)$.

3 Equivariant Cobordism of Locally 1-Standard T-manifolds

In this section, we recall the concept of locally 1-standard T-manifolds and discuss some properties following [27]. Then we prove that any locally 1-standard T-manifold is equivariantly the boundary of an oriented smooth manifold. In particular, good contact toric manifolds and generalized lens spaces are equivariantly cobordant to zero.

Consider the action $\alpha\colon T^{n+1}\times\mathbb{C}^{n+1}\to\mathbb{C}^{n+1}$ of $(n+1)$-dimensional torus T^{n+1} on $\mathbb{C}^{n+1}$ defined by

$$\alpha((t_1,\cdots,t_n,t_{n+1}),(z_1,\cdots,z_n,z_{n+1}))=(t_1z_1,\cdots,t_nz_n,t_{n+1}z_{n+1}).$$

Then $\mathbb{C}^n\times S^1$ is a T^{n+1}-invariant subset of $\mathbb{C}^{n+1}$ and the orbit space $(\mathbb{C}^n\times S^1)/T^{n+1}$ is $(\mathbb{R}_{\geq 0})^n$. The restriction $\alpha|_{T^{n+1}\times(\mathbb{C}^n\times S^1)}$ is called the standard T^{n+1}-action on $\mathbb{C}^n\times S^1$.

Definition 3.1 *A $(2n+1)$-dimensional smooth manifold N with an effective T^{n+1}-action is called a locally 1-standard T-manifold if the orbit space N/T^{n+1} has the structure of an n-dimensional simple polytope and, for each point $p\in N$, there is*

(1) *an automorphism $\theta\in\mathrm{Aut}(T^{n+1})$,*

(2) *a T^{n+1}-invariant neighbourhood U of p in N and a T^{n+1}-invariant open subset U' of $\mathbb{C}^n\times S^1$ such that there is a θ-equivariantly diffeomorphism $f_\theta: U\to U'$(that is, $f_\theta(tx)=\theta(t)f_\theta(x)$ for $(t,x)\in T^{n+1}\times U$).*

Let

$$\mathfrak{q}\colon N\to Q$$

be the orbit map where Q is an n-dimensional simple polytope. Let $\mathcal{F}(Q):=\{E_1,\cdots,E_\ell\}$ be the set of facets of Q. Then each $N_j:=\mathfrak{q}^{-1}(E_j)$ is a $(2n-1)$-dimensional T^{n+1}-invariant submanifold of N. Then, the isotropy subgroup of N_j is a circle subgroup T_j of T^{n+1}. The group T_j is uniquely determined by a primitive vector $\eta_j\in\mathbb{Z}^{n+1}$ for $j=1,2,\cdots,\ell$, that is, we get a natural function

$$\eta\colon\{E_1,\cdots,E_\ell\}\to\mathbb{Z}^{n+1} \tag{3.1}$$

defined by $\eta(E_j)=\eta_j$.

We discuss the constructive definition of $(2n+1)$-dimensional locally 1-standard T-manifolds on simple polytopes following [27]. Let $\mathcal{F}(Q):=\{E_1,\cdots,E_\ell\}$ be the set of facets of an n-dimensional simple polytope Q.

Definition 3.2 *A function $\xi\colon\mathcal{F}(Q)\to\mathbb{Z}^{n+1}$ is called a hyper characteristic function if $\langle\xi_{j_1},\cdots,\xi_{j_n}\rangle$ is a rank n unimodular submodule of $\mathbb{Z}^{n+1}$ whenever $E_{j_1}\cap\cdots\cap E_{j_n}\neq\emptyset$ where $\xi_j:=\xi(E_j)$ for $j=1,\cdots,\ell$. The pair (Q,ξ) is called a hyper characteristic pair.*

Note that a hyper characteristic function was defined on simplices in [28, Definition 2.1] and Definition 3.2 is the case $k=1$ in [27, Definition 2.2].

Let ξ be a hyper characteristic function on an n-dimensional simple polytope Q. For a point $p\in Q$, let $E_{j_1}\cap\cdots\cap E_{j_k}$ be the face of Q containing p in its relative interior. Then $\exp\big(\langle\xi_{j_1},\cdots,\xi_{j_k}\rangle\bigotimes_{\mathbb{Z}}\mathbb{R}\big)$ is a k-dimensional subtorus of T^{n+1}. We denote this subgroup by T_p.

If p belongs to the relative interior of Q, we denote $T_p = 1$, the identity in T^{n+1}. We define the following identification space:

$$N(Q,\xi) := (T^{n+1} \times Q)/\sim', \tag{3.2}$$

where

$$(t,p) \sim' (s,q) \text{ if and only if } p = q \text{ and } t^{-1}s \in T_p.$$

There is a T^{n+1} action on $N(Q,\xi)$ induced by the multiplication on the first factor of $T^{n+1} \times Q$.

Proposition 3.1 *Let (Q,ξ) be a hyper characteristic pair. Then the space $N(Q,\xi)$ in* (3.2) *is a locally* 1*-standard T-manifold.*

Proof Let Z_Q be the moment angle manifold corresponding to Q (see [5, Section 6.2]). Then Z_Q is a smooth manifold and there is a smooth T^ℓ-action on Z_Q where ℓ is the number of facets of Q (see [5, Corollary 6.2.5]). The space $N(Q,\xi)$ has a manifold structure which satisfies conditions (1)–(2) in Definition 3.1 (see [27, Proposition 2.3]). If rank of $\langle \xi_1, \cdots, \xi_\ell \rangle$ is n, then $N(Q,\xi)$ is equivariantly homeomorphic to $M(Q,\lambda_\xi) \times S^1$ for some quasitoric manifold $M(Q,\lambda_\xi)$ (see [27, Proposition 2.6]). If rank of $\langle \xi_1, \cdots, \xi_\ell \rangle$ is $n+1$, then $N(Q,\xi)$ is equivariantly homeomorphic to Z_Q/T_ξ for some $(\ell-n-1)$-dimensional subgroup T_ξ of T^ℓ (see [27, Proposition 2.7]). Also, $M(Q,\lambda_\xi)$ is equivariantly homeomorphic to Z_Q/T_{λ_ξ} for some $(\ell-n)$-dimensional subgroup T_{λ_ξ} of T^ℓ. Therefore, $N(Q,\xi)$ has a smooth structure such that the T^{n+1}-action on $N(Q,\xi)$ is smooth.

Note that the function η defined in (3.1) satisfies Definition 3.2, see the explanation in [27, Subsection 2.1]. Also the orientation of a locally 1-standard T-manifold $N(Q,\xi)$ can be induced from an orientation of Q and T^{n+1}. If θ is identity in Definition 3.1(2), then the map is called an equivariant diffeomorphism. In addition, if it preserves the orientation then it is called an oriented equivariant diffeomorphism.

Proposition 3.2 (see [27]) *Let N be a locally* 1*-standard T-manifold over the n-dimensional simple polytope Q, and η be a function as defined in* (3.1)*. Then N is equivariantly diffeomorphic to $N(Q,\eta)$.*

Proof This follows by similar arguments in the proofs of Lemma 1.4 and [9, Proposition 1.8] and Proposition 3.1.

Example 3.1 (Generalized lens spaces) Let Δ^n be the n-dimensional simplex and ξ be a hyper characteristic function on it. Assume the rank of $\langle \{\xi(E) \mid E \in \mathcal{F}(\Delta^n)\} \rangle \subseteq \mathbb{Z}^{n+1}$ is $(n+1)$. [28] shows that $N(\Delta^n,\xi)$ is equivariantly homeomorphic (hence diffeomorphic) to the orbit space S^{2n+1}/G_ξ for some free action of a finite group G_ξ. This space is called a generalized lens space in [28]. In particular, if $\{\xi(E) \mid E \in \mathcal{F}(\Delta^n)\}$ forms a basis of $\mathbb{Z}^{n+1}$, then $N(\Delta^n,\xi)$ is homeomorphic to S^{2n+1}.

Consider an integer $p > 1$ and n integers $q_1, \cdots, q_n$ such that $\gcd\{p, q_i\} = 1$ for all $i = 1, \cdots, n$. Then $\mathbb{Z}_p$ acts freely on S^{2n+1} by the following

$$g(z_0, z_1, \cdots, z_n) = (gz_0, g^{q_1}z_1, \cdots, g^{q_n}z_n).$$

The lens space $L(p; q_1, \cdots, q_n)$ is defined to be the orbit space $S^{2n+1}/\mathbb{Z}_p$. [28] showed that there is a hyper characteristic function ξ on Δ^n such that $L(p; q_1, \cdots, q_n)$ is equivariantly diffeomorphic to $N(\Delta^n, \xi)$.

Example 3.2 (Good contact toric manifolds) Luo [22, Chapter 2] discussed the construction of good contact toric manifolds which are compact connected contact toric manifolds studied in [20, Section 2]. We briefly recall the construction. Let Q be an n-dimensional simple lattice polytope embedded in $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$. Consider the cone $C(Q)$ on Q with apex $\mathbf{0} \in \mathbb{R}^{n+1}$ and the set $\{\widetilde{E} \mid E \in \mathcal{F}(Q)\}$ of facets of $C(Q)$ where $\widetilde{E} := C(E) \setminus \{0\}$. Let $\xi(E)$ be the primitive outward normal vector on $\widetilde{E}$. This defines a function $\xi \colon \mathcal{F}(Q) \to \mathbb{Z}^{n+1}$. Since the facets of $C(Q) - \{0\}$ intersect transversely, the function ξ satisfies Definition 3.2. Then the space $N(Q, \xi)$ is T^{n+1}-equivariantly homeomorphic to a good contact toric manifold whose moment cone is $C(Q)$. Moreover, any good contact toric manifold can be obtained in this way. For details, we refer to [20, 22].

Lemma 3.1 *Let Q be an n-dimensional simple polytope and*

$$\xi \colon \mathcal{F}(Q) \to \mathbb{Z}^{n+1}$$

be a hyper characteristic function. Then there exists $\mathbf{a} = (a_1, \cdots, a_{n+1}) \in \mathbb{Z}^{n+1}$ such that the set $\{\mathbf{a}, \xi(E_{j_1}), \cdots, \xi(E_{j_n})\}$ is a linearly independent subset in $\mathbb{Z}^{n+1}$ whenever $E_{j_1} \cap \cdots \cap E_{j_n}$ is a vertex of Q.

Proof Let b be a vertex of Q. Then $b = E_{j_1} \cap \cdots \cap E_{j_n}$ for some unique facets $E_{j_1}, \cdots, E_{j_n}$. Let $\mathbb{Z}_b$ be the submodule of $\mathbb{Z}^{n+1}$ generated by $\xi_{j_1}, \cdots, \xi_{j_n}$. So the rank of $\mathbb{Z}_b$ is n for any vertex $b \in V(Q)$. Since there are only finitely many vertices in Q we have $\mathbb{Z}^{n+1} \setminus \bigcup\limits_{b \in V(Q)} \mathbb{Z}_b$ is non-empty. We prove this by contradiction. Suppose that

$$\mathbb{Z}^{n+1} = \bigcup_{b \in V(Q)} \mathbb{Z}_b. \tag{3.3}$$

Let $V_b := \mathbb{Z}_b \bigotimes\limits_{\mathbb{Z}} \mathbb{R}$. Then V_b is an n-dimensional linear subspace of $\mathbb{R}^{n+1}$. From (3.3) it follows that

$$\mathbb{R}^{n+1} = \mathbb{Z}^{n+1} \bigotimes_{\mathbb{Z}} \mathbb{R} = \bigcup_{b \in V(Q)} \Big(\mathbb{Z}_b \bigotimes_{\mathbb{Z}} \mathbb{R}\Big) = \bigcup_{b \in V(Q)} V_b,$$

which is a contradiction as $V(Q)$ is a finite set. Let $\mathbf{a} \in \mathbb{Z}^{n+1} \setminus \bigcup\limits_{b \in V(Q)} \mathbb{Z}_b$ be primitive. Then $\mathbf{a}$ is the desired vector of the lemma.

Theorem 3.1 *Let N be a $(2n+1)$-dimensional locally 1-standard T-manifold. Then there is a smooth oriented T^{n+1}-manifold with boundary M and an orientation preserving equivariant diffeomorphism from ∂M to N.*

Proof There exists a hyper characteristic pair (Q, ξ) such that N is equivariantly diffeomorphic to $N(Q, \xi)$ by Proposition 3.2. Let $\mathcal{F}(Q) := \{E_1, \cdots, E_\ell\}$ and $I = [0, 1]$. Therefore, $Q \times I$ is an $(n+1)$-dimensional simple polytope and the set of facets of $Q \times I$ is $\mathcal{F}(Q \times I) := \{E_1 \times I, \cdots, E_\ell \times I, Q \times \{0\}, Q \times \{1\}\}$. Let $\mathbf{a} \in \mathbb{Z}^{n+1}$ satisfy Lemma 3.1. Define a map $\lambda\colon \mathcal{F}(Q \times I) \to \mathbb{Z}^{n+1}$ by

$$\lambda(F) = \begin{cases} \xi(E_i), & \text{if } F = E_i \times I \text{ for } i = 1, 2, \cdots, l, \\ \mathbf{a}, & \text{if } F = Q \times \{0\},\ Q \times \{1\}. \end{cases} \tag{3.4}$$

Then λ is an $\mathcal{R}$-characteristic function on $P := Q \times I$ and $M(P, \lambda)$ is a quasitoric orbifold. Theorem 2.1 gives a resolution of singularities

$$M(P^{(d)}, \lambda^{(d)}) \to \cdots \to M(P^{(1)}, \lambda^{(1)}) \to M(P, \lambda) \tag{3.5}$$

for $M(P, \lambda)$, where $M(P^{(d)}, \lambda^{(d)})$ is a quasitoric manifold and $M(P^{(i)}, \lambda^{(i)})$ is a blowup of $M(P^{(i-1)}, \lambda^{(i-1)})$ for $i = 1, \cdots, d$.

Note that for any face E of Q, the codimension of E in Q is the same as the codimension of the face $E \times I$ in P. Moreover, if

$$E = \bigcap_{s=1}^{k} E_{j_s}$$

for some unique facets $E_{j_1}, E_{j_2}, \cdots, E_{j_k}$ of Q, then $E_{j_1} \times I, E_{j_2} \times I, \cdots, E_{j_k} \times I$ are facets of P and

$$E \times I = \bigcap_{s=1}^{k} (E_{j_s} \times I).$$

Now $\lambda(E_{j_s} \times I) = \xi(E_{j_s})$ and $\{\xi(E_{j_1}), \xi(E_{j_2}), \cdots, \xi(E_{j_k})\}$ is a direct summand of $\mathbb{Z}^{n+1}$. Then we have

$$|G_{E \times I}(P, \lambda)| = 1$$

for any face $E \times I$ of P, where E is a face of Q. Therefore, if $|G_F(P, \lambda)| \neq 1$ for a face F of P, then either $F \subseteq Q \times \{0\}$ or $F \subseteq Q \times \{1\}$. Thus we have $|G_{(E \times I) \bigcap P^{(j)}}(P^{(j)}, \lambda^{(j)})| = 1$ for every $j = 1, \cdots, d$. Therefore, the blowups in the resolution (3.5) need to be taken only on the faces arising from $Q \times \{0\}$ or $Q \times \{1\} \subset P$. Since P is a simple polytope, one can consider the necessary blowups in the resolution (3.5) such that $Q \times \left[\frac{1}{2} - \delta, \frac{1}{2}\right] \subset P^{(d)}$, for some $\delta > 0$. Let

$$\widetilde{P} := \iota^{-1}\Big(Q \times \Big[0, \frac{1}{2}\Big]\Big) \subset P^{(d)},$$

where ι is the inclusion map

$$\iota\colon P^{(d)} \to P.$$

Then $\widetilde{P}$ is a simple polytope. Let

$$\pi'\colon M(P^{(d)}, \lambda^{(d)}) = (P^{(d)} \times T^{n+1})/\!\sim\, \to P^{(d)}$$

be the quotient map. We prove that $(\pi')^{-1}(\widetilde{P})$ is an oriented T^{n+1}-manifold with boundary, where the boundary is $N(Q, \xi)$.

Let α be a point in $\widetilde{P}$ and $\alpha \notin \big\{\big(x, \frac{1}{2}\big) : x \in Q\big\}$. Since $\widetilde{P}$ is a simple polytope and $\big\{\big(x, \frac{1}{2}\big) : x \in Q\big\}$ is a facet of $\widetilde{P}$, there exists an open neighbourhood U_α of α in $\widetilde{P}$ such that U_α does not intersect $\big\{\big(x, \frac{1}{2}\big) : x \in Q\big\}$. So U_α is open in $P^{(d)}$. Thus $(\pi')^{-1}(U_\alpha) \subset (\pi')^{-1}(\widetilde{P})$ is an open subset of the smooth manifold $M(P^{(d)}, \lambda^{(d)})$.

Now if $\alpha = \big(x, \frac{1}{2}\big) \in \widetilde{P}$ for some $x \in Q$, then consider the tubular neighbourhood $Q \times \big(\frac{1}{2} - \delta, \frac{1}{2}\big]$ of $Q \times \{\frac{1}{2}\}$ in $\widetilde{P}$. Note that $\big\{\big(x, \frac{1}{2}\big) : x \in Q\big\} = Q \times \{\frac{1}{2}\}$ which is nothing but Q. Then

$$\Big(Q \times \Big(\frac{1}{2} - \delta, \frac{1}{2}\Big] \times T^{n+1}\Big)/\!\sim\, = \Big(\Big(Q \times \Big\{\frac{1}{2}\Big\} \times T^{n+1}\Big)/\!\sim'\Big) \times \Big(\frac{1}{2} - \delta, \frac{1}{2}\Big].$$

The characteristic function on $P^{(d)}$ induces a hyper characteristic function on the facets of $Q \times \big\{\frac{1}{2}\big\}$ which is the same as the hyper characteristic function on the facets of Q. This implies that $\big(\big(Q \times \big\{\frac{1}{2}\big\} \times T^{n+1}\big)/\!\sim'\big)$ is equivariantly diffeomorphic to $N(Q, \xi)$ preserving their orientations. Thus the manifold with boundary $(\pi')^{-1}(\widetilde{P})$ satisfies our claim. Hence a locally 1-standard T-manifold is equivariantly cobordant to zero.

Example 3.3 Consider the pentagon Q and the hyper characteristic function

$$\xi\colon \mathcal{F}(Q) \to \mathbb{Z}^3$$

as in Figure 2(a). Then we have the locally 1-standard T-manifold $N(Q, \xi)$. There exists a point $\mathbf{a} \in \mathbb{Z}^3$ which satisfies Lemma 3.1 for this pair (Q, ξ). In this case one can take $\mathbf{a} = (1, 2, 0)$. Then using (3.4) we can define an $\mathcal{R}$-characteristic function λ on the pentagonal prism $P := Q \times I$, see Figure 2(b). Thus, we have the quasitoric orbifold $M(P, \lambda)$.

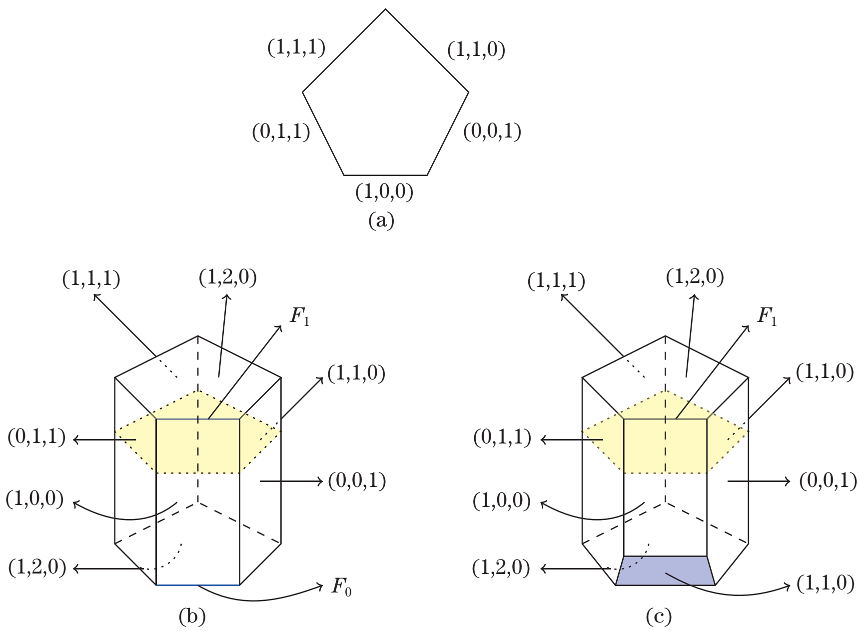

Figure 2 Procedure of constructing a manifold whose boundary is a given locally 1-standard T-manifold.

In the simple polytope P, the only faces F with $|G_F| \neq 1$ are two edges F_0 and F_1 which are coloured with blue in Figure 2(b) and the 4 vertices of these edges. Using the techniques of the proof of Theorem 2.1, first we blow up P along the face F_0 and get the simple polytope P' as in Figure 2(c). We denote the corresponding blowup of quasitoric orbifold $M(P, \lambda)$ by $M(P', \lambda')$. Note that after this blowup the only faces F in the polytope P' with $|G_F| \neq 1$ are F_1 and the two vertices of F_1, see Figure 2(c).

Note that $Q \times \big\{\frac{1}{2}\big\} \subseteq P' \subseteq P$ which is denoted by the dotted pentagon filled with yellow in Figures 2(b) and 2(c). Also $Q \times \big\{\frac{1}{2}\big\}$ can be identified with Q. Let $\widetilde{P}$ denote the lower portion of this pentagon $Q \times \left\{\frac{1}{2}\right\}$ in P', that is $\widetilde{P} = \left(Q \times \left[0, \frac{1}{2}\right]\right) \cap P'$. Then $N(Q, \xi)$ is equivariantly the boundary of the oriented smooth manifold $(\pi')^{-1}(\widetilde{P})$ with boundary for this example.

Corollary 3.1 *Any good contact toric manifold is equivariantly the boundary of an oriented smooth manifold.*

Proof This follows from Example 3.2 and Theorem 3.1.

Corollary 3.2 *Any generalized lens space is equivariantly the boundary of an oriented smooth manifold.*

Proof This follows from Example 3.1 and Theorem 3.1.

We note that the conclusion of Corollary 3.2 can be obtained from [15]. However, our proof is more geometric and explicit. We also note that the article [28] gave a partial answer to Corollary 3.2 under some number theoretic sufficient conditions.

Remark 3.1 Using Theorem 3.1 and [23, Theorem 4.9], we get that all Stiefel-Whitney numbers of locally 1-standard T-manifolds are zero.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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