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On Extensions of Conformal Representations of Orthogonal Lie Algebras*

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Abstract By using Shen's technique of mixed product, Xu and Zhao constructed an inhomogeneous representation on the tensor space of any finite-dimensional irreducible $o(n)$-module with the polynomial space, and found a sufficient condition for this representation to be irreducible. In this paper, the author gives a sufficient and necessary condition for the irreducibility of this representation. Note that the Laplace operator and partial differential operators play important roles in the proof.

Keywords Irreducibility, Polynomial space, Laplace operator, Orthogonal Lie algebra
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1 Introduction

In the 1980s, Shen [5–7] investigated the graded modules of graded Lie algebras of Cartan type. In that paper, the author introduced the notion of "mixed product" and proved that if U is an admissible module of a graded Lie algebra L of Cartan type, and V is a module of the zero-grade term L_0 of L, then there exists an L-module structure on $U \otimes V$, which is called the mixed product.

By using Shen's mixed product for Witt algebras, Zhao and Xu [11] generalized a non-homogenous representation of $sl(n+1)$ to a non-homogenous representation on the tensor space of any finite-dimensional irreducible $sl(n)$-module with the polynomial space. For any given finite-dimensional irreducible $sl(n)$-module, they obtained a new one-parameter family of infinite-dimensional $sl(n+1)$-modules, which are in general not of highest-weight type. Moreover, they gave conditions for the infinite-dimensional $sl(n+1)$-modules to be irreducible.

Xu has also conducted other research on non-homogeneous representations of Lie algebras [8–9].

After that, Xu and Zhao [10] did similar work on $o(n)$. As they said, higher-dimensional conformal field theory is not so well understood partly because of the lack of knowledge about the infinite-dimensional irreducible modules of orthogonal Lie algebras that are compatible with the natural conformal representations. In [10], they generalized the conformal representation

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of $o(n+2,\mathbb{C})$ to an inhomogeneous representation of $o(n+2,\mathbb{C})$ on the tensor space of any finite-dimensional irreducible $o(n,\mathbb{C})$-module with the polynomial space, and found a sufficient condition for the generalized representation to be irreducible.

Inspired by Xu and Zhao's work, Cao, Luo and Ou [1] used Shen's mixed product to study some infinite-dimensional modules with respect to inhomogeneous representations for special linear superalgebras $sl(m|n)$.

In [1, 11], conditions that were found for the irreducibility are sufficient and necessary. However, the conditions in [10] is not necessary. One wonders what the sufficient and necessary condition is for the extensions of conformal representations for orthogonal Lie algebras. This is what this paper concerns.

Here are some notations that are used in this paper:

$\mathbb{N}$: The set of nonnegative integers.

$\mathbb{C}$: The set of complex numbers.

Id_V: The identity map on V.

$E_{i,j}$: The square matrix with 1 as its (i,j)-entry and 0 else where.

∂_i : Abbreviation for ∂_{x_i}.

hwv: A highest weight vector of an $o(n,\mathbb{C})$-module.

$\mathrm{Im}\,\varphi$: $\mathrm{Im}\,\varphi=\{\varphi(v)\mid v\in V\}$, where φ is a linear transformation on V.

$\overline{m,n}$: $\overline{m,n}=\begin{cases}\{m,m+1,\cdots,n\}, & \text{if } m\le n,\\ \emptyset, & \text{otherwise,}\end{cases}$ where m and n are integers.

Let us recall the extensions of the conformal representations for the orthogonal Lie algebras. Let us start with Type D.

Let $n>1$ be an integer. Let $\mathcal{A}=\mathbb{C}[x_1,\cdots,x_{2n}]$ and the Witt algebra $\mathcal{W}_n=\sum\limits_{i=1}^{n}\mathcal{A}\partial_i$. Let $\Im$ be a monomorphism from the Lie algebra $\mathcal{W}_n$ to the Lie algebra of semi-product $\mathcal{W}_n+gl(n,\mathcal{A})$ defined by

$$\Im\Big(\sum_{i=1}^{n}f_i\partial_i\Big)=\sum_{i=1}^{n}f_i\partial_i+\sum_{i,j=1}^{n}\partial_i(f_j)E_{i,j}. \tag{1.1}$$

Let

$$A_{i,j}=x_i\partial_j-x_{n+j}\partial_{n+i},\quad B_{i,j}=x_i\partial_{n+j}-x_j\partial_{n+i},\quad C_{i,j}=x_{n+i}\partial_j-x_{n+j}\partial_i \tag{1.2}$$

for $i,j\in\overline{1,n}$. Then

$$\mathcal{G}_n=\sum_{i,j=1}^{n}\mathbb{C}A_{i,j}+\sum_{1\le i<j\le n}(\mathbb{C}B_{i,j}+\mathbb{C}C_{i,j}) \tag{1.3}$$

forms a Lie algebra, which is isomorphic to $o(2n,\mathbb{C})$. We will omit the subscript n when there is no ambiguity. In the rest of this section and the next, we will identify $o(2n,\mathbb{C})$ with $\mathcal{G}$.

Take the subspaces

$$H=\sum_{i=1}^{n}\mathbb{C}A_{i,i} \tag{1.4}$$

as Cartan subalgebra of $o(2n,\mathbb{C})$. Set

$$o(2n,\mathbb{C})^+ = \sum_{1\le i<j\le n}(\mathbb{C}A_{i,j}+\mathbb{C}B_{i,j}), \qquad o(2n,\mathbb{C})^- = \sum_{1\le j<i\le n}(\mathbb{C}A_{i,j}+\mathbb{C}C_{i,j}). \tag{1.5}$$

Define $\{\varepsilon_i \mid i\in\overline{1,n}\}\subset H^*$ by $\varepsilon_i(A_{j,j})=\delta_{i,j}$. The set of roots is $\Phi_D=\{\pm\varepsilon_i\pm\varepsilon_j \mid 1\le i<j\le n\}$ and the set of positive simple roots is

$$\Pi_D=\{\varepsilon_1-\varepsilon_2,\cdots,\varepsilon_{n-1}-\varepsilon_n,\varepsilon_{n-1}+\varepsilon_n\}. \tag{1.6}$$

The set of dominant integral weights is

$$\begin{aligned}\Lambda_D^+=\{\mu\in L_{\mathbb{Q}} \mid \mu_1\ge\mu_2\ge\cdots\ge\mu_{n-2}\ge\mu_{n-1}\ge|\mu_n|,\\ \mu_i\in\mathbb{Z}/2 \text{ for } i\in\overline{1,n}\}.\end{aligned} \tag{1.7}$$

Note that given $\mu\in\Lambda_D^+$, there exists a unique sequence $S(\mu)=\{n_0,n_1,\cdots,n_s\}$ such that

$$0=n_0<n_1<\cdots<n_s=n, \tag{1.8}$$

$$\mu_i=\mu_j \Leftrightarrow n_r<i,j\le n_{r+1} \quad \text{for some } r\in\overline{0,s-1}. \tag{1.9}$$

The Laplace operators

$$\Delta=\sum_{i=1}^n\partial_i\partial_{n+i}, \qquad \eta=\sum_{i=1}^n x_ix_{n+i}. \tag{1.10}$$

Let

$$D=\sum_{r=1}^{2n}x_r\partial_r, \quad J_i=-\eta\partial_{n+i}+x_iD, \quad J_{n+i}=-\eta\partial_i+x_{n+i}D. \tag{1.11}$$

Then

$$\mathcal{C}_{2n}=\mathbb{C}D+\sum_{r=1}^{2n}(\mathbb{C}\partial_{x_r}+\mathbb{C}J_r)+\mathcal{G} \tag{1.12}$$

is the Lie algebra of the $2n$-dimensional split conformal group, which is isomorphic to $o(2n+2,\mathbb{C})$. Hereafter in this paper, we identify $\mathcal{C}_{2n}$ with $o(2n+2,\mathbb{C})$.

Let M be any $o(2n,\mathbb{C})$-module. Given any constant number $b\in\mathbb{C}$, an $o(2n+2,\mathbb{C})$-module structure on $\widehat{M}=\mathcal{A}\otimes_{\mathbb{C}}M$ was constructed in [10], where the hidden central transformation $(\sum\limits_{i=1}^{2n}E_{i,i})|_{\widehat{M}}$ takes the constant value b. Note that $\widehat{M}$ is the extension of the conformal representation for orthogonal Lie algebras, which is the main object in [10] and also the leading role in this paper. What we want to do now is to find the conditions for the $o(2n+2,\mathbb{C})$-module $\widehat{M}$ to be irreducible.

Here we list the action equations:

$$A_{i,j}|_{\widehat{M}}=A_{i,j}|_{\mathcal{A}}\otimes\mathrm{Id}_M+\mathrm{Id}_{\mathcal{A}}\otimes A_{i,j}|_M, \tag{1.13}$$

$$B_{i,j}|_{\widehat{M}} = B_{i,j}|_{\mathcal{A}} \otimes \mathrm{Id}_M + \mathrm{Id}_{\mathcal{A}} \otimes B_{i,j}|_M, \tag{1.14}$$

$$C_{i,j}|_{\widehat{M}} = C_{i,j}|_{\mathcal{A}} \otimes \mathrm{Id}_M + \mathrm{Id}_{\mathcal{A}} \otimes C_{i,j}|_M, \tag{1.15}$$

$$\partial_k|_{\widehat{M}} = \partial_k|_{\mathcal{A}} \otimes \mathrm{Id}_M, \quad D|_{\widehat{M}} = D|_{\mathcal{A}} \otimes \mathrm{Id}_M + \mathrm{Id}_{\mathcal{A}} \otimes D|_M, \tag{1.16}$$

$$J_i|_{\widehat{M}} = J_i|_{\mathcal{A}} \otimes \mathrm{Id}_M + \sum_{p=1}^{n} x_{n+p}|_{\mathcal{A}} \otimes B_{i,p}|_M + \sum_{q=1}^{n} x_q|_{\mathcal{A}} \otimes A_{i,q}|_M + x_i|_{\mathcal{A}} \otimes D|_M, \tag{1.17}$$

$$J_{n+i}|_{\widehat{M}} = J_{n+i}|_{\mathcal{A}} \otimes \mathrm{Id}_M - \sum_{p=1}^{n} x_{n+p}|_{\mathcal{A}} \otimes A_{p,i}|_M + \sum_{q=1}^{n} x_q|_{\mathcal{A}} \otimes C_{i,q}|_M + x_{n+i}|_{\mathcal{A}} \otimes D|_M, \tag{1.18}$$

where $i, j \in \overline{1, n}$, $k \in \overline{1, 2n}$, $x_k|_{\mathcal{A}}(f) = x_k f$ for $f \in \mathcal{A}$. Here we identify D with the hidden central transformation $\sum\limits_{i=1}^{2n} E_{i,i}$, i.e., $D|_M = b\mathrm{Id}_M$.

We can similarly obtain $\mathcal{C}_{2n+1}$ and the $o(2n+3, \mathbb{C})$-module $\widehat{M}$ for any $o(2n+1, \mathbb{C})$-module M (see Subsection 3.1 for details).

Denote by $V(\mu)$ the finite-dimensional irreducible $o(2n, \mathbb{C})$-module or $o(2n+1, \mathbb{C})$-module with highest weight μ. In [10], Xu and Zhao's main theory indicates that the $o(2n+2, \mathbb{C})$-module $\widehat{V(\mu)}$ is irreducible if $b \in \mathbb{C} \setminus \{n-1-\mathbb{N}/2, \mu_1 + 2n - \iota_\mu - 1 - \mathbb{N}\}$, and the module $\widehat{V(\mu)}$ of $o(2n+3, \mathbb{C})$ is irreducible if $b \in \mathbb{C} \setminus \{n - \mathbb{N}/2, \mu_1 + 2n - \iota_\mu - \mathbb{N}\}$, where $\mu \neq 0$ is a dominant integral weight, ι_μ is defined by $\iota_\mu \in \overline{1, n}$,

$$\mu_1 = \mu_2 = \cdots = \mu_{\iota_\mu} \neq \mu_{\iota_\mu + 1}.$$

Note that in fact ι_μ is $n_1, n_2, \cdots, n_s$ (cf. (1.8)–(1.9)). In addition, they gave the necessary and sufficient condition for $\widehat{V(0)}$ to be irreducible.

We want to modify their condition and obtain a necessary and sufficient one. To do this, we define

$$\begin{aligned} \Theta_D(\mu) = \{&|\mu_n| + n - 1 - k, \mu_{n_i} + 2n - n_i - k_i \mid k \geq 0; i \in \overline{1, s-1}, \\ &k_i \in \overline{1, \mu_{n_i} - |\mu_{n_{i+1}}|}\}, \end{aligned} \tag{1.19}$$

$$\begin{aligned} \Theta_B(\mu) = \{&\mu_{n_i} + 2n - n_i - k_i, \mu_n + n - k_s \mid k_s \in \overline{0, [\mu_n] - 1}; i \in \overline{1, s-1}, \\ &k_i \in \overline{0, \mu_{n_i} - \mu_{n_{i+1}} - 1}\} \\ &\cup \{-\mu_{n_i+1} + n_i - k_i \mid i \in \overline{1, s-1}, k_i \in \overline{0, \mu_{n_i} - \mu_{n_i+1} - 1}\} \\ &\cup \Big\{ -\mu_1 - k, (\mu_n) + n - \frac{1}{2} - k \mid k \in \mathbb{N} \Big\} \\ &\cup \Big\{ n - l, n - l - \frac{1}{2} \mid l \in \overline{0, [\mu_n] - 1} \Big\}, \end{aligned} \tag{1.20}$$

where μ is a dominant integral weight, whether it is nonzero or not, $[\mu_n] = \max\{k \mid k \text{ is an integer}, k \leq \mu_n\}$ is the integer part of μ_n, $(\mu_n) = \mu_n - [\mu_n]$.

Main Theorem *The $o(2n+2, \mathbb{C})$-module $\widehat{V(\mu)}$ is irreducible if and only if $b \notin \Theta_D(\mu)$; the $o(2n+3, \mathbb{C})$-module $\widehat{V(\mu)}$ is irreducible if and only if $b \notin \Theta_B(\mu)$.*

According to our main theorem, b is allowed to take more values. Let us take type D as an example: b is allowed to be half integers when μ_n is an integer, while b is allowed to be integers when μ_n is a half integer. In addition, we allow b to be $\mu_{n_{i+1}} + 2n - n_i - k$, for $i \in \overline{1, s-1}$, $k \in \overline{1, n_{i+1} - n_i}$, which were obsoleted by Xu and Zhao's condition, since

$$\mu_{n_{i+1}} + 2n - n_i - k = \mu_1 + 2n - n_i - 1 - (k - 1 + \mu_1 - \mu_{n_{i+1}}) \in \{\mu_1 + 2n - \iota_\mu - 1 - \mathbb{N}\}.$$

Sections 2 and 3 are devoted to proving the main theorem for type D and type B, respectively.

2 Proof of Type D

In this section, we first introduce some notations and lemmas in Subsection 2.1. The necessity and sufficiency are investigated in Subsections 2.2 and 2.3, respectively.

2.1 Some notations and lemmas

For convenience, we almost use the same notations as in [10]. Throughout this section, we always let M be a finite-dimensional irreducible $o(2n, \mathbb{C})$-module with highest weight $\mu \in \Lambda_D^+$.

According to [2, 4], there is a bijection between the set of hwvs in $\widehat{M} = \mathcal{A} \otimes_{\mathbb{C}} M$ and the set

$$\begin{aligned} V_D = \{f \in \mathcal{A} \mid B_{n-1,n}^{\mu_{n-1}+\mu_n+1}(f) = 0 \text{ and } A_{j,j+1}^{\mu_j-\mu_{j+1}+1}(f) = 0 \\ \text{for } j = 1, 2, \cdots, n-1\}. \end{aligned} \tag{2.1}$$

We recall some notations from [10], but with a little change:

$$x^\alpha = \prod_{i=1}^{2n} x_i^{\alpha_i}, \quad J^\alpha = \prod_{i=1}^{2n} J_i^{\alpha_i}, \quad |\alpha| = \sum_{i=1}^{2n} \alpha_i \quad \text{for } \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{2n}) \in \mathbb{N}^{2n}. \tag{2.2}$$

$$\mathcal{A}_k = \mathrm{Span}_{\mathbb{C}}\{x^\alpha \mid \alpha \in \mathbb{N}^{2n}, |\alpha| = k\}, \quad M_{\langle k\rangle} = \mathcal{A}_k \otimes_{\mathbb{C}} M \quad \text{for } k \in \mathbb{N}. \tag{2.3}$$

We write $|f| = k$ if $f \in \mathcal{A}_k$, and $|u| = k$ if $u \in M_{\langle k\rangle}$.

In [10], a linear transformation φ on $\widehat{M}$ was defined by

$$\varphi(x^\alpha \otimes v) = J^\alpha(1 \otimes v) \quad \text{for } \alpha \in \mathbb{N}^{2n}, v \in M. \tag{2.4}$$

Note that φ can be viewed as an $o(2n, \mathbb{C})$-module homomorphism, which is a key point to us.

According to Lemma 2.3 in [10], if M is a finite-dimensional irreducible $o(2n, \mathbb{C})$-module, then the space $\mathrm{Im}\,\varphi$ is an irreducible $o(2n+2, \mathbb{C})$-submodule of $\widehat{M}$. Therefore, the $o(2n+2, \mathbb{C})$-module $\widehat{M}$ is irreducible if and only if φ is surjective. Note that φ can be restricted to $M_{\langle k\rangle}$, and each $M_{\langle k\rangle}$ is finite-dimensional. Thus φ is surjective if and only if it is bijective.

In $sl(n)$ and $sl(m \mid n)$ cases, the hwvs play a key role, since if the highest weights are of multiplicity 1, then the corresponding hwvs are eigenvectors of φ. It still holds in $so(n)$ case. Therefore, let us investigate the hwvs in $\widehat{M}$.

First, there is a unique hwv in $V(\mu)$ up to a scalar, denoted by v_μ. According to Pieri's formula (cf. [3]),

$$V(\varepsilon_1) \otimes V(\mu) \cong \begin{cases} \bigoplus_{i=1}^{s} V(\mu - \varepsilon_{n_i}) \oplus \bigoplus_{i=0}^{s-1} V(\mu + \varepsilon_{n_i+1}), & \text{if } \mu_{n-1} + \mu_n > 0, \\ \bigoplus_{i=1}^{s-2+\delta_{\mu_n,0}} V(\mu - \varepsilon_{n_i}) \oplus \bigoplus_{i=0}^{s-1} V(\mu + \varepsilon_{n_i+1}), & \text{if } \mu_{n-1} + \mu_n = 0. \end{cases}$$

Then one obtains

$$x_{n_i+1} \otimes v_\mu \in \bigoplus_{j=0}^{i} V(\mu + \varepsilon_{n_j+1}) \quad \text{for } i \in \overline{0, s-1},$$

$$x_{n+n_i} \otimes v_\mu \in \bigoplus_{j=0}^{s-1} V(\mu + \varepsilon_{n_j+1}) \oplus \bigoplus_{j=i}^{s} V(\mu - \varepsilon_{n_i}) \quad \text{for } i \in \overline{1, s}.$$

Thus the hwvs in $V(\mu + \varepsilon_{n_j+1})$ and $V(\mu - \varepsilon_{n_i})$ can be written as

$$v_{\langle 1\rangle} = x_1 \otimes v_\mu, \tag{2.5}$$

$$v_{\langle n_i+1\rangle} = x_{n_i+1} \otimes v_\mu + \sum_{j=0}^{i-1} X_{ij} \cdot v_{\langle n_j+1\rangle} \quad \text{for } i = 1, \cdots, s-1, \tag{2.6}$$

$$v_{\langle -n_i\rangle} = x_{n+n_i} \otimes v_\mu + \sum_{j=i+1}^{s} Y_{ij} \cdot v_{\langle -n_j\rangle} + \sum_{j=0}^{s-1} Z_{ij} \cdot v_{\langle n_j+1\rangle} \quad \text{for } i = 1, \cdots, s, \tag{2.7}$$

where $X_{i,j}, Y_{i,j}, Z_{i,j} \in U(o(2n, \mathbb{C})^-)$. In the following discussion, we always assume there is a $u_{\langle -n\rangle}$. If $\mu_{n-1} + \mu_n = 0$, then we set $u_{\langle -n\rangle} = 0$. We have (cf. [10, (2.81)–(2.82)])

$$\varphi(v_{\langle n_i+1\rangle}) = (b + \mu_{n_i+1} - n_i)v_{\langle n_i+1\rangle} \quad \text{for } i = 0, \cdots, s-1, \tag{2.8}$$

$$\varphi(v_{\langle -n_j\rangle}) = (b - \mu_{n_j} - 2n + 1 + n_j)v_{\langle -n_j\rangle} \quad \text{for } j = 1, \cdots, s. \tag{2.9}$$

Remark 2.1 If we write $v_{\langle n_i+1\rangle} = \sum_{p=1}^{2n} x_p \otimes v_p$, $v_p \in M$, then v_{n_i+1} is a multiple of v_μ and $v_p = 0$ when $p > n_i + 1$. Since the weight of x_p is lower than that of x_{n_i+1}, which means the weight of v_p is higher than the weight of v_μ, but this is impossible, since v_μ is the hwv. Thus, we have

$$\partial_k v_{\langle n_i+1\rangle} = 0 \quad \text{for } k \in \overline{n_i+2, n}, \quad \partial_{n+j} v_{\langle n_i+1\rangle} = 0 \quad \text{for } j \in \overline{1, n}. \tag{2.10}$$

By a similar argument, we have

$$\partial_{n+l} v_{\langle -n_i\rangle} = 0 \quad \text{for } l \in \overline{1, n_i - 1}. \tag{2.11}$$

The structure of $M_{\langle 1\rangle}$ is pretty clear. We can consider $M_{\langle k+1\rangle}$ as a quotient of $\langle x_1\rangle \otimes M_{\langle k\rangle}$ in the following way. Throughout in this section, the notation $\langle u\rangle$ denotes the $o(2n, \mathbb{C})$-module generated by u. Define a linear transformation $\psi : \langle x_1\rangle \otimes M_{\langle k\rangle} \to M_{\langle k+1\rangle}$ by

$$\psi(x_i \otimes (x^\alpha \otimes w)) = x_i x^\alpha \otimes w \tag{2.12}$$

for $i \in \overline{1, 2n}, |\alpha| = k, w \in M$. It is easy to see that ψ is a surjective $o(2n, \mathbb{C})$-module homomorphism.

Suppose u is a hwv in $M_{\langle k\rangle}$ of weight ν, the corresponding sequence $S(\nu) = \{h_0, h_1, \cdots, h_t\}$. Replacing M by $\langle u\rangle$, we define $X_{i,j}$, $Y_{i,j}$, $Z_{i,j}$ corresponding to u and ν, which we denote by $X_{i,j}^{(u)}$, $Y_{i,j}^{(u)}$, $Z_{i,j}^{(u)}$. Then using the notations introduced above, we can get all hwvs in $\langle x_1\rangle \otimes \langle u\rangle$, which we denote by $\widetilde{u}_{\langle h_i+1\rangle}$, $\widetilde{u}_{\langle -h_i\rangle}$. Note that if u' is a hwv in $M_{\langle k'\rangle}$ of weight ν, then $X_{i,j}^{(u')}$,

$Y_{i,j}^{(u')}$, $Z_{i,j}^{(u')}$ should be equal to $X_{i,j}^{(u)}$, $Y_{i,j}^{(u)}$, $Z_{i,j}^{(u)}$, respectively, since $\langle u\rangle$ and $\langle u'\rangle$ are isomorphic $o(2n,\mathbb{C})$-modules.

Denote $u_{\langle h_i+1\rangle}=\psi(\widetilde{u}_{\langle h_i+1\rangle})$, $u_{\langle -h_i\rangle}=\psi(\widetilde{u}_{\langle -h_i\rangle})$. Then all the hwvs in $M_{\langle k+1\rangle}$ should be equal to some $u_{\langle h_k+1\rangle}$ or $u_{\langle -h_i\rangle}$.

Moreover, we have another expression of the hwvs in $M_{\langle k-1\rangle}$, by using the duality between the multiplication operators and the differential operators. Suppose

$$\widetilde{u}_{\langle h_i+1\rangle}=x_{h_i+1}\otimes u+\sum_{j=0}^{i-1}X_{i,j}^{(u)}\cdot\widetilde{u}_{\langle h_j+1\rangle},\tag{2.13}$$

$$\widetilde{u}_{\langle -h_i\rangle}=x_{n+h_i}\otimes u+\sum_{p=i+1}^{t}Y_{i,p}^{(u)}\cdot\widetilde{u}_{\langle -h_p\rangle}+\sum_{j=0}^{t-1}Z_{i,j}^{(u)}\cdot\widetilde{u}_{\langle h_j+1\rangle}.\tag{2.14}$$

Then

$$u_{\langle h_i+1\rangle}=x_{h_i+1}u+\sum_{j=0}^{i-1}X_{i,j}^{(u)}\cdot u_{\langle h_j+1\rangle},\tag{2.15}$$

$$u_{\langle -h_i\rangle}=x_{n+h_i}u+\sum_{p=i+1}^{t}Y_{i,p}^{(u)}\cdot u_{\langle -h_p\rangle}+\sum_{j=0}^{t-1}Z_{i,j}^{(u)}\cdot u_{\langle h_j+1\rangle}.\tag{2.16}$$

Denote

$$u^{\langle h_i+1\rangle}=\partial_{n+h_i+1}u+\sum_{j=0}^{i-1}X_{i,j}^{(u)}\cdot u^{\langle h_j+1\rangle},\tag{2.17}$$

$$u^{\langle -h_i\rangle}=\partial_{h_i}u+\sum_{p=i+1}^{t}Y_{i,p}^{(u)}\cdot u^{\langle -h_p\rangle}+\sum_{j=0}^{t-1}Z_{i,j}^{(u)}\cdot u^{\langle h_j+1\rangle}.\tag{2.18}$$

Lemma 2.1 $\Delta(u_{\langle h_i+1\rangle})=u^{\langle h_i+1\rangle}+(\Delta(u))_{\langle h_i+1\rangle}$, $\Delta(u_{\langle -h_i\rangle})=u^{\langle -h_i\rangle}+(\Delta(u))_{\langle h_i\rangle}$.

Proof We have

$$\Delta(u_{\langle 1\rangle})=\sum_{j=1}^{n}\partial_j\partial_{n+j}(x_1u)=\partial_{n+1}u+x_1\sum_{j=1}^{n}\partial_j\partial_{n+j}u=u^{\langle 1\rangle}+(\Delta(u))_{\langle 1\rangle},$$

and thus the first equation holds for $i=0$. Suppose it holds for all $j<i$, we have

$$\begin{aligned}\Delta(u_{\langle h_i+1\rangle})&=\Delta(x_{h_i+1}u)+\sum_{j=0}^{i-1}X_{i,j}^{(u)}\Delta(u_{\langle h_j+1\rangle})\\&=\partial_{n+h_i+1}u+x_{h_i+1}\Delta(u)+\sum_{j=0}^{i-1}X_{i,j}^{(u)}(u^{\langle h_j+1\rangle}+(\Delta(u))_{\langle h_j+1\rangle})\\&=u^{\langle h_i+1\rangle}+(\Delta(u))_{\langle h_i+1\rangle}.\end{aligned}$$

Thus the first equation holds for $i\in\overline{0,t-1}$. The second equation can be proved similarly.

Therefore, if $u^{\langle a\rangle}\neq 0$ ($a=h_i+1,-h_i$), then $u^{\langle a\rangle}$ is an hwv in $M_{\langle k-1\rangle}$. In addition, we have

$$(\eta^d u)_{\langle a\rangle}=\eta^d u_{\langle a\rangle},\quad (\eta^d u)^{\langle a\rangle}=d\eta^{d-1}u_{\langle a\rangle}+\eta^d u^{\langle a\rangle},\tag{2.19}$$

since η commutes with X for all $X \in U(o(2n,\mathbb{C}))$ and

$$\partial_i(\eta^d u) = d\eta^{d-1}x_{n+i}u + \eta^d\partial_i u, \quad \partial_{n+i}(\eta^d u) = d\eta^{d-1}x_i u + \eta^d\partial_{n+i}u.$$

Although the hwvs in $M_{\langle k\rangle}$ need not be eigenvectors of φ when $k > 1$, there are some connections.

Replace M by $M_{\langle k\rangle}$ and consider the tensor product $\langle x_1\rangle \otimes M_{\langle k\rangle}$. We still have the $o(2n,\mathbb{C})$-module homomorphism φ, defined by

$$\begin{aligned} &\varphi(x_i \otimes (x^\alpha \otimes w)) \\ &= J_i(1 \otimes (x^\alpha \otimes w)) \\ &= \sum_{p=1}^n x_{n+p} \otimes B_{ip}(x^\alpha \otimes w) + \sum_{q=1}^n x_q \otimes A_{iq}(x^\alpha \otimes w) \\ &\quad + (b+k)x_i \otimes (x^\alpha \otimes w), \end{aligned} \tag{2.20}$$

$$\begin{aligned} &\varphi(x_{n+i} \otimes (x^\alpha \otimes w)) \\ &= J_{n+i}(1 \otimes (x^\alpha \otimes w)) \\ &= -\sum_{p=1}^n x_{n+p} \otimes A_{pi}(x^\alpha \otimes w) + \sum_{q=1}^n x_q \otimes C_{iq}(x^\alpha \otimes w) \\ &\quad + (b+k)x_{n+i} \otimes (x^\alpha \otimes w) \end{aligned} \tag{2.21}$$

for $i \in \overline{1,n}$, $\alpha \in \mathbb{N}^{2n}$ ($|\alpha| = k$), $w \in M$. It is different from the original φ, so we denote it by $\varphi^{(k)}$.

It is not difficult to check that

$$\psi J_i(1 \otimes (x^\alpha \otimes w)) = (J_i - \eta\partial_{n+i})(x^\alpha \otimes w) + kx_i x^\alpha \otimes w, \tag{2.22}$$

$$\psi J_{n+i}(1 \otimes (x^\alpha \otimes w)) = (J_{n+i} - \eta\partial_i)(x^\alpha \otimes w) + kx_{n+i}x^\alpha \otimes w. \tag{2.23}$$

Therefore,

$$\begin{aligned} &\varphi(x_i x^\alpha \otimes w) \\ &= J_i\varphi(x^\alpha \otimes w) \\ &= \psi\varphi^{(k)}(x_i \otimes \varphi(x^\alpha \otimes w)) + \eta\partial_{n+i}\varphi(x^\alpha \otimes w) - kx_i\varphi(x^\alpha \otimes w), \end{aligned} \tag{2.24}$$

$$\begin{aligned} &\varphi(x_{n+i} x^\alpha \otimes w) \\ &= J_{n+i}\varphi(x^\alpha \otimes w) \\ &= \psi\varphi^{(k)}(x_{n+i} \otimes \varphi(x^\alpha \otimes w)) + \eta\partial_i\varphi(x^\alpha \otimes w) - kx_{n+i}\varphi(x^\alpha \otimes w). \end{aligned} \tag{2.25}$$

Lemma 2.2 *Assume u is an hwv in $M_{\langle k\rangle}$ of weight ν with sequence $S(\nu) = \{h_0, h_1, \cdots, h_t\}$. Then*

$$\varphi(u_{\langle h_i+1\rangle}) = (b + \nu_{h_i+1} - h_i)\varphi(u)_{\langle h_i+1\rangle} + \eta\varphi(u)^{\langle h_i+1\rangle}, \tag{2.26}$$

$$\varphi(u_{\langle -h_i\rangle}) = (b - \nu_{h_i} - 2n + 1 + h_i)\varphi(u)_{\langle -h_i\rangle} + \eta\varphi(u)^{\langle -h_i\rangle}. \tag{2.27}$$

Proof First, $\varphi(u)$ is an hwv in $M_{\langle k\rangle}$ of weight ν, since φ is an $o(2n,\mathbb{C})$-homomorphism. Applying (2.8)–(2.9) to $\langle x_1\rangle\otimes\langle\varphi(u)\rangle$ and $\varphi^{(k)}$, we get

$$\varphi^{(k)}(\widetilde{\varphi(u)}_{\langle h_i+1\rangle}) = (b+k+\nu_{h_i+1}-h_i)\widetilde{\varphi(u)}_{\langle h_i+1\rangle},$$

$$\varphi^{(k)}(\widetilde{\varphi(u)}_{\langle -h_i\rangle}) = (b+k-\nu_{h_i}-2n+1+h_i)\widetilde{\varphi(u)}_{\langle -h_i\rangle}.$$

Thus

$$\begin{aligned}\varphi(u_{\langle 1\rangle}) &= \varphi(x_1u) = J_1\varphi(u)\\ &= \psi\varphi^{(k)}(x_1\otimes\varphi(u)) + \eta\partial_{n+1}\varphi(u) - kx_1\varphi(u)\\ &= (b+k+\nu_1)x_1\varphi(u) + \eta\varphi(u)^{\langle 1\rangle} - kx_1\varphi(u)\\ &= (b+\nu_1)x_1\varphi(u) + \eta\varphi(u)^{\langle 1\rangle}.\end{aligned}\tag{2.28}$$

Assume that (2.26) holds for $i'<i$. We have

$$\begin{aligned}\varphi(x_{h_i+1}u) &= \psi\varphi^{(k)}(x_{h_i+1}\otimes\varphi(u)) + \eta\partial_{n+h_i+1}\varphi(u) - kx_{h_i+1}\varphi(u)\\ &= \psi\varphi^{(k)}\Big(\widetilde{\varphi(u)}_{\langle h_i+1\rangle} - \sum_{j=0}^{i-1}X_{i,j}^{(u)}.\widetilde{\varphi(u)}_{\langle h_j+1\rangle}\Big) + \eta\partial_{n+h_i+1}\varphi(u) - kx_{h_i+1}\varphi(u)\\ &= \psi\Big((b+k+\nu_{h_i+1}-h_i)\widetilde{\varphi(u)}_{\langle h_i+1\rangle} - \sum_{j=0}^{i-1}(b+k+\nu_{h_j+1}-h_j)X_{i,j}^{(u)}.\widetilde{\varphi(u)}_{\langle h_j+1\rangle}\Big)\\ &\quad + \eta\partial_{n+h_i+1}\varphi(u) - kx_{h_i+1}\varphi(u)\\ &= (b+k+\nu_{h_i+1}-h_i)\varphi(u)_{\langle h_i+1\rangle} - \sum_{j=0}^{i-1}(b+\nu_{h_j+1}-h_j)X_{i,j}^{(u)}.\varphi(u)_{\langle h_j+1\rangle}\\ &\quad + \eta\partial_{n+h_i+1}\varphi(u) - k\varphi(u)_{\langle h_i+1\rangle},\end{aligned}\tag{2.29}$$

$$\begin{aligned}\varphi(u_{\langle h_i+1\rangle}) &= \varphi(x_{h_i+1}u) + \sum_{j=0}^{i-1}X_{i,j}^{(u)}.\varphi(u_{\langle h_j+1\rangle})\\ &= \varphi(x_{h_i+1}u) + \sum_{j=0}^{i-1}X_{i,j}^{(u)}((b+\nu_{h_j+1}-h_j)\varphi(u)_{\langle h_j+1\rangle} + \eta\varphi(u)^{\langle h_j+1\rangle})\\ &= (b+k+\nu_{h_i+1}-h_i)\varphi(u)_{\langle h_i+1\rangle} + \eta\varphi(u)^{\langle h_i+1\rangle} - k\varphi(u)_{\langle h_i+1\rangle}\\ &= (b+\nu_{h_i+1}-h_i)\varphi(u)_{\langle h_i+1\rangle} + \eta\varphi(u)^{\langle h_i+1\rangle}.\end{aligned}\tag{2.30}$$

Thus (2.26) holds for all i. (2.27) can be proved similarly.

2.2 Necessity for type *D*

This subsection is devoted to proving the necessity for type D. From now on, we fix $M=V(\mu)$, where $\mu\in\Lambda_D^+$, $S(\mu)=\{n_0,n_1,\cdots,n_s\}$ such that (1.8)–(1.9) hold.

As mentioned above, the difficulty in our case is some highest weights are not of multiplicity 1. For example, μ is one of them, since η and $\sum\limits_{j=n_i+1}^{n_{i+1}}x_jx_{n+j}$, $i=1,\cdots,s-1$ are all in V_D. In fact, all highest weights that are not of multiplicity 1 have something to do with η and

$\sum\limits_{j=n_i+1}^{n_{i+1}} x_j x_{n+j}$. This makes it complicated, since $\eta \otimes v_\mu$ is not an eigenvector of φ, thus we cannot obtain the condition of irreducibility only by eigenvalues. Our approach is using some special vector and making $\sum\limits_{j=n_i+1}^{n_{i+1}} x_j x_{n+j}$ "go away".

Let

$$f = \prod_{i=1}^{n-1} x_{n+i}^{\mu_i - |\mu_{i+1}|} = \prod_{i=1}^{s-1} x_{n+n_i}^{\mu_{n_i} - |\mu_{n_{i+1}}|}, \quad \nu = \mu - \sum_{i=1}^{n-1} (\mu_i - |\mu_{i+1}|)\varepsilon_i. \tag{2.31}$$

By straightforward computation, we have $f \in V_D$ (cf. (2.1)). Thus there is an hwv in form of

$$u = f \otimes v_\mu + \cdots \tag{2.32}$$

with weight ν. By the similar argument to Lemma 3.2.3 in [11], we have

$$\varphi(u) = a_u u, \tag{2.33}$$

where

$$a_u = \prod_{i=1}^{s-1} \prod_{j=1}^{\mu_{n_i} - |\mu_{n_i+1}|} (b - \mu_{n_i} - 2n + n_i + j). \tag{2.34}$$

Thus we obtain the first necessary condition for type D.

Theorem 2.1 *If φ is bijective, then $b \neq \mu_{n_i} + 2n - n_i - j$ for $i \in \overline{1, s-1}$, $j \in \overline{1, \mu_{n_i} - |\mu_{n_i+1}|}$.*

Note that $\eta u \in M_{\langle \mu_1 - |\mu_n| + 2\rangle}$. Since x^α has weight $\sum\limits_{i=1}^{n} (\alpha_i - \alpha_{n+i})\varepsilon_i$, we conclude that the hwvs in $M_{\langle \mu_1 - |\mu_n| + 2\rangle}$ of weight ν takes the form $g \otimes v_\mu + \cdots$, where $g = (\sum\limits_{i=1}^{n} a_i x_i x_{n+i}) f$ for some $a_i \in \mathbb{C}$, $i = 1, \cdots, n$, and g should be in V_D. By straightforward computation, we get g must be a multiple of ηf. That is to say, ν is of multiplicity 1 in $M_{\langle \mu_1 - |\mu_n| + 2\rangle}$. Thus, ηu is an eigenvector of φ. Let us work out the corresponding eigenvalue.

Lemma 2.3 *$\varphi(\eta u) = a_u (b + \mu_1)(b - |\mu_n| - n + 1)\eta u$.*

Proof The following equation will be used:

$$[\partial_{n+1}, \widetilde{\eta}] = \sum_{i=1}^{n} (J_{n+i} B_{i,1} + \delta_{i,1} J_1 - J_1 + J_i(\delta_{1,i} D - A_{1,i})). \tag{2.35}$$

One sees that $\varphi(\eta u) = \widetilde{\eta}\varphi(u) = a_u \widetilde{\eta} u$. Assume

$$\widetilde{\eta} u = a \eta u.$$

We have the following two cases.

Case 1 $\mu_1 = \mu_2$.

In this case, we have $n_1 > 1$, so $\partial_{n+1} f = 0$, hence $\partial_{n+1} u = 0$. On one hand,

$$\partial_{n+1} \widetilde{\eta} u = a x_1 u. \tag{2.36}$$

On the other hand,

$$\begin{aligned}\partial_{n+1}\widetilde{\eta}u &= [\partial_{n+1}, \widetilde{\eta}]u + \widetilde{\eta}\partial_{n+1}u \\ &= \sum_{i=1}^{n}(\delta_{i,1}J_1 - J_1 + J_i(\delta_{1,i}D - A_{1,i}))(u) \\ &= (-(n-1) + (\mu_1 - |\mu_n| + b) - \mu_1)J_1 \cdot u \\ &= (b+\mu_1)(b - |\mu_n| - n + 1)x_1 u, \end{aligned} \tag{2.37}$$

since $A_{1,i} \cdot u = 0$, $B_{1,i} \cdot u = 0$ for $i > 1$.

Comparing the coefficients in (2.36)–(2.37), we get $a = (b+\mu_1)(b - |\mu_n| - n + 1)$.

Case 2 $\mu_1 > \mu_2$.

It is a little complicated in this case, since $\partial_{n+1}u \neq 0$ and $\nu - \varepsilon_1$ is not of multiplicity 1 in $M_{\langle \mu_1 - |\mu_n| + 1\rangle}$.

Let

$$u^i = x_{n+1}^i \prod_{j=2}^{n-1} x_{n+j}^{\mu_j - |\mu_{j+1}|} \otimes v_\mu + \cdots \tag{2.38}$$

be the hwv in $M_{\langle \mu_2 - |\mu_n| + i\rangle}$ with highest weight $\nu + (\mu_1 - \mu_2 - i)\varepsilon_1$, $i \in \overline{0, \mu_1 - \mu_2}$. Note $u^i = u$ when $i = \mu_1 - \mu_2$.

By straightforward computing, one sees that the highest weight $\nu + (\mu_1 - \mu_2 - i)\varepsilon_1$, $i \in \overline{0, \mu_1 - \mu_2}$ is of multiplicity 1 in $M_{\langle \mu_2 - |\mu_n| + i\rangle}$, but of multiplicity 2 in $M_{\langle \mu_2 - |\mu_n| + i + 2\rangle}$ because ηu^i and $x_1 u^{i+1}$ are both hwvs in $M_{\langle \mu_2 - |\mu_n| + i + 2\rangle}$. Since $\partial_{n+1}u^{i+1}$ is an hwv in $M_{\langle \mu_2 - |\mu_n| + i\rangle}$ with the highest weight $\nu + (\mu_1 - \mu_2 - i)\varepsilon_1$ and

$$\partial_{n+1}u^{i+1} = (i+1)x_{n+1}^i \prod_{j=2}^{n-1} x_{n+j}^{\mu_j - |\mu_{j+1}|} \otimes v_\mu + \cdots,$$

we have

$$\partial_{n+1}u^{i+1} = (i+1)u^i, \quad i \in \overline{0, \mu_1 - \mu_2 - 1}. \tag{2.39}$$

Assume

$$\widetilde{\eta}u^i = a_i \eta u^i + b_i x_1 u^{i+1}, \quad i \in \overline{0, \mu_1 - \mu_2 - 1}. \tag{2.40}$$

We get

$$\partial_{n+1}\widetilde{\eta}u^0 = (a_0 + b_0)x_1 u^0, \tag{2.41}$$

$$\partial_{n+1}\widetilde{\eta}u^i = a_i x_1 u^i + i a_i \eta u^{i-1} + (i+1)b_i x_1 u^i, \quad i \in \overline{1, \mu_1 - \mu_2 - 1}, \tag{2.42}$$

$$\partial_{n+1}\widetilde{\eta}u = a x_1 u + a(\mu_1 - \mu_2)\eta u^{\mu_1 - \mu_2 - 1}. \tag{2.43}$$

On the other hand,

$$\partial_{n+1}\widetilde{\eta}u^0 = [\partial_{n+1}, \widetilde{\eta}]u^0 + \widetilde{\eta}\partial_{n+1}u^0$$

$$= (b + \mu_2 - |\mu_n| - n + 1 - \mu_1)(b + \mu_1)x_1 u^0, \tag{2.44}$$

and by (2.22),

$$\begin{aligned}
J_1 \cdot u^i &= \psi J_1(1 \otimes u^i) + \eta \partial_{n+1} u^i - |u^i| x_1 u^i \\
&= (b + \mu_1 - i)x_1 u^i + i\eta u^{i-1}, \\
\partial_{n+1} \widetilde{\eta} u^i &= [\partial_{n+1}, \widetilde{\eta}]u^i + \widetilde{\eta}\partial_{n+1} u^i \\
&= (b + \mu_2 - |\mu_n| + i - n + 1 - \mu_1 + i)((b + \mu_1 - i)x_1 u^i + i\eta u^{i-1}) \\
&\quad + i(a_{i-1}\eta u^{i-1} + b_{i-1} x_1 u^i)
\end{aligned} \tag{2.45}$$

for $i \in \overline{1, \mu_1 - \mu_2}$. Comparing the coefficients in (2.41)–(2.45), we obtain

$$\begin{gathered}
a_0 + b_0 = (b + \mu_2 - |\mu_n| - n + 1 - \mu_1)(b + \mu_1), \\
(i + 1)(a_i + b_i) = (b + \mu_2 - |\mu_n| - n + 1 - \mu_1 + 2i)(b + \mu_1) + i(a_{i-1} + b_{i-1}), \\
(\mu_1 - \mu_2 + 1)a = (b - |\mu_n| - n + 1 + \mu_1 - \mu_2)(b + \mu_1) + (\mu_1 - \mu_2)(a_{\mu_1-\mu_2-1} + b_{\mu_1-\mu_2-1}).
\end{gathered}$$

Thus we get

$$a_i + b_i = (b + \mu_2 - |\mu_n| - n + 1 - \mu_1 + i)(b + \mu_1)$$

and $a = (b - |\mu_n| - n + 1)(b + \mu_1)$.

Next we will show $\eta^k u$, $k \in \mathbb{N}$ are eigenvectors of φ and work out the corresponding eigenvalues.

Lemma 2.4 *For any $w \in M_{\langle l \rangle}$, we have $\widetilde{\eta}(\eta^k w) = k(2b - n + k + 1 + l)\eta^{k+1} w + \eta^k \widetilde{\eta} w$.*

Proof

$$\begin{aligned}
J_i(\eta^k w) &= J_i(\eta^k)w + \eta^k J_i(w) = k\eta^k x_i w + \eta^k J_i(w), \\
J_{n+i} J_i(\eta^k w) &= J_{n+i}(k\eta^k x_i w + \eta^k J_i(w)) \\
&= kJ_{n+i}(\eta^k)x_i w + k\eta^k J_{n+i}(x_i w) + J_{n+i}(\eta^k)J_i(w) \\
&\quad + \eta^k J_{n+i} J_i(w) \\
&= k^2 \eta^k x_i x_{n+i} w + k\eta^k J_{n+i}(x_i w) + k\eta^k x_{n+i} J_i(w) \\
&\quad + \eta^k J_{n+i} J_i(w).
\end{aligned} \tag{2.46}$$

Note

$$J_i(x_{n+i} w) = J_i(x_{n+i})w + x_{n+i} J_i(w) = x_i x_{n+i} w - \eta w + x_{n+i} J_i(w),$$

which implies

$$x_{n+i} J_i(w) = J_i(x_{n+i} w) - x_i x_{n+i} w + \eta w.$$

Therefore

$$\begin{aligned}
J_{n+i} J_i(\eta^k w) &= k^2 \eta^k x_i x_{n+i} w + k\eta^k J_{n+i}(x_i w) \\
&\quad + k\eta^k (J_i(x_{n+i} w) - x_i x_{n+i} w + \eta w) + \eta^k J_{n+i} J_i(w)
\end{aligned}$$

$$= k(k-1)x_ix_{n+i}\eta^kw + k\eta^k(J_{n+i}x_i + J_ix_{n+i})(w)$$
$$+ k\eta^{k+1}w + \eta^kJ_{n+i}J_i(w). \tag{2.47}$$

Taking summation on i, we get

$$\begin{aligned}
&\sum_{i=1}^{n} J_{n+i}J_i(\eta^kw)\\
&= k(k-1)\eta^{k+1}w + k(2b+2-2n+l)\eta^{k+1}w + nk\eta^{k+1}w + \eta^k\sum_{i=1}^{n} J_{n+i}J_i(w)\\
&= k(2b-n+k+1+l)\eta^{k+1}w + \eta^k\widetilde{\eta}w.
\end{aligned} \tag{2.48}$$

Note that

$$\sum_{i=1}^{n}(J_{n+i}x_i + J_ix_{n+i})|_{M_{\langle l\rangle}} = (2b+2-2n+l)\eta|_{M_{\langle l\rangle}}$$

according to Lemma 2.5 in [10].

Lemma 2.5 $\varphi(\eta^ku) = a_u\prod_{t=0}^{k-1}(b+\mu_1+t)(b-|\mu_n|-n+1+t)\eta^ku$ *for all* $k\in\mathbb{N}$.

Proof It is the consequence of Lemma 2.3 when $k=1$.

Assume the lemma holds for k. We have

$$\begin{aligned}
&\widetilde{\eta}(\eta^ku)\\
&= k(k+2b+1-n+\mu_1-|\mu_n|)\eta^{k+1}u + (b+\mu_1)(b-|\mu_n|-n+1)\eta^{k+1}u\\
&= (b+\mu_1+k)(b-|\mu_n|-n+1+k)\eta^{k+1}u.
\end{aligned} \tag{2.49}$$

Therefore

$$\begin{aligned}
\varphi(\eta^{k+1}u) &= \widetilde{\eta}\varphi(\eta^ku)\\
&= a_u\prod_{t=0}^{k}(b+\mu_1+t)(b-|\mu_n|-n+1+t)\eta^{k+1}u.
\end{aligned} \tag{2.50}$$

We obtain the second necessary condition for type D.

Theorem 2.2 *If* φ *is bijective, then* $b-|\mu_n|-n+1+k\neq 0$ *for all* $k\in\mathbb{N}$.

In fact, we also obtain $b+\mu_1+k\neq 0$ for all $k\in\mathbb{N}$. But

$$b+\mu_1+k = b-|\mu_n|-n+1+(\mu_1+|\mu_n|+n-1+k),$$

and thus $b+\mu_1+k\neq 0$ can be obtained by Theorem 2.2.

2.3 Sufficiency for type *D*

We will first show $\eta^d\otimes v_\mu\in\operatorname{Im}\varphi$ if $b\notin\Theta_D(\mu)$, and then show $\{\eta^d\otimes v_\mu\mid d\in\mathbb{N}\}$ generate the whole $\widehat{M}$.

Let

$$f_s = 1, \quad f_i = x_{n+n_i}^{\mu_{n_i} - |\mu_{n_{i+1}}|} f_{i+1} \quad \text{for } 0 < i < s. \tag{2.51}$$

There is a unique hwv in $M_{\langle \mu_{n_i+1} - |\mu_n| + k \rangle}$ in the form of

$$u_i^k = x_{n+n_i}^k f_{i+1} \otimes v_\mu + \cdots \tag{2.52}$$

for $0 < i < s$, $0 \le k \le \mu_{n_i} - |\mu_{n_{i+1}}|$ up to a scalar. One sees that $u_{i-1}^0 = u_i^{\mu_{n_i} - |\mu_{n_{i+1}}|}$, f_1 is equal to f in (2.31) and $u_1^{\mu_{n_1} - |\mu_{n_2}|}$ is equal to u in (2.32).

Assume the sequence corresponding to u_i^k is

$$0 = h_0 < h_1 < h_2 < \cdots < h_{i-1} < h_i < h_{i+1} < \cdots < h_t = n. \tag{2.53}$$

Note that (1) $h_j = n_j$ for $0 < j < i$; (2) $n_i - 1 = h_i$ if $n_{i-1} < n_i - 1$ while $n_i - 1 = h_{i-1}$ if $n_{i-1} = n_i - 1$, anyway, $n_i - 1 \in \{h_0, h_1, \cdots, h_t\}$.

Lemma 2.6 $(u_i^{k_i})^{\langle n_i \rangle} = k_i u_i^{k_i - 1}$ *for* $i \in \overline{1, s-1}$, $k_i \in \overline{1, \mu_{n_i} - |\mu_{n_{i+1}}|}$.

Proof We have $(u_i^{k_i})^{\langle n_j + 1 \rangle} = 0$, $j < i$, since $\partial_{n+p} u_i^{k_i} = 0$ for all $p \in \overline{1, n_i - 1}$. Thus

$$(u_i^{k_i})^{\langle n_i \rangle} = \partial_{n+n_i}(u_i^{k_i}) = k_i x_{n+n_i}^{k_i - 1} f_{i+1} \otimes v_\mu + \cdots, \tag{2.54}$$

where the ellipsis part are terms unrelated to v_μ. Hence we get $(u_i^{k_i})^{\langle n_i \rangle} = k_i u_i^{k_i - 1}$, since $\mu - \sum_{p=n_i+1}^{n} (\mu_p - |\mu_{p+1}|)\varepsilon_p - (k_i - 1)\varepsilon_{n_i}$ is of multiplicity 1 in $M_{\langle \mu_{n_i+1} - |\mu_n| + k_i - 1 \rangle}$.

Lemma 2.7 *If* $b \notin \Theta_D(\mu)$, *then* $\eta^d u_i^{k_i} \in \operatorname{Im} \varphi$ *for all* $d \in \mathbb{N}$, $i \in \overline{1, s-1}$, $k_i \in \overline{0, \mu_{n_i} - |\mu_{n_{i+1}}|}$.

Proof Take induction on i. First of all, we have $\eta^d u \in \operatorname{Im} \varphi$ when $b \notin \Theta_D(\mu)$ by Lemma 2.5. That is to say $\eta^d u_1^k \in \operatorname{Im} \varphi$ when $k = \mu_{n_1} - |\mu_{n_2}|$. Assume that $\eta^d u_1^k \in \operatorname{Im} \varphi$ and $\varphi(w_1^{d,k}) = \eta^d u_1^k$ ($d \in \mathbb{N}$) for some $k \in \overline{1, \mu_{n_1} - |\mu_{n_2}|}$. We have

$$(\eta^d u_1^k)^{\langle n_1 \rangle} = d\eta^{d-1}(u_1^k)_{\langle n_1 \rangle} + k\eta^d u_1^{k-1} \tag{2.55}$$

by (2.19) and Lemma 2.6. On the other hand, we have

$$\begin{aligned} \varphi((w_1^{d-1,k})_{\langle n_1 \rangle}) &= (b + (\mu_{n_1} - k) - n_1 + 1)\eta^{d-1}(u_1^k)_{\langle n_1 \rangle} + \eta(\eta^{d-1} u_1^k)^{\langle n_1 \rangle} \\ &= (b + (\mu_{n_1} - k) - n_1 + d)\eta^{d-1}(u_1^k)_{\langle n_1 \rangle} + k\eta^d u_1^{k-1} \end{aligned} \tag{2.56}$$

by Lemma 2.2. Subtracting (2.55) from (2.56), we get

$$(b + \mu_{n_1} - k - n_1)\eta^{d-1}(u_1^k)_{\langle n_1 \rangle} \in \operatorname{Im} \varphi.$$

Since

$$b + \mu_{n_1} - k - n_1 = b - |\mu_n| - n + 1 + (\mu_{n_1} + |\mu_n| + n - n_1 + 1)$$

and $\mu_{n_1} + |\mu_n| \ge 0$, $n - n_1 \ge 0$, thus $\mu_{n_1} + |\mu_n| + n - n_1 + 1 > 0$, which implies $|\mu_n| + n - 1 - (\mu_{n_1} + |\mu_n| + n - n_1 + 1) \in \Theta_D$, we get

$$\eta^{d-1}(u_1^k)_{\langle n_1 \rangle}, \ \eta^d u_1^{k-1} \in \operatorname{Im} \varphi. \tag{2.57}$$

Therefore, we implies $\eta^d u_1^k \in \operatorname{Im}\varphi$ for all $d \in \mathbb{N}$, $k \in \overline{0, \mu_{n_1} - |\mu_{n_1+1}|}$.

Now assume $\eta^d u_{i'}^{k'} \in \operatorname{Im}\varphi$ for all $i' < i$, $k' \in \overline{0, \mu_{n_{i'}} - |\mu_{n_{i'}+1}|}$.

We have $\eta^d u_i^{\mu_{n_i} - |\mu_{n_i+1}|} \in \operatorname{Im}\varphi$ since $u_i^{\mu_{n_i} - |\mu_{n_{i+1}}|} = u_{i-1}^0$. Assume $\eta^d u_i^k \in \operatorname{Im}\varphi$ $(d \in \mathbb{N})$ and $\varphi(w_i^{d,k}) = \eta^d u_i^k$ for some $k \in \overline{1, \mu_{n_i} - |\mu_{n_{i+1}}|}$. Repeating the discussion from (2.55) to (2.57), we get $\eta^d (u_i^k)_{\langle n_i \rangle}$, $\eta^d u_i^{k-1} \in \operatorname{Im}\varphi$. This completes the proof.

When $i = s-1, k = 0$, we get $\eta^d \otimes v_\mu \in \operatorname{Im}\varphi$.

Lemma 2.8 *The $o(2n+2, \mathbb{C})$-module $\widehat{M}$ can be generated by $\{\eta^d \otimes v_\mu \mid d \in \mathbb{N}\}$.*

Proof Denote the $o(2n+2, \mathbb{C})$-module generated by $\{\eta^d \otimes v_\mu \mid d \in \mathbb{N}\}$ by M'. We will show

$$\eta^d M_{\langle k \rangle} \subset M', \quad \forall d, k \in \mathbb{N} \tag{2.58}$$

by taking induction on k.

For $k = 0$, we have $\eta^d M_{\langle 0 \rangle} = \langle \eta^d \otimes v_\mu \rangle \subset M'$. Assume $\eta^d M_{\langle t \rangle} \subset M'$ for all $d \in \mathbb{N}$ and $0 \le t < k$. Note that

$$M_{\langle k \rangle} = \sum_{i=1}^n x_i M_{\langle k-1 \rangle} + \sum_{i=1}^n x_{n+i} M_{\langle k-1 \rangle},$$

and thus it is sufficient to show $\eta^d x_i w$, $\eta^d x_{n+i} w \in M'$ for all $w \in M_{\langle k-1 \rangle}$ and $i \in \overline{1, n}$.

Note that

$$\begin{aligned} \partial_i(\eta^{d+1} w) &= (d+1)\eta^d x_{n+i} w + \eta^{d+1} \partial_i(w), \\ \partial_{n+i}(\eta^{d+1} w) &= (d+1)\eta^d x_i w + \eta^{d+1} \partial_{n+i}(w). \end{aligned}$$

Since $\eta^{d+1} w \in M'$, we get $\partial_i(\eta^{d+1} w)$, $\partial_{n+i}(\eta^{d+1} w) \in M'$. We have

$$\eta^{d+1} \partial_i(w),\ \eta^{d+1} \partial_{n+i}(w) \in \eta^{d+1} M_{\langle k-2 \rangle} \subset M'$$

by inductive assumption. Thus, we get $\eta^d x_{n+i} w, \eta^d x_i w \in M'$. Thus (2.58) holds for all $d, k \in \mathbb{N}$. For $d = 0$, we have $M_{\langle k \rangle} \subset M'$, $\forall k \in \mathbb{N}$, that is to say $M' = \widehat{M}$.

By Lemma 2.7 $M' \subset \operatorname{Im}\varphi$ when $b \notin \Theta_D(\mu)$. Therefore, we obtain the following theorem, by which we can finish our proof for sufficiency.

Theorem 2.3 *When $b \notin \Theta_D(\mu)$, the $o(2n, \mathbb{C})$-homomorphism φ is bijective.*

3 Proof of Type B

In this section, we recall the extensions of the conformal representations for Lie algebras of type B in Subsection 3.1. Then introduce some notations and give some lemmas in Subsection 3.2. The necessity and the sufficiency are investigated in Subsections 3.3–3.4, respectively. The corresponding proof is omitted when a lemma is just an analogue to some lemma in Section 2.

3.1 Extensions of the conformal representations for B_{n+1}

Let $n \geq 1$ be an integer. In this section, we set $\mathcal{A} = \mathbb{C}[x_0, x_1, \cdots, x_{2n}]$. The corresponding Laplace operator and dual invariant are

$$\Delta = \partial_0^2 + \sum_{i=1}^{n} \partial_i \partial_{n+i}, \qquad \eta = \frac{1}{2}x_0^2 + \sum_{i=1}^{n} x_i x_{n+i}. \tag{3.1}$$

We still use $A_{i,j}$, $B_{i,j}$, $C_{i,j}$, J_i and J_{n+i} as defined in (1.2) and (1.11), but redefine D as the following

$$D = \sum_{i=0}^{2n} x_i \partial_i. \tag{3.2}$$

Denote

$$J_0 = x_0 D - \eta \partial_0, \quad K_i = x_0 \partial_i - x_{n+i} \partial_0, \quad K_{n+i} = x_0 \partial_{n+i} - x_i \partial_0 \tag{3.3}$$

for all $i \in \overline{1, n}$. In this section, we set

$$\mathcal{G} = \sum_{j=1}^{2n} \mathbb{C}K_j + \sum_{p,q=1}^{n} \mathbb{C}A_{p,q} + \sum_{1 \leq p < q \leq n} (\mathbb{C}B_{p,q} + \mathbb{C}C_{p,q}). \tag{3.4}$$

One sees that $\mathcal{G}$ is a Lie algebra isomorphic to $o(2n+1, \mathbb{C})$. In addition,

$$\mathcal{C}_{2n+1} = \mathbb{C}D + \sum_{i=0}^{2n} \mathbb{C}J_i + \sum_{j=0}^{2n} \mathbb{C}\partial_j + \mathcal{G} \tag{3.5}$$

is the Lie algebra of $(2n+1)$-dimensional conformal group over $\mathbb{C}$ with η in (3.1) as the metric, which is isomorphic to the Lie algebra $o(2n+3, \mathbb{C})$. Hereafter in this section, we identify $o(2n+1, \mathbb{C})$ and $o(2n+3, \mathbb{C})$ with $\mathcal{G}$ and $\mathcal{C}_{2n+1}$, respectively.

Set

$$\mathcal{G}^+ = \sum_{1 \leq i < j \leq n} (\mathbb{C}A_{i,j} + \mathbb{C}B_{i,j}) + \sum_{k=1}^{n} \mathbb{C}K_{n+i}, \quad \mathcal{G}^- = \sum_{1 \leq j < i \leq n} (\mathbb{C}A_{i,j} + \mathbb{C}C_{i,j}) + \sum_{k=1}^{n} \mathbb{C}K_i. \tag{3.6}$$

We still use the notations defined in (1.4). The set of positive simple roots is

$$\Pi_B = \{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_n \mid 1 \leq i < n\}. \tag{3.7}$$

The set of dominant integral weights is

$$\Lambda_B^+ = \Big\{\mu = \sum_{i=1}^{n} \mu_i \varepsilon_i \mid \mu_i \in \mathbb{N}/2; \mu_i - \mu_{i+1} \in \mathbb{N} \text{ for } i \in \overline{1, n-1}\Big\}. \tag{3.8}$$

For each $\mu \in \Lambda_B^+$, there is a unique sequence $S(\mu) = \{n_0, n_1, \cdots, n_s\}$ such that (1.8)–(1.9) hold.

Let M be an $o(2n+1, \mathbb{C})$-module and let $b \in \mathbb{C}$ be a fixed constant. Then $\widehat{M} = \mathcal{A} \otimes_{\mathbb{C}} M$ becomes an $o(2n+3, \mathbb{C})$-module with the actions (1.13)–(1.16) and

$$K_t|_{\widehat{M}} = K_t|_{\mathcal{A}} \otimes \mathrm{Id}|_M + \mathrm{Id}_{\mathcal{A}} \otimes K_t|_M, \tag{3.9}$$

$$J_0 = J_0|_{\mathcal{A}} \otimes \mathrm{Id}_M + \sum_{s=1}^{2n} x_s \otimes K_s|_M + x_0 \otimes D|_M, \tag{3.10}$$

$$J_i|_{\widehat{M}} = J_i|_{\mathcal{A}} \otimes \mathrm{Id}_M + \sum_{p=1}^{n} x_{n+p} \otimes B_{i,p}|_M - x_0 \otimes K_{n+i}|_M + \sum_{q=1}^{n} x_q \otimes A_{i,q}|_M + x_i \otimes D|_M, \tag{3.11}$$

$$J_{n+i}|_{\widehat{M}} = J_{n+i}|_{\mathcal{A}} \otimes \mathrm{Id}_M - \sum_{p=1}^{n} x_{n+p} \otimes A_{p,i}|_M + \sum_{q=1}^{n} x_q \otimes C_{i,q}|_M - x_0 \otimes K_i|_M + x_{n+i} \otimes D|_M, \tag{3.12}$$

where $t \in \overline{1, 2n}$, $i \in \overline{1, n}$.

According to [2, 4], there is a bijection between the set of hwvs in $\widehat{M}$ and the set

$$V_B = \{f \in \mathcal{A} \mid K_{2n}^{2\mu_n+1}(f) = 0 \text{ and } A_{j,j+1}^{\mu_j-\mu_{j+1}+1}(f) = 0 \text{ for } j = 1, 2, \cdots, n-1\}. \tag{3.13}$$

The notations x^α, J^α, $\mathcal{A}_k$ and $M_{\langle k\rangle}$ can be similarly defined as in (2.2)–(2.3), but all the summations and products should be taken from 0 to $2n$ in the type B case.

In [10], a linear transformation φ on $\widehat{M}$ is defined by

$$\varphi(x^\alpha \otimes v) = J^\alpha(1 \otimes v) \quad \text{for } \alpha \in \mathbb{N}^{2n+1},\ v \in M. \tag{3.14}$$

Moreover, φ can also be viewed as an $o(2n+1, \mathbb{C})$-module homomorphism. According to Lemma 3.3 in [10], $\mathrm{Im}\,\varphi$ is an irreducible $o(2n+3, \mathbb{C})$-module if M is a finite-dimensional irreducible $o(2n+1, \mathbb{C})$-module. Therefore, like the case of type D, the $o(2n+3, \mathbb{C})$-module $\widehat{M}$ is irreducible if and only if φ is bijective.

3.2 Some notations and lemmas

Throughout in this section, we always let M be a finite-dimensional irreducible $o(2n+1, \mathbb{C})$-module with highest weight $\mu \in \Lambda_B^+$. Like in the case of type D, we investigate the hwvs in $\widehat{M}$ first.

By Pieri's formula (cf. [3]), given $\mu \in \Lambda_B^+$ with $S(\mu) = \{n_0, n_1, \cdots, n_s\}$, we have

$$V(\varepsilon_1) \otimes V(\mu) \cong \begin{cases} V(\mu) \oplus \bigoplus_{i=1}^{s-\delta_{\mu_n,\frac{1}{2}}} V(\mu - \varepsilon_{n_i}) \oplus \bigoplus_{r=0}^{s-1} V(\mu + \varepsilon_{n_r+1}), & \text{if } \mu_n > 0, \\ \bigoplus_{i=1}^{s-1} V(\mu - \varepsilon_{n_i}) \oplus \bigoplus_{r=0}^{s-1} V(\mu + \varepsilon_{n_r+1}), & \text{if } \mu_n = 0. \end{cases} \tag{3.15}$$

Denote by v_μ the hwv in $V(\mu)$. Using arguments similar to those in Subsection 2.2, we can write the hwvs in $V(\mu + \varepsilon_{n_i+1})$ as $v_{\langle n_i+1\rangle}$ (cf. (2.5)–(2.6)) and write the hwv in $V(\mu)$ as

$$v_{\langle 0\rangle} = x_0 \otimes v_\mu + \sum_{j=0}^{s-1} X_j \cdot v_{\langle n_j+1\rangle}, \tag{3.16}$$

when $\mu_n > 0$. We set $v_{\langle 0\rangle} = 0$ when $\mu_n = 0$ for convenience. We write the hwvs in $V(\mu - \varepsilon_{n_i})$ as

$$v_{\langle -n_i\rangle} = x_{n+n_i} \otimes v_\mu + \sum_{j=i+1}^{s} Y_{ij} \cdot v_{\langle -n_j\rangle} + \sum_{j=0}^{s-1} Z_{ij} \cdot v_{\langle n_j+1\rangle} + Z_i \cdot v_{\langle 0\rangle}, \tag{3.17}$$

where $X_{i,j}, X_j, Y_{i,j}, Z_{i,j}, Z_i \in \mathcal{G}^-$. We have (cf. [10, (3.67)])

$$\varphi(v_{\langle n_i+1\rangle}) = (b+\mu_{n_i+1} - n_i)v_{\langle n_i+1\rangle} \quad \text{for } i = 0, \cdots, s-1. \tag{3.18}$$

$$\varphi(v_{\langle 0\rangle}) = (b - n)v_{\langle 0\rangle}, \tag{3.19}$$

$$\varphi(v_{\langle -n_i\rangle}) = (b - \mu_{n_i} - 2n + n_i)v_{\langle -n_i\rangle}. \tag{3.20}$$

Suppose that u is an hwv in $M_{\langle k\rangle}$ of weight ν, and the sequence

$$S(\nu) = \{h_0, h_1, \cdots, h_{t-1}, h_t\}.$$

We denote the hwvs in $\langle x_1\rangle \otimes M_{\langle k\rangle}$ by $\widetilde{u}_{\langle h_i+1\rangle}$, $\widetilde{u}_{\langle -h_i\rangle}$, $\widetilde{u}_{\langle 0\rangle}$ (throughout in this section, the notation $\langle w\rangle$ always stands for the $o(2n+1,\mathbb{C})$-module generated by w), and define $\psi:\ \langle x_1\rangle \otimes M_{\langle k\rangle} \to M_{\langle k+1\rangle}$ as in (2.12). Denote $u_{\langle h_i+1\rangle} = \psi(\widetilde{u}_{\langle h_i+1\rangle})$, $u_{\langle 0\rangle} = \psi(\widetilde{u}_{\langle 0\rangle})$, $u_{\langle -h_i\rangle} = \psi(\widetilde{u}_{\langle -h_i\rangle})$. Then all the hwvs in $M_{\langle k+1\rangle}$ should be equal to some $u_{\langle h_k+1\rangle}$, $u_{\langle 0\rangle}$ or $u_{\langle -h_i\rangle}$.

We can also define $u^{\langle h_i+1\rangle}$, $u^{\langle 0\rangle}$ and $u^{\langle -h_i\rangle}$ in the same way as in (2.17)–(2.18). Analogous to Lemma 2.1, we have the following lemma.

Lemma 3.1 $\Delta(u_{\langle h_i+1\rangle}) = u^{\langle h_i+1\rangle} + (\Delta(u))_{\langle h_i+1\rangle}$, $\Delta(u_{\langle 0\rangle}) = u^{\langle 0\rangle} + (\Delta(u))_{\langle 0\rangle}$, $\Delta(u_{\langle -h_i\rangle}) = u^{\langle -h_i\rangle} + (\Delta(u))_{\langle -h_i\rangle}$.

Therefore, as in the case of type D, if $u^{\langle a\rangle} \neq 0\,(a = h_i + 1, 0, -h_i)$, then $u^{\langle a\rangle}$ is an hwv in $M_{\langle k-1\rangle}$. We also have (2.19).

Moreover, (2.22)–(2.25) still hold. In addition, we have

$$\psi J_0(1 \otimes (x^\alpha \otimes w)) = (J_0 - \eta\partial_0)(x^\alpha \otimes w) + kx_0x^\alpha \otimes w, \tag{3.21}$$

$$\begin{aligned}\varphi(x_0x^\alpha \otimes w) &= J_0\varphi(x^\alpha \otimes w)\\ &= \psi\varphi^{(k)}(x_0 \otimes \varphi(x^\alpha \otimes w)) + \eta\partial_0\varphi(x^\alpha \otimes w) - kx_0\varphi(x^\alpha \otimes w),\end{aligned} \tag{3.22}$$

where $\alpha \in \mathbb{N}^{2n+1}(|\alpha| = k)$, $w \in M$.

The following lemma is analogous of Lemma 2.2.

Lemma 3.2 *Assume u is an hwv in $M_{\langle k\rangle}$ of weight ν with sequence $S(\nu) = \{h_0, h_1, \cdots, h_t\}$. Then*

$$\varphi(u_{\langle h_i+1\rangle}) = (b + \nu_{h_i+1} - h_i)\varphi(u)_{\langle h_i+1\rangle} + \eta\varphi(u)^{\langle h_i+1\rangle}, \tag{3.23}$$

$$\varphi(u_{\langle 0\rangle}) = (b - n)\varphi(u)_{\langle 0\rangle} + \eta\varphi(u)^{\langle 0\rangle}, \tag{3.24}$$

$$\varphi(u_{\langle -h_i\rangle}) = (b - \nu_{h_i} - 2n + h_i)\varphi(u)_{\langle -h_i\rangle} + \eta\varphi(u)^{\langle -h_i\rangle}. \tag{3.25}$$

3.3 Necessity for type B

In this subsection, we discuss the necessity for type B. There is some difference from Subsection 2.4, since we have x_0 in this case.

The following discussion is just like Subsection 2.4, but with a little change. Let

$$f = x_{2n}^{[\mu_n]}\prod_{i=1}^{n-1}x_{n+i}^{\mu_i-\mu_{i+1}}, \quad \nu = \mu - \sum_{i=1}^{n-1}(\mu_i-\mu_{i+1})\varepsilon_i - [\mu_n]\varepsilon_n, \tag{3.26}$$

where $[\mu_n]$ stands for the integer part of μ_n. It is straightforward to check $f \in V_B$ (cf. (3.13)), so there is an hwv in form of

$$u = f \otimes v_\mu + \cdots . \tag{3.27}$$

We can conclude that u is an eigenvector of φ, since ν is of multiplicity 1 in $M_{\langle \mu_1-(\mu_n)\rangle}$, where $(\mu_n) = \mu_n - [\mu_n]$. We have

$$\varphi(u) = a_u u, \tag{3.28}$$

where

$$a_u = \prod_{k_s=0}^{[\mu_n]-1}(b-\mu_n-n+k_s)\prod_{i=1}^{s-1}\prod_{k_i=0}^{\mu_{n_i}-\mu_{n_i+1}-1}(b-\mu_{n_i}-2n+n_i+k_i). \tag{3.29}$$

Theorem 3.1 *If φ is bijective, then*
(1) $b \neq \mu_{n_i}+2n-n_i-k_i$ *for* $k_i \in \overline{0, \mu_{n_i}-\mu_{n_i+1}-1}$;
(2) $b \neq \mu_n + n - k_s$ *for* $k_s \in \overline{0, [\mu_n]-1}$.

Denote

$$\widetilde{\eta} = \sum_{i=1}^{n} J_i J_{n+i} + \frac{1}{2}J_0^2. \tag{3.30}$$

Then we have

$$\begin{aligned}[\partial_{n+1}, \widetilde{\eta}] &= \Big[\partial_{n+1}, \sum_{i=1}^{n} J_i J_{n+i} + \frac{1}{2}J_0^2\Big] \\ &= \Big(-n+\frac{1}{2}\Big)J_1 + J_0K_{n+1} + \sum_{i=1}^{n}(J_{n+i}B_{i,1} + J_i(\delta_{1,i}D - A_{1,i})).\end{aligned} \tag{3.31}$$

Using arguments analogous to those in Lemmas 2.4–2.5, we have the following two lemmas.

Lemma 3.3 *Assume $w \in M_{\langle l\rangle}$. Then $\widetilde{\eta}(\eta^k w) = k(2b-n+k+l+\frac{1}{2})\eta^{k+1}w + \eta^k\widetilde{\eta}(w)$.*

Lemma 3.4 $\varphi(\eta^k u) = a_u \prod_{t=0}^{k-1}(b+\mu_1+t)(b-(\mu_n)-n+\frac{1}{2}+t)\eta^k u$ *for all* $k \in \mathbb{N}$.

Therefore, we obtain the second necessary condition for type B.

Theorem 3.2 *If φ is bijective, then* $b \neq -\mu_1 - k, (\mu_n)+n-k-\frac{1}{2}$ *for all* $k \in \mathbb{N}$.

One sees that unlike the case of type D, we cannot include $b+\mu_1+k\neq 0$ from $b-(\mu_n)-n+\frac{1}{2}+k\neq 0$, since μ_1 and $(\mu_n)-\frac{1}{2}$ cannot be integers or half integers at the same time. In addition, the following condition can be obtained by $b-|\mu_n|-n+1+k\neq 0$ in type D case, but cannot be obtained by Theorem 3.2 in type B case. Let

$$g=\prod_{i=2}^{n} x_i^{\mu_{i-1}-\mu_i}=\prod_{i=1}^{s-1} x_{n_i+1}^{\mu_{n_i}-\mu_{n_i+1}}. \tag{3.32}$$

Then $g\in V_B$. Let

$$v=g\otimes v_\mu+\cdots \tag{3.33}$$

be an hwv. Then we have

$$\varphi(v)=a_v v, \tag{3.34}$$

where (cf. [11])

$$a_v=\prod_{i=1}^{s-1}\prod_{k_i=0}^{\mu_{n_i}-\mu_{n_{i+1}}-1}(b+\mu_{n_i+1}-n_i+k_i). \tag{3.35}$$

Therefore, we have $a_v\neq 0$ when φ is bijective.

Theorem 3.3 *If φ is bijective, then $b\neq -\mu_{n_i+1}+n_i-k_i$ for all*

$$i\in\overline{1,s-1},\quad k_i\in\overline{0,\mu_{n_i}-\mu_{n_i+1}-1}.$$

It is time to let x_0 show its colour. For each $k\in\overline{1,[\mu_n]}$, the weight $\mu+\sum\limits_{i=2}^{n}(\mu_{i-1}-\mu_i)\varepsilon_i$ has multiplicity $k+1$ in $M_{\langle\mu_1-\mu_n+2k\rangle}$, because $x_0^{2i}\eta^{k-i}g\in V_B$, $\forall i\in\overline{0,k}$. Let $\nu=\mu+\sum\limits_{i=2}^{n}(\mu_{i-1}-\mu_i)\varepsilon_i$. Let w be any hwv in $M_{\langle\mu_1-\mu_n+2k\rangle}$ ($k\in\overline{0,[\mu_n]}$) with highest weight ν. Then w must be in form of

$$\sum_{i=0}^{k} a_i x_0^{2i}\eta^{k-i} g\otimes v_\mu+\cdots, \tag{3.36}$$

where $a_i\in\mathbb{C}$. Suppose the sequence corresponding to w and ν is $S(\nu)=\{h_0,\cdots,h_t\}$. In fact, $h_i=n_i+1$, $i\in\overline{1,s-1}$,

$$t=\begin{cases} s-1, & n_{s-1}=n-1,\\ s, & n_{s-1}<n-1.\end{cases}$$

Consider the $\mathcal{G}$-module $\langle x_1\rangle\otimes\langle w\rangle$. We have hwvs $\widetilde{w}_{\langle h_i+1\rangle}$, $\widetilde{w}_{\langle 0\rangle}$ and $\widetilde{w}_{\langle -h_i\rangle}$. But $\psi(\widetilde{w}_{\langle h_i+1\rangle})$ has to be 0 for all $i\in\overline{1,t-1}$, since we do not have corresponding hwvs in $M_{\langle\mu_1-\mu_n+2k+1\rangle}$. Thus we can assume

$$w_{\langle 0\rangle}=x_0 w+X\cdot(w_{\langle 1\rangle})=x_0 w+X\cdot(x_1 w) \tag{3.37}$$

for some $X\in U(\mathcal{G}^-)$.

On one hand, we have

$$\begin{aligned}
&\varphi(w_{\langle 0\rangle}) \\
&= J_0\varphi(w) + X.\varphi(x_1 w) \\
&= \sum_{i=1}^n x_i K_i\varphi(w) + \sum_{i=1}^n x_{n+i}K_{n+i}\varphi(w) + bx_0\varphi(w) + \eta\partial_0\varphi(w) \\
&\quad + X\cdot((b+\mu_1)x_1\varphi(w) + \eta\partial_{n+1}\varphi(w)) \\
&= \sum_{i=1}^n K_i(x_i\varphi(w)) + (b-n)x_0\varphi(w) + \eta\varphi(w)^{\langle 0\rangle} + (b+\mu_1)X\cdot x_1\varphi(w) \\
&= \sum_{i=1}^n K_i(x_i\varphi(w)) + (b-n)\varphi(w)_{\langle 0\rangle} + \eta\varphi(w)^{\langle 0\rangle} + (n+\mu_1)X\cdot x_1\varphi(w)
\end{aligned} \tag{3.38}$$

by (3.22). Note that $K_{n+i}\cdot\varphi(w) = 0$, because $\varphi(w)$ is an hwv. Also note that $\varphi(w)_{\langle 0\rangle} = x_0\varphi(w) + X\cdot x_1\varphi(w)$, because $\langle w\rangle$ and $\langle\varphi(w)\rangle$ are isomorphic $\mathcal{G}$-modules.

On the other hand, by Lemma 3.2 we have

$$\varphi(w_{\langle 0\rangle}) = (b-n)\varphi(w)_{\langle 0\rangle} + \eta\varphi(w)^{\langle 0\rangle}. \tag{3.39}$$

Comparing the two equations above, we obtain

$$(n+\mu_1)X\cdot x_1\varphi(w) + \sum_{i=1}^n K_i(x_i\varphi(w)) = 0, \tag{3.40}$$

hence

$$(n+\mu_1)X\cdot x_1 w + \sum_{i=1}^n K_i(x_i w) = 0, \tag{3.41}$$

since $\langle w\rangle \cong \langle\varphi(w)\rangle$. Therefore, we have

$$\begin{aligned}
w_{\langle 0\rangle} &= x_0 w - \frac{1}{n+\mu_1}\sum_{i=1}^n K_i(x_i w), \\
(n+\mu_1)w_{\langle 0\rangle} &= \mu_1 x_0 w - \sum_{i=1}^n x_i K_i w.
\end{aligned} \tag{3.42}$$

Denote $(w_{\langle 0\rangle})_{\langle 0\rangle}$ by $w_{\langle 0,0\rangle}$. By the similar argument, we have

$$(n+\mu_1)w_{\langle 0,0\rangle} = \mu_1 x_0 w_{\langle 0\rangle} - \sum_{i=1}^n x_i K_i w_{\langle 0\rangle}. \tag{3.43}$$

Since $w_{\langle 0,0\rangle}$ is a hwv in $M_{\langle \mu_1-\mu_n+2k+2\rangle}$, $(n+\mu_1)^2 w_{\langle 0,0\rangle}$ takes the form of

$$\sum_{i=0}^{k+1} c_i x_0^{2i}\eta^{k+1-i} g\otimes v_\mu + \cdots . \tag{3.44}$$

Let us work out the relations between a_i and c_i.

We have

$$\begin{aligned}
&(n+\mu_1)w_{\langle 0\rangle}\\
&= \mu_1 x_0 \sum_{i=0}^{k} a_i x_0^{2i}\eta^{k-i} g\otimes v_\mu + \cdots\\
&\quad - \sum_{p=1}^{n}\Big(x_p\Big(K_p\sum_{i=0}^{k} a_i x_0^{2i}\eta^{k-i}g\Big)\otimes v_\mu + x_p\sum_{i=0}^{k} a_i x_0^{2i}\eta^{k-i} g\otimes K_p v_\mu\Big) + \cdots\\
&= g_1\otimes v_\mu + \cdots\,, \qquad (3.45)
\end{aligned}$$

where

$$g_1 = \mu_1 x_0\sum_{i=0}^{k} a_i x_0^{2i}\eta^{k-i} g - \sum_{p=1}^{n} x_p K_p \sum_{i=0}^{k} a_i x_0^{2i}\eta^{k-i} g.$$

By straightforward computing, we get

$$\begin{aligned}
g_1 &= \mu_1 x_0 \sum_{i=0}^{k} a_i x_0^{2i}\eta^{k-i} g - \sum_{i=0}^{k} a_i\eta^{k-i}\sum_{p=1}^{n} x_p K_p(x_0^{2i} g)\\
&= \mu_1 x_0 \sum_{i=0}^{k} a_i x_0^{2i}\eta^{k-i} g - \sum_{i=0}^{k} a_i\eta^{k-i}\sum_{p=1}^{n} x_0^{2i+1} x_p\partial_p g\\
&\quad + \sum_{i=0}^{k} a_i\eta^{k-i}\sum_{p=1}^{n} 2i x_p x_{n+p} x_0^{2i-1} g\\
&= \mu_1 \sum_{i=0}^{k} a_i x_0^{2i+1}\eta^{k-i} g - \sum_{i=0}^{k} a_i(\mu_1-\mu_n)\eta^{k-i} x_0^{2i+1} g\\
&\quad + \sum_{i=0}^{k} 2i a_i \eta^{k-i}\Big(\eta - \frac{1}{2}x_0^2\Big) x_0^{2i-1} g\\
&= \sum_{i=0}^{k} b_i x_0^{2i+1}\eta^{k-i} g, \qquad (3.46)
\end{aligned}$$

where $b_i = (\mu_n - i)a_i + 2(i+1)a_{i+1}$, for all $i=\overline{0,k}$. Here we set $a_{k+1}=0$ for convenience.

$$\begin{aligned}
(n+\mu_1)^2 w_{\langle 0,0\rangle} &= (n+\mu_1)\mu_1 x_0 w_{\langle 0\rangle} - (n+\mu_1)\sum_{i=1}^{n} x_i K_i w_{\langle 0\rangle}\\
&= \mu_1 x_0 g_1\otimes v_\mu + \cdots\\
&\quad - \sum_{i=1}^{n}(x_i K_i g_1)\otimes v_\mu + \cdots\\
&= g_2\otimes v_\mu + \cdots\,, \qquad (3.47)
\end{aligned}$$

where

$$g_2 = \mu_1 x_0 g_1 - \sum_{i=1}^{n} x_i (K_i g_1)$$

$$
\begin{aligned}
&= \mu_1 x_0 \sum_{i=0}^{k} b_i x_0^{2i+1}\eta^{k-i} g - \sum_{p=1}^{n} x_p K_p \sum_{i=0}^{k} b_i x_0^{2i+1}\eta^{k-i} g \\
&= \mu_1 \sum_{i=0}^{k} b_i x_0^{2i+2}\eta^{k-i} g - \sum_{i=0}^{k} b_i x_0^{2i+2}\eta^{k-i} \sum_{p=1}^{n} x_p \partial_p g \\
&\quad + \sum_{i=0}^{k} (2i+1) b_i \eta^{k-i} \sum_{p=1}^{n} x_p x_{n+p} x_0^{2i} g \\
&= \mu_1 \sum_{i=0}^{k} b_i x_0^{2i+2}\eta^{k-i} g - \sum_{i=0}^{k} b_i (\mu_1 - \mu_n) x_0^{2i+2}\eta^{k-i} g \\
&\quad + \sum_{i=0}^{k} (2i+1) b_i \eta^{k-i} \Big(\eta - \frac{1}{2} x_0^2\Big) x_0^{2i} g \\
&= \sum_{i=0}^{k+1} c_i x_0^{2i} \eta^{k+1-i} g, \qquad (3.48)
\end{aligned}
$$

$$
\begin{aligned}
c_i &= \Big(\mu_n - i + \frac{1}{2}\Big) b_{i-1} + (2i+1) b_i \\
&= \Big(\mu_n - i + \frac{1}{2}\Big)((\mu_n - i + 1) a_{i-1} \\
&\quad + 2i a_i) + (2i+1)((\mu_n - i) a_i + 2(i+1) a_{i+1}) \\
&= \Big(\mu_n - i + \frac{1}{2}\Big)(\mu_n - i + 1) a_{i-1} \\
&\quad + (\mu_n + 4i\mu_n - 4i^2) a_i + 2(i+1)(2i+1) a_{i+1} \qquad (3.49)
\end{aligned}
$$

for all $i \in \overline{0, k+1}$. We set $b_{-1} = a_{-1} = a_{k+2} = 0$ for convenience.

When $w = v$, we have $k = 0$, $a_0 = 1$ and

$$
(n + \mu_1)^2 v_{\langle 0,0\rangle} = \Big(\mu_n \eta + \mu_n \Big(\mu_n - \frac{1}{2}\Big) x_0^2\Big) g \otimes v_\mu + \cdots .
$$

Therefore, $v_{\langle 0,0\rangle}$ and ηv are linearly independent.

Lemma 3.5 $\widetilde{\eta} v = ((b + \mu_1)(b - \mu_n - n + \frac{1}{2}) - \frac{1}{2}\mu_n)\eta v + \frac{1}{2}(n + \mu_1)^2 v_{\langle 0,0\rangle}$.

Proof Assume

$$
\widetilde{\eta} v = A\eta v + B(n + \mu_1)^2 v_{\langle 0,0\rangle}, \qquad (3.50)
$$

where $A, B \in \mathbb{C}$. On one hand, we have

$$
\partial_{n+1}\widetilde{\eta} v = (A + B\mu_n) x_1 v, \qquad (3.51)
$$

$$
\partial_0 \widetilde{\eta} v = (A + 2B\mu_n^2) x_0 g \otimes v_\mu + \cdots . \qquad (3.52)
$$

On the other hand, we have

$$
\begin{aligned}
\partial_{n+1}\widetilde{\eta} v &= ([\partial_{n+1}, \widetilde{\eta}] + \widetilde{\eta}\partial_{n+1})(v) \\
&= \Big(\Big(-n + \frac{1}{2}\Big) J_1 + J_1 (D - A_{1,1})\Big) v
\end{aligned}
$$

$$= (b+\mu_1)\Big(b-\mu_n-n+\frac{1}{2}\Big)x_1v. \tag{3.53}$$

In addition, we have

$$[\partial_0,\widetilde{\eta}] = J_0D+\frac{1}{2}J_0-\sum_{i=1}^{n}(J_{n+i}K_{n+i}+K_iJ_i), \tag{3.54}$$

$$\begin{aligned}J_0(v) &= \psi\varphi^{(|v|)}(x_0\otimes v)+\eta\partial_0v-(\mu_1-\mu_n)x_0v\\ &= \sum_{i=1}^{n}x_iK_i(v)+bx_0v\\ &= (b+\mu_1-\mu_n)x_0g\otimes v_\mu+\cdots .\end{aligned} \tag{3.55}$$

We have

$$\begin{aligned}K_1J_1(v) &= (b+\mu_1)K_1(x_1v)\\ &= (b+\mu_1)x_0g\otimes v_\mu+\cdots\end{aligned} \tag{3.56}$$

and

$$\begin{aligned}K_iJ_i(v) &= K_i(\psi\varphi^{(|v|)}(x_i\otimes v)+\eta\partial_{n+i}v-(\mu_1-\mu_n)x_iv)\\ &= K_i\Big(\sum_{q=1}^{i-1}x_qA_{i,q}(v)+(b+\mu_{i-1})x_iv\Big)\\ &= K_i\Big(\Big(x_i\sum_{q=1}^{i-1}x_q\partial_q(g)+(b+\mu_{i-1})x_ig\Big)\otimes v_\mu+\cdots\Big)\\ &= K_i((b+\mu_1)x_ig\otimes v_\mu+\cdots)\\ &= (b+\mu_1)(\mu_{i-1}-\mu_i+1)x_0g\otimes v_\mu+\cdots\end{aligned} \tag{3.57}$$

for all $i\in\overline{2,n}$. Thus, we obtain

$$\begin{aligned}&\partial_0\widetilde{\eta}v\\ &= \Big((b+\mu_1-\mu_n)\Big(b+\mu_1-\mu_n+\frac{1}{2}\Big)-(b+\mu_1)(n+\mu_1-\mu_n)\Big)x_0g\otimes v_\mu+\cdots\\ &= \Big((b+\mu_1)\Big(b-\mu_n-n+\frac{1}{2}\Big)+\mu_n\Big(\mu_n-\frac{1}{2}\Big)\Big)x_0g\otimes v_\mu+\cdots .\end{aligned} \tag{3.58}$$

Comparing the coefficients of x_1v in (3.51), (3.53) and the coefficients of $x_0g\otimes v_\mu$ in (3.52), (3.58), we get

$$A=(b+\mu_1)\Big(b-\mu_n-n+\frac{1}{2}\Big)-\frac{1}{2}\mu_n,\quad B=\frac{1}{2}.$$

Lemma 3.6 *When φ is bijective, we assume $\varphi(w^k)=\eta^kv$ for all $k\in\overline{0,[\mu_n]-1}$. Then ηw^k and $w^k_{\langle 0,0\rangle}$ are linearly independent.*

Proof We have already shown that it holds for $k=0$. Assume the lemma holds for some k. Then $\varphi(\eta w^k)$ and $\varphi(w^k_{\langle 0,0\rangle})$ are linearly independent. We have

$$\varphi(\eta w^k)=\widetilde{\eta}\varphi(w^k)=\widetilde{\eta}(\eta^kv)$$

$$
\begin{aligned}
&= k\Big(2b - n + k + \mu_1 - \mu_n + \frac{1}{2}\Big)\eta^{k+1}v + \eta^k\widetilde{\eta}v \\
&= \Big((b+\mu_1+k)\Big(b-\mu_n-n+\frac{1}{2}+k\Big) - \frac{1}{2}\mu_n\Big)\eta^{k+1}v \\
&\quad + \frac{1}{2}(n+\mu_1)^2\eta^k v_{\langle 0,0\rangle},
\end{aligned} \tag{3.59}
$$

$$
\begin{aligned}
\varphi(w^k_{\langle 0\rangle}) &= (b-n)\varphi(w^k)_{\langle 0\rangle} + \eta\varphi(w^k)^{\langle 0\rangle} \\
&= (b-n)\eta^k v_{\langle 0\rangle} + \eta(\eta^k v)^{\langle 0\rangle} \\
&= (b-n+k)\eta^k v_{\langle 0\rangle}
\end{aligned} \tag{3.60}
$$

(note that $(\eta^k v)^{\langle 0\rangle} = k\eta^k v_{\langle 0\rangle} + \eta^k v^{\langle 0\rangle}$ and $v^{\langle 0\rangle} = 0$),

$$
\begin{aligned}
\varphi(w^k_{\langle 0,0\rangle}) &= (b-n)\varphi(w^k_{\langle 0\rangle})_{\langle 0\rangle} + \eta\varphi(w^k_{\langle 0\rangle})^{\langle 0\rangle} \\
&= (b-n)(b-n+k)\eta^k v_{\langle 0,0\rangle} + (b-n+k)\eta(\eta^k v_{\langle 0\rangle})^{\langle 0\rangle} \\
&= (b-n+k)^2\eta^k v_{\langle 0,0\rangle} + (b-n+k)\eta^{k+1}(v_{\langle 0\rangle})^{\langle 0\rangle}.
\end{aligned} \tag{3.61}
$$

We have

$$
\begin{aligned}
(v_{\langle 0\rangle})^{\langle 1\rangle} &= \partial_{n+1}v_{\langle 0\rangle} = 0, \\
(v_{\langle 0\rangle})^{\langle 0\rangle} &= \partial_0 v_{\langle 0\rangle} = v - \frac{1}{n+\mu_1}\sum_{i=1}^{n}\partial_0 K_i(x_i v) \\
&= v - \frac{1}{n+\mu_1}\sum_{i=1}^{n}([\partial_0, K_i] + K_i\partial_0)(x_i v) \\
&= v - \frac{1}{n+\mu_1}\sum_{i=1}^{n}\partial_i(x_i v) = v - \frac{n+\mu_1-\mu_n}{n+\mu_1}v \\
&= \frac{\mu_n}{n+\mu_1}v.
\end{aligned} \tag{3.62}
$$

Thus

$$
\varphi(w^k_{\langle 0,0\rangle}) = (b-n+k)^2\eta^k v_{\langle 0,0\rangle} + \frac{(b-n+k)\mu_n}{n+\mu_1}\eta^{k+1}v. \tag{3.63}
$$

Set

$$
\begin{aligned}
D_k &= \begin{vmatrix} (b+\mu_1+k)\Big(b-\mu_n-n+\frac{1}{2}+k\Big) - \frac{1}{2}\mu_n & \frac{1}{2}(n+\mu_1)^2 \\ \frac{(b-n+k)\mu_n}{n+\mu_1} & (b-n+k)^2 \end{vmatrix} \\
&= (b-n+k)\Big(b-n+k+\frac{1}{2}\Big)(b+\mu_1+k)(b-n+k-\mu_n).
\end{aligned} \tag{3.64}
$$

We get $D_k \neq 0$ and

$$
w^{k+1} = \frac{1}{D_k}\Big((b-n+k)^2\eta w^k - \frac{1}{2}(n+\mu_1)^2 w^k_{\langle 0,0\rangle}\Big). \tag{3.65}
$$

Assume that

$$
w^k = \sum_{i=0}^{k} a_{k,i}x_0^{2i}\eta^{k-i}g\otimes v_\mu + \cdots, \qquad w^k_{\langle 0,0\rangle} = \sum_{i=0}^{k+1} c_{k,i}x_0^{2i}\eta^{k+1-i}g\otimes v_\mu + \cdots. \tag{3.66}
$$

Then $a_{0,0} = \frac{1}{a_v}$. Note that $\varphi(w^0) = v$, and thus $w^0 = \frac{1}{a_v} v$. So

$$a_{k+1,k+1} = -\frac{1}{2D_k}\Big(\mu_n - k - \frac{1}{2}\Big)(\mu_n - k)a_{k,k} \tag{3.67}$$

by (3.49) and (3.65). For $k+1 < [\mu_n]$, we have $a_{k+1,k+1} \neq 0$.

Moreover, we have

$$\begin{aligned} & c_{k+1,i} \\ &= \Big(\mu_n - i + \frac{1}{2}\Big)(\mu_n - i + 1)a_{k+1,i-1} + (\mu_n + 4i\mu_n - 4i^2)a_{k+1,i} + 2(i+1)(2i+1)a_{k+1,i+1} \end{aligned}$$

by (3.49). Here we set $a_{k+1,-1} = a_{k+1,k+2} = a_{k+1,k+3} = 0$. In particular,

$$c_{k+1,k+2} = \Big(\mu_n - k - \frac{3}{2}\Big)(\mu_n - k - 1)a_{k+1,k+1} \neq 0.$$

Assume that $A\eta w^{k+1} + Bw^{k+1}_{\langle 0,0\rangle} = 0$. Then

$$\Big(A\eta \sum_{i=0}^{k+1} a_{k+1,i}x_0^{2i}\eta^{k+1-i} + B\sum_{i=0}^{k+2} c_{k+1,i}x_0^{2i}\eta^{k+2-i}\Big)g \otimes v_\mu + \cdots = 0.$$

Thus we obtain

$$\begin{aligned} & A\eta \sum_{i=0}^{k+1} a_{k+1,i}x_0^{2i}\eta^{k+1-i} + B\sum_{i=0}^{k+2} c_{k+1,i}x_0^{2i}\eta^{k+2-i} \\ &= \sum_{i=0}^{k+2}(Aa_{k+1,i} + Bc_{k+1,i})x_0^{2i}\eta^{k+2-i} = 0, \end{aligned}$$

which implies $Aa_{k+1,i} + Bc_{k+1,i} = 0$ for all $i \in \overline{0, k+2}$. In particular, $Aa_{k+1,k+2} + Bc_{k+1,k+2} = 0$, which means $A = B = 0$ since $a_{k+1,k+2} = 0$, $c_{k+1,k+2} \neq 0$. The lemma is proved.

From the proof of Lemma 3.6, we obtain the following theorem.

Theorem 3.4 *If φ is bijective, then $b \neq n-k, n-k-\frac{1}{2}$ for all $k \in \overline{0, [\mu_n]-1}$.*

3.4 Sufficiency for type *B*

This subsection is almost a copy of Subsection 2.4. So we omit the proofs.

Let

$$f_s = x_{2n}^{[\mu_n]}, \quad f_i = x_{n+n_i}^{\mu_{n_i} - \mu_{n_{i+1}}} f_{i+1} \quad \text{for } 0 < i < s. \tag{3.68}$$

There is a unique hwv in $M_{\langle k\rangle}$ in the form of

$$u_s^k = x_{2n}^k \otimes v_\mu + \cdots \tag{3.69}$$

for $0 \le k \le [\mu_n]$ up to a scalar, and there is a unique hwv in $M_{\langle \mu_{n_i+1} - (\mu_n) + k\rangle}$ in the form of

$$u_i^k = x_{n+n_i}^k f_{i+1} \otimes v_\mu + \cdots \tag{3.70}$$

for $0 < i < s$, $0 \leq k \leq \mu_{n_i} - \mu_{n_{i+1}}$ up to a scalar. Note $u_{i-1}^0 = u_i^{\mu_{n_i}-\mu_{n_{i+1}}}$. One should also note f_1 is equal to f defined in (3.26) and $u_1^{\mu_{n_1}-\mu_{n_2}}$ is equal to u in (3.27).

The following lemma can be obtained almost by the same argument as Lemma 2.7. The only difference is that we need the condition given by Theorem 3.4 when $i = s$.

Lemma 3.7 *If $b \notin \Theta_B(\mu)$, then $\eta^d u_i^{k_i} \in \operatorname{Im}\varphi$ for all $d \in \mathbb{N}$, $i \in \overline{1, s-1}$, $k_i \in \overline{0, \mu_{n_i} - \mu_{n_{i+1}}}$, $k_s \in \overline{0, [\mu_n]}$.*

For $n_i = n, k = 0$, we get $\eta^d \otimes v_\mu \in \operatorname{Im}\varphi$.

By the similar argument as in Lemma 2.8, we have the following lemma.

Lemma 3.8 *The $o(2n+3, \mathbb{C})$-module $\widehat{M}$ can be generated by $\{\eta^d \otimes v_\mu \mid d \in \mathbb{N}\}$.*

The investigation about the sufficiency can be finished by the following theorem.

Theorem 3.5 *If $b \notin \Theta_B(\mu)$, then the $o(2n+1, \mathbb{C})$-homomorphism φ is bijective.*

Remark 3.1 One who has read [1, 11] may have noted that the Jordan-Hölder series have been given in those cases. However, the Jordan-Hölder series for orthogonal Lie algebras are not given in [10] or this paper. Here is why. In their cases, condition for the irreducibility is given by the eigenvalues of φ. They know exactly what the kernel and image of φ are when φ is not bijective.

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Declarations

Conflicts of interest The authors declare no conflicts of interest.

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