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A CLASS MATRIX EVOLUTION EQUATIONS

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§ 1. Introduction

Since the pioneer work of GGKM[1], Lax[2], Zakharov and Shabat[3] and AKNS[4] extended the inverse scattering method to solve much more evolution equations.

In ref. [5] a class of evolution equations were deduced from the eigenvalue problem

$$\varphi_x = M\varphi,\ \varphi_x = \frac{\partial\varphi}{\partial x},\ \varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \tag{1.1}$$

$$M = \begin{pmatrix} -i\xi & q(x,\ t)\\ r(x,\ t) & i\xi\end{pmatrix} \tag{1.2}$$

When ξ deforms according to some law, where two special classes of evolution equations associated with $r=1$, $q=u(x,\ t)$ and $r=q=V(x,\ t)$, may be viewed as generalized KDV and MKDV equations; The author proved that the Miura transformation not only exists between the KDV equation and the MKDV equation but also exists between the generalized KDV equation and generalized MKDV equation and got the Bäcklund transformation for the generalized KDV equation.

In this paper, we exchange M as follows

$$M = \begin{pmatrix} -i\xi I & Q(x,\ t)\\ R(x,\ t) & i\xi I\end{pmatrix}. \tag{1.3}$$

where Q, R are $N\times N$ matrix. I is an $N\times N$ unit matrix (now In (1.1), φ is $2N$–dimension vector) and introduce the matrix form of Miura transformation. We can extend the results of ref. [5] to a class of matrix evolution equations.

§ 2. Evolution equations

Consider the eigenvalue problem

$$\varphi_x = M\varphi,\ M = \begin{pmatrix} -i\xi I & Q\\ R & i\xi I\end{pmatrix},\ \varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\\ \cdots\\ \varphi_{2N}\end{pmatrix}, \tag{2.1}$$

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with an associated time evolution equation

$$\varphi_t = S\varphi, \quad S = \begin{pmatrix} A & B \\ C & E \end{pmatrix}, \tag{2.2}$$

where Q, R; A, B, C, E are $N\times N$ matrices. The elements of Q and R are function of x, t, and A, B, C and E depend on x, t and ξ. I is a $N\times N$ unit matrix.

Let

$$\xi_t = \sum_{j=0}^{n} l_j(t)\xi^{n-j}, \tag{2.3}$$

$$S = \sum_{j=0}^{n} S_j \xi^{n-j}, \quad S_j = \begin{pmatrix} A_j & B_j \\ C_j & E_j \end{pmatrix}. \tag{2.4}$$

Without any conditions imposed on the matrices Q, R at infinity we can deduce the following matrix evolution equations

$$\sigma P_t = 2i \sum_{k=0}^{n} L^{n-k}\sigma \left[P, \begin{pmatrix} \alpha_k & 0 \\ 0 & \delta_k \end{pmatrix}\right] - 2i \sum_{k=0}^{n} l_k L^{n-k}(xP), \tag{2.5}$$

where

$$\sigma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad P = \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}, \tag{2.6}$$

α_k, δ_k are arbitrary $N\times N$ matrix, The elements of which only depend on variable t.

Operater L is determined as follows

$$L = \frac{1}{2i}\sigma([P, D^{-1}[P, \cdot]] - D), \quad D = \frac{\partial}{\partial x}, \quad D^{-1}D = DD^{-1} = 1, \tag{2.7}$$

$[A, B]$ is Poisson bracket, $[A, B] = AB - BA$, $P_t = \frac{\partial}{\partial t}P$.

Now we derive the equation (2.5).

From $\varphi_{xt} = \varphi_{tx}$, we can get $S_x - M_t = [M, S]$. If we write

$$M = \begin{pmatrix} -i\xi I & 0 \\ 0 & i\xi I \end{pmatrix} + P, \quad \text{where} \quad \varphi = \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}$$

then

$$S_x - M_t = \left[\begin{pmatrix} -i\xi I & 0 \\ 0 & i\xi I \end{pmatrix}, S\right] + [P, S].$$

By an easy computation, we have

$$\left[\begin{pmatrix} -i\xi I & 0 \\ 0 & i\xi I \end{pmatrix}, S\right] = 2i\xi \begin{pmatrix} 0 & -B \\ C & 0 \end{pmatrix},$$

hence

$$S_x - M_t = 2i\xi \begin{pmatrix} 0 & -B \\ C & 0 \end{pmatrix} + [P, S]. \tag{2.8}$$

Substituting (2.3) and (2.4) into (2.8), it follows

$$\sum_{j=0}^{n} S_{j,x}\xi^{n-j} + \sum_{j=0}^{n} il_j\xi^{n-j}\sigma - P_t = 2i\xi \sum_{j=0}^{n} \begin{pmatrix} 0 & -B_j \\ C_j & 0 \end{pmatrix}\xi^{n-j} + \sum_{j=0}^{n} [P, S_j]\xi^{n-j}. \tag{2.9}$$

Comparing with the coefficient of same powers of ξ in (2.9), we have

$$\begin{cases} B_0 = C_0 = 0, \\ S_{j,x} + il_j\sigma = 2i\begin{pmatrix} 0 & -B_{j+1} \\ C_{j+1} & 0 \end{pmatrix} + [P, S_j], \quad j = 0, 1, \cdots, n-1, \\ S_{n,x} + il_n\sigma = \begin{pmatrix} 0 & Q_t \\ R_t & 0 \end{pmatrix} + [P, S_n]. \end{cases} \tag{2.10}$$

If we introduce B_{n+1}, C_{n+1} as follows

$$\begin{cases} Q_t = -2iB_{n+1}, \\ R_t = 2iC_{n+1}, \end{cases} \tag{2.11}$$

then (2.10) can be written as

$$\begin{cases} B_0 = C_0 = 0, \\ S_{j,x} + il_j\sigma = 2i\begin{pmatrix} 0 & -B_{j+1} \\ C_{j+1} & 0 \end{pmatrix} + [P, S_j], \quad j = 0, 1, \cdots, n. \end{cases} \tag{2.12}$$

Writing

$$S_j = \begin{pmatrix} A_j & B_j \\ C_j & E_j \end{pmatrix} = \begin{pmatrix} A_j & 0 \\ 0 & E_j \end{pmatrix} + \begin{pmatrix} 0 & B_j \\ C_j & 0 \end{pmatrix},$$

from (2.12) it follows

$$\begin{pmatrix} A_{j,x} & B_{j,x} \\ C_{j,x} & E_{j,x} \end{pmatrix} + il_j\sigma = 2i\begin{pmatrix} 0 & -B_{j+1} \\ C_{j+1} & 0 \end{pmatrix} + \left[P, \begin{pmatrix} A_j & 0 \\ 0 & E_j \end{pmatrix}\right] + \left[P, \begin{pmatrix} 0 & B_j \\ C_j & 0 \end{pmatrix}\right].$$

But a Poisson bracket of a non-diagonal block matrix with a diagonal block matrix results a non-diagonal block matrix, while that of a non-diagonal with a non-diagonal results a diagonal.

Thus, we have

$$\begin{cases} \begin{pmatrix} 0 & B_{j,x} \\ C_{j,x} & 0 \end{pmatrix} = 2i\begin{pmatrix} 0 & -B_{j+1} \\ C_{j+1} & 0 \end{pmatrix} + \left[P, \begin{pmatrix} A_j & 0 \\ 0 & E_j \end{pmatrix}\right], \\ \begin{pmatrix} A_{jx} & 0 \\ 0 & E_{jx} \end{pmatrix} = -il_j\sigma + \left[P, \begin{pmatrix} 0 & B_j \\ C_j & 0 \end{pmatrix}\right], \quad j = 0, 1, \cdots, n. \end{cases} \tag{2.13}$$

From the second equation of (2.13), we have

$$\begin{pmatrix} A_j & 0 \\ 0 & E_j \end{pmatrix} = -il_jx\sigma + D^{-1}\left[P, \begin{pmatrix} 0 & B_j \\ C_j & 0 \end{pmatrix}\right] - 2i\begin{pmatrix} \alpha_j & 0 \\ 0 & \delta_j \end{pmatrix}, \tag{2.14}$$

where α_j, δ_j are integral constant matrices, the elements of which only depend on t.

We rewrite the first equation of (2.13) as

$$\begin{pmatrix} 0 & B_{j+1} \\ C_{j+1} & 0 \end{pmatrix} = \frac{1}{2i}\sigma\left[P, \begin{pmatrix} A_j & 0 \\ 0 & E_j \end{pmatrix}\right] - \frac{1}{2i}\sigma D\begin{pmatrix} 0 & B_j \\ C_j & 0 \end{pmatrix}.$$

Substituting (2.14) into above equation, we have

$$\begin{pmatrix} 0 & B_{j+1} \\ C_{j+1} & 0 \end{pmatrix} = \frac{1}{2i}\sigma\left[P, D^{-1}\left[P, \begin{pmatrix} 0 & B_j \\ C_j & 0 \end{pmatrix}\right]\right] - \frac{1}{2i}\sigma D\begin{pmatrix} 0 & B_j \\ C_j & 0 \end{pmatrix}$$
$$- \sigma\left[P, \begin{pmatrix} \alpha_j & 0 \\ 0 & \delta_j \end{pmatrix}\right] - \frac{1}{2}\sigma[P, l_jx\sigma].$$

Since

$$-\frac{1}{2}\sigma[P,\ l_j x\sigma]=l_j xP,$$

and we note the definition of operator L in (2.7), we have

$$\begin{pmatrix}0 & B_{j+1}\\ C_{j+1} & 0\end{pmatrix}=L\begin{pmatrix}0 & B_j\\ C_j & 0\end{pmatrix}-\sigma\left[P,\ \begin{pmatrix}\alpha_j & 0\\ 0 & \delta_j\end{pmatrix}\right]+l_j xP,\quad j=0,\ 1,\ \cdots,\ n. \tag{2.15}$$

From (2.15) and by induction it follows

$$\begin{pmatrix}0 & B_{j+1}\\ C_{j+1} & 0\end{pmatrix}=L^{j+1}\begin{pmatrix}0 & B_0\\ C_0 & 0\end{pmatrix}-\sum_{k=0}^{j}L^{j-k}\sigma\left[P,\ \begin{pmatrix}\alpha_j & 0\\ 0 & \delta_j\end{pmatrix}\right]+\sum_{k=0}^{j}L^{j-k}(l_k xP).$$

In particuler, taking $j=n$, and using the equation (2.11) and $B_0=C_0=0$, we can get equation (2.5).

§ 3. Alternative form of the evolution equation

Let $\sigma_1=I,\ \sigma_2,\ \sigma_3,\ \cdots,\ \sigma_{N^2}$ be the orthogonal basis in the linear space, constructed by all $N\times N$ matrices (when $N=2$, this just is Pauli matrices).

Let

$$\alpha_k=\alpha_k^{(m)}\sigma_m,\quad \delta_k=\delta_k^{(m)}\sigma_m.$$

In above equation super and below index m means summation for all m

$$m=1,\ 2,\ \cdots,\ N^2,$$

where $\alpha_k^{(m)},\ \delta_k^{(m)}$ are functions of t.

We introduce the antipoisson bracket $\{A,\ B\}=AB+BA$. Write

$$\sigma\left[P,\ \begin{pmatrix}\alpha_k & 0\\ 0 & \delta_k\end{pmatrix}\right]=\nu_k^{(m)}\begin{pmatrix}0 & \{Q,\ \sigma_m\}\\ \{R,\ \sigma_m\} & 0\end{pmatrix}+\mu_k^{(m)}\begin{pmatrix}0 & [Q,\ \sigma_m]\\ [\sigma_m,\ R] & 0\end{pmatrix}, \tag{3.1}$$

where

$$\nu_k^{(m)}=\frac{\delta_k^{(m)}-\alpha_k^{(m)}}{2},\quad \mu_k^{(m)}=\frac{\delta_k^{(m)}+\alpha_k^{(m)}}{2}. \tag{3.2}$$

Since $\alpha_k,\ \delta_k$ are arbitrary, $\nu_k^{(m)}$ and $\mu_k^{(m)}(m=1,\ 2,\cdots,\ N^2,\ k=1,\ \cdots,\ n)$ are arbitrary also. Substituting (3.1) into (2.5) we obtain the alternative form of evolution equation as follows

$$\begin{pmatrix}0 & Q_t\\ -R_t & 0\end{pmatrix}=2iP_{1,n}^{(m)}(L)\begin{pmatrix}0 & \{Q,\ \sigma_m\}\\ \{R,\ \sigma_m\} & 0\end{pmatrix}+2iP_{2,n}^{(m)}(L)\begin{pmatrix}0 & [Q,\ \sigma_m]\\ [\sigma_m,\ R] & 0\end{pmatrix}$$

$$-2iP_{3,n}(L)\begin{pmatrix}0 & xQ\\ xR & 0\end{pmatrix}, \tag{3.3}$$

where

$$P_{1,n}^{(m)}(\lambda)=\sum_{k=0}^{n}\nu_k^{(m)}\lambda^{n-k},\quad P_{2,n}^{(m)}(\lambda)=\sum_{k=0}^{n}\mu_k^{(m)}\lambda^{n-k},\quad P_{3,n}(\lambda)=\sum_{k=0}^{n}l_k\lambda^{n-k}. \tag{3.4}$$

§ 4. Two special cases

(1) $R=I,\ Q=U.$

In this case

$$\begin{pmatrix} 0 & [Q, \sigma_m] \\ [\sigma_m, R] & 0 \end{pmatrix} = \begin{pmatrix} 0 & [U, \sigma_m] \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \{Q, \sigma_m\} \\ \{R, \sigma_m\} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \{U, \sigma_m\} \\ 2\sigma_m & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & xQ \\ xR & 0 \end{pmatrix} = \begin{pmatrix} 0 & xU \\ xI & 0 \end{pmatrix}.$$

We can prove

$$L\begin{pmatrix} 0 & \{U, \sigma_m\} \\ 2\sigma_m & 0 \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} 0 & -\underline{G}\sigma_m \\ 0 & 0 \end{pmatrix}, \tag{4.1}$$

$$L^2\begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & (-T)E \\ 0 & 0 \end{pmatrix}, \tag{4.2}$$

where E is arbitrary $N \times N$ matrix. $T = \frac{1}{4}\underline{L}_s$ and operators $\underline{G}$ and $\underline{L}_s$ are given as follows

$$\underline{L}_s F = F_{xx} - 2\{U, F\} - \underline{G}D^{-1}F, \tag{4.3}$$

$$\underline{G}F = \{U_x, F\} - [U, D^{-1}[U, F]]. \tag{4.4}$$

These two operators conform with ref. [6].

Furthermore, we have

$$2iL(xP) = -\sigma D(xP) = \begin{pmatrix} 0 & -(xQ_x + Q) \\ xR_x + R & 0 \end{pmatrix}, \tag{4.5}$$

hence

$$L\begin{pmatrix} 0 & xU \\ xI & 0 \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} 0 & U \\ I & 0 \end{pmatrix} + \frac{1}{2i}\begin{pmatrix} 0 & -xU_x - 2U \\ 0 & 0 \end{pmatrix}$$

$$= \frac{1}{4i}\begin{pmatrix} 0 & \{U, \sigma_1\} \\ 2\sigma_1 & 0 \end{pmatrix} + \frac{1}{2i}\begin{pmatrix} 0 & -xU_x - 2U \\ 0 & 0 \end{pmatrix}. \tag{4.6}$$

Note that, left hand side of (3.3) becames

$$\begin{pmatrix} 0 & U_t \\ 0 & 0 \end{pmatrix},$$

on account of the compatibility and properties of L in (4.1), (4.2), and (4.6), and we take $P_{3,n}$ to be odd polynomial, $P_{2,n}^{(m)}$ even polynomial, $P_{1,n}^{(m)}$ odd polynomial, except $P_{1,n}^{(1)}$, which was taken to be a special form (see (4.10)). Suppose

$$P_{3,n}(L) = \sum_{j=0}^{n} l_{2j+1} L^{2(n-j)+1}, \tag{4.7}$$

$$P_{2,n}^{(m)}(L) = \sum_{j=0}^{n} \mu_{2j}^{(m)} L^{2(n-j)}, \quad m = 1, 2, \cdots, N^2, \tag{4.8}$$

$$P_{1,n}^{(m)}(L) = \sum_{j=0}^{n} \nu_{2j+1}^{(m)} L^{2(n-j)+1}, \quad m = 2, 3, \cdots, N^2, \tag{4.9}$$

$$P_{1,n}^{(1)}(L) = \sum_{j=0}^{n} \nu_{2j+1}^{(1)} L^{2(n-j)+1} + \frac{1}{4i}\sum_{j=0}^{n} l_{2j+1} L^{2(n-j)}. \tag{4.10}$$

From (4.6), we got

$$L^3\begin{pmatrix}0 & xU\\ xI & 0\end{pmatrix}=\frac{1}{4i}L^2\begin{pmatrix}0 & \{U,\sigma_1\}\\ 2\sigma_1 & 0\end{pmatrix}+\frac{1}{2i}L^2\begin{pmatrix}0 & -xU_x-2U\\ 0 & 0\end{pmatrix}$$

$$=\frac{1}{4i}L^2\begin{pmatrix}0 & \{U,\sigma_1\}\\ 2\sigma_1 & 0\end{pmatrix}+\frac{1}{2i}\begin{pmatrix}0 & (-T)(-xU_x-2U)\\ 0 & 0\end{pmatrix}$$

$$\cdots\cdots\cdots,$$

$$L^{2j+1}\begin{pmatrix}0 & xU\\ xI & 0\end{pmatrix}=\frac{1}{4i}L^{2j}\begin{pmatrix}0 & \{U,\sigma_1\}\\ 2\sigma_1 & 0\end{pmatrix}+\frac{1}{2i}\begin{pmatrix}0 & (-T)^j(-xU_x-2U)\\ 0 & 0\end{pmatrix}.$$

Thus

$$P_{3,n}(L)\begin{pmatrix}0 & xU\\ xI & 0\end{pmatrix}=\sum_{j=0}^{n} l_{2j+1}L^{2(n-j)+1}\begin{pmatrix}0 & xU\\ xI & 0\end{pmatrix}$$

$$=\frac{1}{4i}\sum_{j=0}^{n} l_{2j+1}L^{2(n-j)}\begin{pmatrix}0 & \{U,\sigma_1\}\\ 2\sigma_1 & 0\end{pmatrix}+\frac{1}{2i}\sum_{j=0}^{n} l_{2j+1}\begin{pmatrix}0 & (-T)^{n-j}(-xU_x-2U)\\ 0 & 0\end{pmatrix}.$$

Since

$$P^{(1)}_{1,n}(L)\begin{pmatrix}0 & \{U,\sigma_1\}\\ 2\sigma_1 & 0\end{pmatrix}=\sum_{j=0}^{n}\nu^{(1)}_{2j+1}L^{2(n-j)+1}\begin{pmatrix}0 & \{U,\sigma_1\}\\ 2\sigma_1 & 0\end{pmatrix}$$

$$+\frac{1}{4i}\sum_{j=0}^{n} l_{2j+1}L^{2(n-j)}\begin{pmatrix}0 & \{U,\sigma_1\}\\ 2\sigma_1 & 0\end{pmatrix},$$

hence

$$P^{(1)}_{1,n}(L)\begin{pmatrix}0 & \{U,\sigma_1]\\ 2\sigma_1 & 0\end{pmatrix}-P_{3,n}(L)\begin{pmatrix}0 & xU\\ xI & 0\end{pmatrix}$$

$$=\frac{1}{2i}\sum_{j=0}^{n}\nu^{(1)}_{2j+1}\begin{pmatrix}0 & (-T)^{n-j}(-\underline{G}\sigma_1)\\ 0 & 0\end{pmatrix}+\frac{1}{2i}\sum_{j=0}^{n} l_{2j+1}\begin{pmatrix}0 & (-T)^{n-j}(2U+xU_x)\\ 0 & 0\end{pmatrix}, \tag{4.11}$$

for $m\geqslant 2$

$$P^{(m)}_{1,n}(L)\begin{pmatrix}0 & \{U,\sigma_m\}\\ 2\sigma_m & 0\end{pmatrix}=\frac{1}{2i}\sum_{j=0}^{n}\nu^{(m)}_{ij+1}\begin{pmatrix}0 & (-T)^{n-j}(-\underline{G}\sigma_m)\\ 0 & 0\end{pmatrix}, \tag{4.12}$$

for $m\geqslant 1$

$$P^{(m)}_{2,n}(L)\begin{pmatrix}0 & [U,\sigma_m]\\ 0 & 0\end{pmatrix}=\sum_{j=0}^{n}\mu^{(m)}_{2j}\begin{pmatrix}0 & (-T)^{n-j}[U,\sigma_m]\\ 0 & 0\end{pmatrix}. \tag{4.13}$$

Substituting (4.11), (4.12), (4.13) into (3.3), we have

$$\begin{pmatrix}0 & U_t\\ 0 & 0\end{pmatrix}=\sum_{j=0}^{n}\nu^{(m)}_{2j+1}\begin{pmatrix}0 & (-T)^{n-j}(-\underline{G}\sigma_m)\\ 0 & 0\end{pmatrix}+\sum_{j=0}^{n} l_{2j+1}\begin{pmatrix}0 & (-T)^{n-j}(2U+xU_x)\\ 0 & 0\end{pmatrix}$$

$$+2i\sum_{j=0}^{n}\mu^{(m)}_{2j}\begin{pmatrix}0 & (-T)^{n-j}[U,\sigma_m]\\ 0 & 0\end{pmatrix},$$

i. e.

$$U_t=\sum_{j=0}^{n}(-\nu^{(m)}_{2j+1})(-T)^{n-j}\underline{G}\sigma_m+2i\sum_{j=0}^{n}\mu^{(m)}_{2j}(-T)^{n-j}[U,\sigma_m]$$

$$+\sum_{j=0}^{n} l_{2j+1}(-T)^{n-j}(2U+xU_x). \tag{4.14}$$

(2) $R=Q=V$.

In this case

$$\begin{pmatrix} 0 & [Q, \sigma_m] \\ [\sigma_m, R] & 0 \end{pmatrix} = \begin{pmatrix} 0 & [V, \sigma_m] \\ -[V, \sigma_m] & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & \{Q, \sigma_m\} \\ \{R, \sigma_m\} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \{V, \sigma_m\} \\ \{V, \sigma_m\} & 0 \end{pmatrix},$$

$$P=\begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix}, \begin{pmatrix} 0 & xQ \\ xR & 0 \end{pmatrix} = \begin{pmatrix} 0 & xV \\ xV & 0 \end{pmatrix},$$

we can prove

$$L\begin{pmatrix} 0 & \{V, \sigma_m\} \\ \{V, \sigma_m\} & 0 \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} 0 & -\underline{G}_1\sigma_m \\ \underline{G}_1\sigma_m & 0 \end{pmatrix}, \tag{4.15}$$

$$L^2\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} = \begin{pmatrix} 0 & (-S)E \\ SE & 0 \end{pmatrix}, \tag{4.16}$$

where E is an arbitrary $N\times N$ matrix, $S=\frac{1}{4}\underline{L}_s^{(1)}$ and operators $\underline{G}_1$ and $\underline{L}_s^{(1)}$ are given as follows

$$\underline{L}_s^{(1)}F=F_{xx}-[V, D^{-1}[V, F_x]]-\{V, \{V, F\}\}-\underline{G}_1D^{-1}\{V, F\}, \tag{4.17}$$

$$\underline{G}_1F=\{V_x, F\}-[V, D^{-1}[V^2, F]], \tag{4.18}$$

where F is an arbitrary $N\times N$ matrix.

Furthermore, we have (from (4.5))

$$L\begin{pmatrix} 0 & xV \\ xV & 0 \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} 0 & -(xV)_x \\ (xV)_x & 0 \end{pmatrix}. \tag{4.19}$$

Note that left hand side of (3.3) becames

$$\begin{pmatrix} 0 & V_t \\ -V_t & 0 \end{pmatrix},$$

on account of the compatibility and properties of L in (4.15), (4.16), and (4.19), and it is sufficient to take $P_{1,u}^{(m)}$, $P_{3,n}$ to be odd polynomiai, $P_{2,n}^{(m)}$ to be even polynomial, $P_{3,n}$, $P_{2,n}^{(m)}$, $P_{1,n}^{(m)}$ is just as (4.7), (4.8), (4.9) but in (4.9) index m is from 1 to N^2.

Similar to case (1) we have the following evolution equations

$$\begin{pmatrix} 0 & V_t \\ -V_t & 0 \end{pmatrix} = \sum_{j=0}^{n} \nu_{2j+1}^{(m)}\begin{pmatrix} 0 & (-S)^{n-j}(-\underline{G}_1\sigma_m) \\ (-S)^{n-j}\underline{G}_1\sigma_m & 0 \end{pmatrix}$$

$$+2i\sum_{j=0}^{n}\mu_{2j}^{(m)}\begin{pmatrix} 0 & (-S)^{n-j}[V, \sigma_m] \\ -(-S)^{n-j}[V, \sigma_m] & 0 \end{pmatrix}$$

$$+\sum_{j=0}^{n} l_{2j+1}\begin{pmatrix} 0 & (-S)^{n-j}(xV)_x \\ -(-S)^{n-j}(xV)_x & 0 \end{pmatrix}.$$

From this we have two equations, but it is only one and the same equation, practically

$$V_t=\sum_{j=0}^{n}(-\nu_{2j+1}^{(m)})(-S)^{n-j}\underline{G}_1\sigma_m+2i\sum_{j=0}^{n}\mu_{2j}^{(m)}(-S)^{n-j}[V,\sigma_m]$$
$$+\sum_{j=0}^{n}l_{2j+1}(-S)^{n-j}(xV)_x. \tag{4.20}$$

§ 5. Miura transformation and Bäklund transformation

Matrix Miura transformation is defined by

$$U=V_x+V^2. \tag{5.1}$$

Derivate with respect to x, it yeilds

$$U_x=(D+\{V,\ \cdot\})V_x, \tag{5.2}$$

we can prove that

$$(D+\{V,\ \cdot\})[V,\sigma_m]=[U,\sigma_m], \tag{5.3}$$
$$(D+\{V,\ \cdot\})(xV)_x=2U+xU_x. \tag{5.4}$$

In the appendix we have proved

$$(D+\{V,\ \cdot\})\underline{G}_1\sigma_m=\underline{G}\sigma_m \tag{5.5}$$

and

$$(D+\{V,\ \cdot\})S(V)=T(U)(D+\{V,\ \cdot\}). \tag{5.6}$$

The equation (5.6) is important for finding Miura transformation. From (5.6) and by induction, we can prove that for any positive integer n,

$$(D+\{V,\ \cdot\})S^n(V)=T^n(U)(D+\{V,\ \cdot\}). \tag{5.7}$$

Derivating with respect to t on both sides of equation (5.1), it yeilds

$$U_t=(D+\{V,\ \cdot\})V_t. \tag{5.8}$$

The both sides of (5.7) are operated on $[V,\sigma_m]$, $\underline{G}_1\sigma_m$ and $(xV)_x$ respectively, we have

$$(D+\{V,\ \cdot\})S^n(V)[V,\sigma_m]=T^n(U)[U,\sigma_m], \tag{5.9}$$
$$(D+\{V,\ \cdot\})S^n(V)\underline{G}_1\sigma_m=T^n(U)\underline{G}\sigma_m, \tag{5.10}$$
$$(D+\{V,\ \cdot\})S^n(V)(xV)_x=T^n(U)(2U+xU_x), \tag{5.11}$$

operating $(D+\{V,\ \cdot\})$ on both sides of (4.20), and substituting (5.8), (5.9), (5.10), (5.11) into (4.20), we have

$$U_t-\sum_{j=0}^{n}(-\nu_{2j+1}^{(m)})(-T)^{n-j}\underline{G}\sigma_m-2i\sum_{j=0}^{n}\mu_{2j}^{(m)}(-T)^{n-j}[U,\sigma_m]$$
$$-\sum_{j=0}^{n}l_{2j+1}(-T)^{n-j}(2U+xU_x)$$
$$=(D+\{V,\ \cdot\})\Big\{V_t-\sum_{j=0}^{n}(-\nu_{2j+1}^{(m)})(-S)^{n-j}\underline{G}_1\sigma_m-2i\sum_{j=0}^{n}\mu_{2j}^{(m)}(-S)^{n-j}[V,\sigma_m]$$
$$-\sum_{j=0}^{n}l_{2j+1}(-S)^{n-j}(xV)_x\Big\}. \tag{5.12}$$

Thus, if V is a solution of (4.20), then U, given by (5.1), is a solution of (4.14). But then $-V$ is also a solution (4.20), so that (4.14) has a new solution given by

$$\tilde{U}=-V_x+V^2. \tag{5.13}$$

This passage from V to U is called a Miura transformation and that from U to $\tilde{U}$ a Bäklund transformation.

To pass from U to $\tilde{U}$ we set

$$U=W_x,\ \tilde{U}=\tilde{W}_x. \tag{5.14}$$

Then

$$U-\tilde{U}=(W-\tilde{W})_x=2V_x,$$

and we can suppose

$$2V=W-\tilde{W}. \tag{5.15}$$

From (5.14), (5.1),(5.13), we have

$$U+\tilde{U}=(W+\tilde{W})_x=2V^2. \tag{5.16}$$

From (5.15) and (5.16), it follws

$$(W+\tilde{W})_x=\frac{1}{2}(W-\tilde{W})^2. \tag{5.17}$$

Exchanging V for $\frac{1}{2}(W-\tilde{W})$ in (4.20), it follows

$$\left(\frac{W-\tilde{W}}{2}\right)_t=\sum_{j=0}^{n}(-\nu_{2j+1}^{(m)})(-S)^{n-j}\underline{G}_1\sigma_m+2i\sum_{j=0}^{n}\mu_{2j}^{(m)}(-S)^{n-j}\left[\frac{W-\tilde{W}}{2},\ \sigma_m\right]$$
$$+\sum_{j=0}^{n}l_{2j+1}(-S)^{n-j}\left(\frac{x(W-\tilde{W})}{2}\right)_x, \tag{5.18}$$

with W given, such that $U=W_x$ is a solution of (4.14), and solving the equation (5.17), (5.18), we get $\tilde{W}$, and it gives a new solution $\tilde{U}=\tilde{W}_x$ of (4.14).

§ 6. It is clear that when $N=1$

U, V, are considered, not two matrices but two functions u, v, then from (4.3) (4.4) and (4.17), (4.18) we get

$$T=\frac{D^2}{4}-u-\frac{1}{2}u_xD^{-1}, \tag{6.1}$$

$$S=\frac{1}{4}D^2-v^2-v_xD^{-1}v, \tag{6.2}$$

when $N=1$, among σ_1, σ_2, $\cdots$, σ_{N^2} there is only $\sigma_1=1$, summation of index m in (4.14), (4.20) only contains one term. Note that second terms in both equations ought to be vanish (since $[U,\ \sigma_1]=0$, $[V,\ \sigma_1]=0$), while

$$\begin{cases}\underline{G}(1)=2u_x,\\ \underline{G}_1(1)=2v_x,\end{cases} \tag{6.3}$$

therefore (4.14) becomes

$$u_t=2\sum_{j=0}^{n}(-\nu_{2j+1}^{(1)})(-T)^{n-j}u_x+\sum_{j=0}^{n}l_{2j+1}(-T)^{n-j}(2u+xu_x), \tag{6.4}$$

and (4.20) becomes

$$v_t=2\sum_{j=0}^{n}(-\nu_{2j+1}^{(1)})(-S)^{n-j}v_x+\sum_{j=0}^{n}l_{2j+1}(-S)^{n-j}(xv_x)_x. \tag{6.5}$$

These results conform with ref. [5].

T and S, given by (6.1) and (6.2) respectively, conform with ref. [7].

§ 7. Appendix Proof of (5.5) and (5.6)

(1) Some identical relations

$$\{V,\ [V,\ \cdot]\}=[V^2,\ \cdot], \tag{7.1}$$

$$[V,\ [V^2,\ \cdot]]=[V^2,\ [V,\ \cdot]], \tag{7.2}$$

$$\{\{A,\ B\},\ C\}-\{A,\ \{B,\ C\}\}=[B,\ [A,\ C]]. \tag{7.3}$$

(2) Proof of (5.5)

From definition

$$\begin{aligned}(D+\{V,\ \cdot\})\underline{G}_1\sigma_m&=D\underline{G}_1\sigma_m+\{V,\ \underline{G}_1\sigma_m\}\\ &=\{V_{xx},\ \sigma_m\}-[V_x,\ D^{-1}[V^2,\ \sigma_m]]-[V,\ [V^2,\ \sigma_m]]+\{V,\ \{V_x,\ \sigma_m\}\}\\ &\quad-\{V,\ [V,\ D^{-1}[V^2,\ \sigma_m]]\}.\end{aligned}$$

Using (7.1), (7.2), (7.3), we get

$$\begin{aligned}(D+\{V,\ \cdot\})\underline{G}_1\sigma_m&=\{V_{xx},\ \sigma_m\}-[V_x,\ D^{-1}[V^2,\ \sigma_m]]-[V^2,\ [V,\ \sigma_m]]\\ &\quad+\{\{V,\ V_x\},\ \sigma_m\}-[V_x,\ [V,\ \sigma_m]]-[V^2,\ D^{-1}[V^2,\ \sigma_m]]\\ &=\{V_{xx}+\{V,\ V_x\},\ \sigma_m\}-[V_x+V^2,\ D^{-1}[V^2,\ \sigma_m]]-[V_x+V^2,\ [V,\ \sigma_m]]\\ &=\{U_x,\ \sigma_m\}-[U,\ D^{-1}[V^2,\ \sigma_m]]-[U,\ D^{-1}[V_x,\ \sigma_m]]\\ &=\{U_x,\ \sigma_m\}-[U,\ D^{-1}[U,\ \sigma_m]]=\underline{G}\sigma_m.\end{aligned}$$

(3) Proof of (5.6)

By definition

$$\begin{aligned}&4(D+\{V,\ \cdot\})S(V)F\\ &=(D+\{V,\ \cdot\})(F_{xx}-[V,\ D^{-1}[V,\ F_x]]-\{V,\ \{V,\ F\}\}-\underline{G}_1D^{-1}\{V,\ F\})\\ &=F_{xxx}+\{V,\ F_{xx}\}-[V,\ [V,\ F_x]]-[V_x+V^2,\ D^{-1}[V,\ F_x]]-D\{V,\ \{V,\ F\}\}\\ &\quad-\{V,\ \{V,\ \{V,\ F\}\}\}-\{V_{xx},\ D^{-1}\{V,\ F\}\}-\{V_x,\ \{V,\ F\}\}\\ &\quad+[V_x+V^2,\ D^{-1}[V^2,\ D^{-1}\{V,\ F\}]]+[V,\ [V^2,\ D^{-1}\{V,\ F\}]]\\ &\quad-\{V,\ \{V_x,\ D^{-1}\{V,\ F\}\}\},\end{aligned} \tag{7.4}$$

we can prove the following equation

$$\begin{aligned}[V_x+V^2,\ D^{-1}[V,\ F_x]]&=[V_x,\ [V,\ F]]+[V^2,\ [V,\ F]]\\ &\quad-[V_x+V^2,\ D^{-1}[V_x,\ F]],\end{aligned} \tag{7.5}$$

$$\{V,\ \{V,\ \{V,\ F\}\}\}=2\{V^2,\ \{V,\ F\}\}-[V^2,\ [V,\ F]], \tag{7.6}$$

$$\begin{aligned}[V,\ [V,\ F_x]]+D\{V,\ \{V,\ F\}\}&=\{V_x,\ \{V,\ F\}\}+\{\{V,\ V_x\},\ F\}\\ &\quad-[V_x,\ [V,\ F]]+2\{V^2,\ F_x\},\end{aligned} \tag{7.7}$$

$$[V,\ [V^2,\ D^{-1}\{V,\ F\}]]-\{V,\ \{V_x,\ D^{-1}\{V,\ F\}\}\}$$
$$=-\{\{V,\ V_x\},\ D^{-1}\{V,\ F\}\}+[V_x+V^2,\ D^{-1}[V_x,\ D^{-1}\{V,\ F\}]]$$
$$+[V_x+V^2,\ D^{-1}[V^2,\ F]]. \tag{7.8}$$

Substituting (7.5), (7.6), (7.7), (7.8) into (7.4), adding and subtracting the term $2\{V_x,\ F_x\}+\{V_{xx},\ F\}$, we get

$$4(D+\{V,\ \cdot\})S(V)F=F_{xxx}+\{V,\ F_{xx}\}+(2\{V_x,\ F_x\}+\{V_{xx},\ F\}$$
$$-2\{V_x+V^2,\ F_x\}-\{V_{xx},\ F\}+2\{V^2,\ F_x\})-\{V_x,\ \{V,\ F\}\}-\{\{V,\ V_x\},\ F\}$$
$$+[V_x,\ [V,\ F]]-2\{V^2,\ F_x\}+(-[V_x,\ [V,\ F]]-[V^2,\ [V,\ F]]$$
$$+[V_x+V^2,\ D^{-1}[V_x,\ F]])+(-2\{V^2,\ \{V,\ F\}\}+[V^2,\ [V,\ F]])$$
$$-\{V_{xx},\ D^{-1}\{V,\ F\}\}-\{V_x,\ \{V,\ F\}\}+[V_x+V^2,\ D^{-1}[V^2,\ D^{-1}\{V,\ F\}]]$$
$$+(-\{\{V,\ V_x\},\ D^{-1}\{V,\ F\}\}+[V_x+V^2,\ D^{-1}[V_x,\ D^{-1}\{V,\ F\}]]$$
$$+[V_x+V^2,\ D^{-1}[V^2,\ F]])$$
$$=D^2(F_x+\{V,\ F\})-2\{V_x+V^2,\ F_x+\{V,\ F\}\}-\{V_{xx}+\{V,\ V_x\},\ F+D^{-1}\{V,\ F\}\}$$
$$+[V_x+V^2,\ D^{-1}[V_x+V^2,\ F+D^{-1}\{V,\ F\}]]$$
$$=D^2(F_x+\{V,\ F\})-2\{U,\ F_x+\{V,\ F\}\}-\{U_x,\ D^{-1}(F_x+\{V,\ F\})\}$$
$$+[U,\ D^{-1}[U,\ D^{-1}(F_x+\{V,\ F\})]]$$
$$=4T(U)(F_x+\{V,\ F\})=4T(U)(D+\{V,\ \cdot\})F.$$

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一类矩阵发展方程

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摘 要

在文献[5]中，考虑了如下特征值问题

$$\varphi_x = M\varphi,\ \varphi_x = \frac{\partial\varphi}{\partial x},$$

其中

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{1}$$

$$M = \begin{pmatrix} -i\xi & q(x,\ t) \\ r(x,\ t) & i\xi \end{pmatrix}, \tag{2}$$

这里假定特征值 ξ 以某种规律随着时间变化而变化. 文章中得出了一类发展方程，其中两个特殊情形：$r=1,\ q=u(x,\ t)$ 和 $r=q=v(x,\ t)$ 分别可以当作推广的 KDV 方程和推广的 MKDV 方程. 并证明了不仅在 KDV 方程和 MKDV 方程之间存在 Miura 变换，而且在推广的 KDV 方程和推广的 MKDV 方程之间也存在 Miura 变换. 又证明了对推广的 KDV 方程存在 Bäcklund 变换.

本文将[5]的结果推广至矩阵情形：

设

$$M = \begin{pmatrix} -i\xi I & Q(x,\ t) \\ R(x,\ t) & i\xi I \end{pmatrix}, \tag{3}$$

这里 $Q,\ R$ 为 $N\times N$ 矩阵，I 是 $N\times N$ 单位阵，相应的在(1)式中的向量 φ 是 $2N$ 维向量. 我们引进矩阵型的 Miura 变换，并得到了与[5]相平行的结果.