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A REMARK ON HARMONIC MAPS

XIN YUANLONG
(*Institute of Mathematics, Fudan University*)

It is of interest to study harmonic maps from spheres in many different contexts. In [6] the author proved nonexistence of nontrivial stable harmonic maps from high dimentional sphere to any Riemannian manifold. It is natural to ask what the situation is in the case of two dimensional spheres. If the target manifold is a compact Riemann surface, then any harmonic map must be either holomorphic or antiholomorphic as a consequence of Elles-Wood's theorem[2]. If the target manifold is a Kähler manifold with certain nonnegative holomorphic bisectional curvature, then it follows from Siu-Yau's theorem[4, 5] that any stable harmonic map is either holomorphic or antiholomorphic.

In this paper we consider the stable harmonic maps from 2-dimensional sphere to any Kähler manifold (This is interesting to some physicists who consider it to be a chiral model of the field theory with values in arbitrary Kähler manifold[3]). We obtain an integral inequality from which the well know Siu-Yau's theorem[4] can be established.

The method of obtaining this inequality is rather similar to the one in the previous paper[6] with certain variations.

Let M and N be Riemannian manifolds. Compactness and boundlessness of M are assumed. Any smooth map ϕ: $M\to N$ induces a map ϕ_*: $TM\to TN$, where TM and TN are the tangent bundles of M and N, respectively. The induced vector bundle $\phi^{-1}TN$ over M inherits a fibre metric and a Riemannian connection $\tilde{\nabla}$ from the metric of N and the canonical connection ∇ in N. The energy integral and the tension field of a given map ϕ from M to N are respectively defined by

$$E(\phi)=\int_M e(\phi)*1=\frac{1}{2}\int_M\langle\phi_*e_i,\ \phi_*e_i\rangle*1 \tag{1}$$

and

$$\tau(\phi)=\tilde{\nabla}_{e_i}\phi_*e_i-\phi_*(\nabla_{e_i}e_i), \tag{2}$$

where $e(\phi)$ denotes the energy density of ϕ, $\{e_i\}$ an local orthonormal frame in M and $*1$ the volume form of M. Here and henceforth we use the summation convention.

A smooth map with vanishing tension field is called a harmonic map. The first variation formula shows that harmonic maps are critical points of the energy integral

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(1).

The index form for a harmonic map is defined by

$$I(V, W)=\int_M \langle -\tilde{\nabla}^*\tilde{\nabla}V - R^N(\phi_* e_i, V)\phi_* e_i, W\rangle * 1 \tag{3}$$

for any cross sections V and W in $\phi^{-1}TN$ over M, $\tilde{\nabla}^*\tilde{\nabla}$ denoting trace Laplace operator in the Riemannian vector bundle $\phi^{-1}TN$ over M and $R^N(\cdot, \cdot)\cdot$ the curvature tensor of N. A harmonic map with $I(V, V)\geqslant 0$ for any $V\in\Gamma(\phi^{-1}TN)$ is said to be stable.

The basic formulas can be found in the paper [1] written by Eells, J. and Lemaire, L.

Let M and N be Kähler manifolds and $\{e_k, Je_k\}$ be Hermitian frame of M with canonical complex structure J. We denote TM^c (resp. TN^c) for the complexification of TM (resp. TN). There is a direct sum decomposition $TM^c=TM'\oplus TM''$, $\varepsilon_k=\frac{\sqrt{2}}{2}\times(e_k-iJe_k)$ and $\bar{\varepsilon}_k=\frac{\sqrt{2}}{2}(e_k+iJe_k)$ are bases of TM' and TM'', respectively. We still use the notation ϕ_*: $TM^c\to TN^c$ for the complexification of ϕ_*.

We define

$$\begin{aligned}\phi_*\varepsilon_k&=\phi'_*\varepsilon_k+\phi''_*\varepsilon_k,\\ \phi_*\bar{\varepsilon}_k&=\phi''_*\bar{\varepsilon}_k+\phi'_*\bar{\varepsilon}_k\end{aligned} \tag{4}$$

and

$$(d'\phi)x=\phi'_*x,\quad (d''\phi)x=\phi''_*x \quad \text{for } x\in TM^c,$$

where

$$\phi''_*\bar{\varepsilon}_k=\overline{\phi'_*\varepsilon_k}=\frac{\sqrt{2}}{4}(\phi_*e_k+i\phi_*Je_k+iJ\phi_*e_k-J\phi_*Je_k)$$

and

$$\phi'_*\bar{\varepsilon}_k=\overline{\phi''_*\varepsilon_k}=\frac{\sqrt{2}}{4}(\phi_*e_k+i\phi_*Je_k-iJ\phi_*e_k+J\phi_*Je_k).$$

The complex vector bundle $\phi^{-1}TN^c$ over M inherits the metric $\langle , \rangle$ and the connection $\tilde{\nabla}$ by complex linearity.

The partial energy densities are defined as follows

$$e'(\phi)=\langle\phi'_*\varepsilon_k, \overline{\phi'_*\varepsilon_k}\rangle=\frac{1}{4}(\langle\phi_*e_k, \phi_*e_k\rangle+\langle\phi_*Je_k, \phi_*Je_k\rangle+2\langle\phi_*Je_k, J\phi_*e_k\rangle) \tag{5}$$

and

$$e''(\phi)=\langle\phi''_*\varepsilon_k, \overline{\phi''_*\varepsilon_k}\rangle=\frac{1}{4}(\langle\phi_*e_k, \phi_*e_k\rangle+\langle\phi_*Je_k, \phi_*Je_k\rangle-2\langle\phi_*Je_k, J\phi_*e_k\rangle). \tag{6}$$

Therefore

$$e(\phi)=e'(\phi)+e''(\phi). \tag{7}$$

It is easy to see that $e''(\phi)=0$ if and only if ϕ is holomorphic and $e'(\phi)=0$ if and only if ϕ is antiholomorphic.

For convenience, let us express the condition of the harmonicity in complex form. By direct computation it is easy to check that

$$4(\tilde{\nabla}_{\bar{\varepsilon}_k}\phi'_*)\varepsilon_k = 4(\tilde{\nabla}_{\varepsilon_k}\phi'_*)\bar{\varepsilon}_k = (1-iJ)\tau(\phi)$$

and

$$4(\tilde{\nabla}_{\bar{\varepsilon}_k}\phi''_*)\varepsilon_k = 4(\tilde{\nabla}_{\varepsilon_k}\phi'')\bar{\varepsilon}_k = (1+iJ)\tau(\phi). \tag{8}$$

On the other hand, since

$$\begin{aligned} -\tilde{\nabla}^*\tilde{\nabla} &= -\tilde{\nabla}_{e_k}\tilde{\nabla}_{e_k} - \tilde{\nabla}_{Je_k}\tilde{\nabla}_{Je_k} + \tilde{\nabla}_{\nabla_{e_k}e_k+\nabla_{Je_k}Je_k} \\ &= -\frac{1}{2}(\tilde{\nabla}_{\varepsilon_k+\bar{\varepsilon}_k}\tilde{\nabla}_{\varepsilon_k+\bar{\varepsilon}_k} - \tilde{\nabla}_{\varepsilon_k-\bar{\varepsilon}_k}\tilde{\nabla}_{\varepsilon_k-\bar{\varepsilon}_k}) + \frac{1}{2}\tilde{\nabla}_{\nabla_{\varepsilon_k+\bar{\varepsilon}_k}(\varepsilon_k+\bar{\varepsilon}_k)-\nabla_{\varepsilon_k-\bar{\varepsilon}_k}(\varepsilon_k-\bar{\varepsilon}_k)} \\ &= -(\tilde{\nabla}_{\bar{\varepsilon}_k}\tilde{\nabla}_{\varepsilon_k} - \tilde{\nabla}_{\nabla_{\bar{\varepsilon}_k}\varepsilon_k}) - (\tilde{\nabla}_{\varepsilon_k}\tilde{\nabla}_{\bar{\varepsilon}_k} - \tilde{\nabla}_{\nabla_{\varepsilon_k}\bar{\varepsilon}_k}) = -2(\tilde{\nabla}_{\bar{\varepsilon}_k}\tilde{\nabla}_{\varepsilon_k} - \tilde{\nabla}_{\nabla_{\bar{\varepsilon}_k}\varepsilon_k}) - R^N(\phi_*\bar{\varepsilon}_k, \phi_*\varepsilon_k) \end{aligned}$$

and

$$\begin{aligned} &-\langle R^N(\phi_*e_k, V)\phi_*e_k, \bar{V}\rangle_N - \langle R^N(\phi_*Je_k, V)\phi_*Je_k, \bar{V}\rangle_N \\ &= -\frac{1}{2}\langle R^N(\phi_*\varepsilon_k+\phi_*\bar{\varepsilon}_k, V)(\phi_*\varepsilon_k+\phi_*\bar{\varepsilon}_k), \bar{V}\rangle_N \\ &\quad + \frac{1}{2}\langle R^N(\phi_*\varepsilon_k-\phi_*\bar{\varepsilon}_k, V)(\phi_*\varepsilon_k-\phi_*\bar{\varepsilon}_k), \bar{V}\rangle_N \\ &= -\langle R^N(\phi_*\varepsilon_k, V)\phi_*\bar{\varepsilon}_k, \bar{V}\rangle_N - \langle R^N(\phi_*\bar{\varepsilon}_k, V)\phi_*\varepsilon_k, \bar{V}\rangle_N. \end{aligned}$$

Thus we obtain the complex form of the 2nd variation formula

$$I(V, \bar{V}) = -2\int_M (\langle(\tilde{\nabla}_{\bar{\varepsilon}_k}\tilde{\nabla}_{\varepsilon_k} - \tilde{\nabla}_{\nabla_{\bar{\varepsilon}_k}\varepsilon_k})V, \bar{V}\rangle_N + \langle R^N(\phi_*\bar{\varepsilon}_k, V)\phi_*\varepsilon_k, \bar{V}\rangle_N) * 1. \tag{9}$$

Now we prove the following theorem.

Theorem 1. *Let $\phi: S^2 \to N$ be a stable harmonic map from 2-sphere to any Kähler manifold. The following inequality is valid*

$$\int_{S^2} B(\phi_*Je - J\phi_*e, \phi_*Je + J\phi_*e) * 1 \leq 0, \tag{10}$$

where $B(X, Y)$ stands for $\langle R^N(X, JX)Y, JY\rangle_N$.

Proof Given a point $p \in S^2$ at which $\nabla_\varepsilon\varepsilon = \nabla_\varepsilon\bar{\varepsilon} = \nabla_{\bar{\varepsilon}}\varepsilon = \nabla_{\bar{\varepsilon}}\bar{\varepsilon} = 0$. Suppose $\nabla_e\varepsilon = a\varepsilon$, $\nabla_e\bar{\varepsilon} = -a\bar{\varepsilon}$, $\nabla_e\bar{\varepsilon} = b\bar{\varepsilon}$ and $\nabla_{\bar{\varepsilon}}\varepsilon = -b\varepsilon$. Then at the point p

$$\begin{aligned} \nabla_{\bar{\varepsilon}}a &= \nabla_{\bar{\varepsilon}}\langle\nabla_e\varepsilon, \bar{\varepsilon}\rangle = \langle\nabla_{\bar{\varepsilon}}\nabla_e\varepsilon, \bar{\varepsilon}\rangle = \frac{1}{2}\langle\nabla_{e+iJe}\nabla_{e-iJe}\varepsilon, \bar{\varepsilon}\rangle \\ &= \frac{i}{2}\langle(\nabla_{Je}\nabla_e - \nabla_e\nabla_{Je})(e-iJe), (e+iJe)\rangle \\ &= \frac{1}{4}\langle(\nabla_{Je}\nabla_e - \nabla_e\nabla_{Je})Je, e\rangle - \frac{1}{4}\langle(\nabla_{Je}\nabla_e - \nabla_e\nabla_{Je})e, Je\rangle = -\frac{1}{2}K, \end{aligned}$$

where K is the gaussian curvature. In our case $K=1$, $\nabla_{\bar{\varepsilon}}a = -\frac{1}{2}$. Similarly, $\nabla_\varepsilon b = -\frac{1}{2}$.

Let u be a vector field on S^2 which is the gradient vector field of the restriction of a linear function in R^3. In other words, $u \in \Theta = \{\text{grad } f|_{S^2}: f \text{ is linear in } R^3\}$. Choosing $v = \frac{\sqrt{2}}{2}(u - iJu)$, then we have

$$v = \langle v, \bar{\varepsilon}\rangle\varepsilon, \quad \nabla_\varepsilon v = -\sqrt{2}f\varepsilon, \quad \nabla_{\bar{\varepsilon}}\nabla_\varepsilon v = -v \text{ and } \nabla_{\bar{\varepsilon}}v = 0,$$

noticing the facts: $\nabla_e u = -fe$ and $\nabla_e\nabla_e u + \nabla_{Je}\nabla_{Je}u = -u$.

Now using the condition for harmonicity, we have at the point p

$$-\tilde{\nabla}_{\bar{\varepsilon}}\tilde{\nabla}_{\varepsilon}\phi''_*v=-\tilde{\nabla}_{\bar{\varepsilon}}\tilde{\nabla}_{\varepsilon}\phi''_*\langle v, \bar{\varepsilon}\rangle\varepsilon=-\tilde{\nabla}_{\bar{\varepsilon}}(\langle\nabla_{\varepsilon}v, \bar{\varepsilon}\rangle\phi''_*\varepsilon+\langle v, \nabla_{\varepsilon}\bar{\varepsilon}\rangle\phi''_*\varepsilon+\langle v, \bar{\varepsilon}\rangle\tilde{\nabla}_{\varepsilon}\phi''_*\varepsilon)$$

$$=\langle-\nabla_{\bar{\varepsilon}}\nabla_{\varepsilon}v, \bar{\varepsilon}\rangle\phi''_*\varepsilon-\langle v, \nabla_{\bar{\varepsilon}}\nabla_{\varepsilon}\bar{\varepsilon}\rangle\phi''_*\varepsilon-\langle v, \bar{\varepsilon}\rangle\tilde{\nabla}_{\bar{\varepsilon}}\tilde{\nabla}_{\varepsilon}\phi''_*\varepsilon.$$

At the point p

$$-\langle v, \nabla_{\bar{\varepsilon}}\nabla_{\varepsilon}\bar{\varepsilon}\rangle=\langle v, \nabla_{\bar{\varepsilon}}a\bar{\varepsilon}\rangle=-\frac{1}{2}\langle v, \bar{\varepsilon}\rangle,$$

$$\tilde{\nabla}_{\bar{\varepsilon}}\tilde{\nabla}_{\varepsilon}\phi''_*\varepsilon=\tilde{\nabla}_{\varepsilon}\tilde{\nabla}_{\bar{\varepsilon}}\phi''_*\varepsilon+R^N(\phi_*\varepsilon, \phi_*\bar{\varepsilon})\phi''_*\varepsilon=\frac{1}{2}\phi''_*\varepsilon+R^N(\phi_*\varepsilon, \phi_*\bar{\varepsilon})\phi''_*\varepsilon.$$

Therefore

$$-\tilde{\nabla}_{\bar{\varepsilon}}\tilde{\nabla}_{\varepsilon}\phi''_*v=-R^N(\phi_*\varepsilon, \phi_*\bar{\varepsilon})\phi''_*v.$$

Substituting the above expression into the 2^{nd} variation formula (9), we obtain the following relation

$$I(\phi''_*v, \overline{\phi''_*v})=2\int_{S^2}\langle R^N(\phi''_*v, \phi_*\varepsilon)\phi_*\bar{\varepsilon}, \overline{\phi''_*v}\rangle_N*1.$$

Choosing v such that $v(p)=\varepsilon(p)$, the stability of the harmonic map $\phi: S^2\to N$ means

$$\int_{S^2}\langle R^N(\phi''_*\varepsilon, \phi_*\varepsilon)\phi_*\bar{\varepsilon}, \phi'_*\bar{\varepsilon}\rangle_N*1=\int_{S^2}\langle R^N(\phi''_*\varepsilon, \phi'_*\varepsilon)\phi''_*\bar{\varepsilon}, \phi'_*\bar{\varepsilon}\rangle_N*1\geqslant0.$$

From the Bianchi identity and $\langle R^N(\phi''_*\varepsilon, \phi''_*\bar{\varepsilon})\phi'_*\bar{\varepsilon}, \phi'_*\varepsilon\rangle=0$ (it can be verified by direct computation) we obtain

$$\int_{S^2}\langle R^N(\phi''_*\varepsilon, \overline{\phi''_*\varepsilon})\phi'_*\varepsilon, \overline{\phi'_*\varepsilon}\rangle_N*1=-\int_{S^2}\langle R^N(\overline{\phi''_*\varepsilon}, \phi''_*\varepsilon)\phi'_*\varepsilon, \overline{\phi'_*\varepsilon}\rangle_N*1\leqslant0. \quad (11)$$

The algebra gives

$$\langle R^N(\overline{\phi''_*\varepsilon}, \phi''_*\varepsilon)\phi'_*\varepsilon, \overline{\phi'_*\varepsilon}\rangle_N$$
$$=-\langle R^N(\phi_*e+J\phi_*Je, J\phi_*e-\phi_*Je)(\phi_*e-J\phi_*Je), (J\phi_*e+\phi_*Je)\rangle_N$$
$$=-B(\phi_*Je-J\phi_*e, \phi_*Je+J\phi_*e). \quad (12)$$

Substituting (12) into (11), we have inequality (10). Q. E. D.

Corollary 2. *Let N be a Kähler manifold with nonnegative holomorphic bisectional curvature and $\phi: S^2\to N$ a stable harmonic map. Then*

$$B(\phi_*Je-J\phi_*e, \phi_*Je+J\phi_*e)=0. \quad (13)$$

Furthermore, if N has positive holomorphic bisectional curvature, then ϕ is either holomorphic or antiholomorphic.

Proof For any $p\in S^2$ at which $\phi_*Je-J\phi_*e\neq0$ and $\phi_*Je+J\phi_*e\neq0$

$$B(\phi_*Je-J\phi_*e, \phi_*Je+J\phi_*e)=16e'(\phi)e''(\phi)B(X, Y)\geqslant0 \quad (14)$$

where $X=\frac{\phi_*Je-J\phi_*e}{\sqrt{4e''(\phi)}}$ and $Y=\frac{\phi_*Je+J\phi_*e}{\sqrt{4e'(\phi)}}$ are unit vectors. From (10) and (14)

$$\int_{S^2}B(\phi_*Je-J\phi_*e, \phi_*Je+J\phi_*e)*1=0.$$

Then we have (13).

Suppose that ϕ is neither holomorphic nor antiholomorphic. Let Z_1 and Z_2 are zero sets of $e'(\phi)$ and $e''(\phi)$, respectively. Then Z_1 and Z_2 are finite. Denote $Z=Z_1+Z_2$. At $p\in S^2\backslash Z$

$$B(\phi_* Je-J\phi_* e,\ \phi_* Je+J\phi_* e)=16e'(\phi)e''(\phi)B(X,\ Y)>0$$

by the positivity of holomorphic bisectional curvature, which is a contradiction in view of (10). Q. E. D.

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调和映照的一个注记

忻 元 龙

(复旦大学数学研究所)

摘 要

本文以二维球面到 Kähler 流形的调和映照为研究对象. 这个课题是近年来引起很多数学家和物理学家注目的课题.

利用稳定性条件,我们获得了一个积分不等式. 作为它的一个应用,我们可以证明肖荫堂和邱成桐的一个关于调和映照全纯性的著名定理.