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AIMOST PERIODIC LINEAR SYSTEM AND EXPONENTIAL DICHOTOMIES

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§ 1. Introduction

Let us consider the linear differential system

$$\frac{dx}{dt}=A(t)x, \tag{1.1}$$

where $A(t)$ is $n\times n$ matrix which is continuous and bounded on the real axis. If the fundamental matrix $X(t)$ of (1.1) is decomposed as follows

$$X(t)=X_1(t)+X_2(t),\ X^{-1}(s)=Z_1(s)+Z_2(s),$$
$$X(t)X^{-1}(s)=X_1(t)Z_1(s)+X_2(t)Z_2(s),$$

and there are two positive constants α and β such that

$$\|X_1(t)Z_1(s)\|\leqslant\beta\exp(-\alpha(t-s)),\ t\geqslant s,$$
$$\|X_2(t)Z_2(s)\|\leqslant\beta\exp(\alpha(t-s)),\ s\geqslant t,$$

then we say that (1.1) admits an exponential dichotomy.

In this paper we shall establish the condition of admitting exponential dichotomies for almost periodic linear system.

§ 2. The structure of quasi periodic linear system

In [2] or [3], under suitable assumptions the author established the reducibility theorem for the quasi periodic linear system

$$\frac{dx}{dt}=A(\omega t)x,\ A(\omega t)=A(\omega_1 t,\ \omega_2 t,\ \cdots,\ \omega_m t). \tag{2.1}$$

In general, we can transform (2.1) into a triangular system by performing a quasi periodic unitary transformation.

Theorem 2. 1. *There is a quasi periodic unitary transformation*

$$y=Q(\omega t)x,\quad Q(\omega t)=Q(\omega_1 t,\ \omega_2 t,\ \cdots,\ \omega_m t)$$

which reduces (2.1) *into a triangular linear system*

$$\frac{dy}{dt}=C(\omega t)y,$$

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Where $C(\omega t)$ is a quasi periodic and triangular matrix.

Proof By the approximate theorem of almost periodic functions, there are matrices $A_r(\omega_1^{(r)}t)=A_r(\omega_1^{(r)}t, \omega_2^{(r)}t, \cdots, \omega_m^{(r)}t)$ tending to $A(\omega t)$ uniformly, where the elements of $A_r(\omega^{(r)}t)$ are trigonometric polynomails with frequencies $\omega_1^{(r)}, \omega_2^{(r)}, \cdots, \omega_m^{(r)}$. At the same time we can assume that the frequencies $\omega_1^{(r)}, \omega_2^{(r)}, \cdots, \omega_m^{(r)}$ submit to the irrationality conditions

$$\left|\sum_{j=1}^{m} k_j\omega_j^{(r)}\right| \geqslant K(\omega^{(r)})\left(\sum_{j=1}^{m}|k_j|\right)^{-(m+1)}, \tag{2.2}$$

where $K(\omega^{(r)})>0$, and $k_1, k_3, \cdots, k_m$ are integers but not all zero, since the points submitted to the conditions (2.2) are dense everywhere in the m-dimensional Euclidian space.

We have proved in [3] that the Theorem 2.1 holds for $A(\omega t)\in C^{(m+1)}$, so that the linear system

$$\frac{dx}{dt}=A_r(\omega^{(r)}t)x \tag{2.3}$$

can be reduced into a triangular and quasi periodic linear system

$$\frac{dy}{dt}=C_r(\omega^{(r)}t)$$

by performing a quasi periodic uuiary transformation

$$y=Q_r(\omega^{(r)}t),\ Q_r(\omega^{(r)}t)=Q_r(\omega_1^{(r)}t, \omega_2^{(r)}t, \cdots, \omega_m^{(r)}t),$$

where

(i) $C_r(\omega^{(r)}t)=Q_r(\omega^{(r)}t)A_r(\omega^{(r)}t)\bar{Q}_r^*(\omega^{(r)}t)+\frac{d}{dt}Q_r(\omega^{(r)}t)\bar{Q}_r^*(\omega^{(r)}t)$, $\bar{Q}_r^*(\omega^{(r)}t)$ is the transposed and conjugate matrix of $Q_r(\omega^{(r)}t)$.

(ii) The imaginary part of each diagonal element of $C_r(\omega^{(r)}t)$ is constant.

(iii) $\|C_r(\omega^{(r)}t)\|\leqslant 3M, \left\|\frac{d}{dt}Q_r(\omega^{(r)}t)\right\|\leqslant 4M$, when $\|A_r(\omega^{(r)}t)\|\leqslant M$. Therefore $Q_r(\omega^{(r)}t)$ and $C_r(\omega^{(r)}t+\bar{C}_r^*(\omega^{(r)}t)=Q_r(\omega^{(r)}t)(A_r(\omega^{(r)}t)+A_r^*(\omega^{(r)}t))\bar{Q}_r^*(\omega^{(r)}t)$ are equicontinuous and uniformly bounded. Since the triangular matrix $C_r(\omega^{(r)}t)$ has the property (ii) as stated above, the equicontinuity and uniformly boundedness of $C_r(\omega^{(r)}t)+\bar{C}_r^*(\omega^{(r)}t)$ implies that $C_r(\omega^{(r)}t)$ has the same property. Now we assume that $Q_r(\omega^{(r)}t)\to Q(\omega t)$ and $C_r(\omega^{(r)}t)\to C(\omega t)$ uniformly on any finite interval.

Theorem (*Kronecker*). *If $\omega_1, \omega_2, \cdots, \omega_m$ are independent with respect to integers, then for any $\delta>0$, and any point $u^0=(u_1^0, u_2^0, \cdots, u_m^0)$ there are integer vector $k=(k_1, k_2, \cdots, k_m)$ and a real number t_0 such that*

$$\|\omega t_0-u^0-2\pi k\|<\delta.$$

Proof (cf., Hardy, G. H. and Wright, E. M., An Introduction to the Theory of Numbers (Oxford 1933).)

By the quasi periodicity of $Q_r(\omega^{(r)}t)$ and the relation

$$\frac{d}{dt} Q_r(\omega^{(r)}t) = C_r(\omega^{(r)}t) Q_r(\omega^{(r)}t) - Q_r(\omega^{(r)}t) A_r(\omega^{(r)}t),$$

we have $\sum_{j=1}^{m} \frac{\partial}{\partial u_j} Q_r(\omega^{(r)}t)\omega_j^{(r)} = C_r(\omega^{(r)}t) Q_r(\omega^{(r)}t) - Q_r(\omega^{(r)}t) A_r(\omega^{(r)}t).$

Since $\frac{\partial}{\partial u_j} Q_r(u)$ is continuous and periodic with respect to the components $u_1, u_2, \cdots, u_m$ of u with period 2π, it implies that $\frac{\partial}{\partial u_j} Q_r(u)$ is uniformly continuous. i. e., for any $\varepsilon > 0$, there is a positive number $\delta(\varepsilon) > 0$ such that

$$\left\| \frac{\partial}{\partial u_j} Q_r(u') - \frac{\partial}{\partial u_j} Q_r(u'') \right\| < \varepsilon, \text{ whenever } \|u' - u''\| < \delta(\varepsilon).$$

By the Kronecker's Theorem we see that for any point u^0 and any positive number $\delta(\varepsilon) > 0$, there are t_0 and k such that

$$\|\omega^{(r)}t_0 - u^0 - 2\pi k\| < \delta(\varepsilon).$$

Therefore we have

$$\left\| \sum_{j=1}^{m} \frac{\partial}{\partial u_j} Q_r(u^0)\omega_j^{(r)} \right\| \leqslant \left\| \sum_{j=1}^{m} \left(\frac{\partial}{\partial u_j} Q_r(u^0 + 2\pi k) - \frac{\partial}{\partial u_j} Q_r(\omega^{(r)}t_0) \omega_j^{(r)} \right\| \right.$$
$$+ \|C_r(\omega^{(r)}t_0) Q_r(\omega^{(r)}t_0) - Q_r(\omega^{(r)}t_0) A_r(\omega^{(r)}t_0)\| \leqslant \varepsilon \|\omega^{(r)}\| + 3M.$$

Then we have $\left\| \sum_{j=1}^{m} Q_r(u)\omega_j^{(r)} \right\| \leqslant 3M$, for the number ε and the point u^0 are arbitrary.

We denote $\omega^{(r)}$ by $\omega^{(r)}(1)$, and take the vectors $\omega^{(r)}(2), \omega^{(r)}(3), \cdots, \omega^{(r)}(m)$ such that

(a) $\omega^{(r)}(1), \omega^{(r)}(2), \cdots, \omega^{(r)}(m)$ are independent of each other with respect to real number, and $\|\omega^{(r)}(j)\| = 1$, $j = 2, \cdots, m$;

(b) The components of $\omega^{(r)}(j) = (\omega_{j1}^{(r)}, \omega_{j2}^{(r)}, \cdots, \omega_{jm}^{(r)})$ submit to the irrationality conditions. $j = 2, 3, \cdots, m$;

(c) $|\det(\omega^{(r)}(1), \omega^{(r)}(2), \cdots, \omega^{(r)}(m))| \geqslant \alpha^* > 0$, where α^* is independent of r, $r = 1, 2, 3, \cdots$.

By the same argument above-mentioned, we have that $\sum_{j=1}^{m} \frac{\partial}{\partial u_j} Q_r(u)\omega_{sj}^{(r)}$, $s = 1, 2, \cdots, m$, are bounded uniformly, which implies that $\frac{\partial}{\partial u_j} Q_r(u)$ are bounded uniformly. Therefore $Q_r(u)$ are equicontinuous and uniformly bounded, and $Q_r(u)$ tends to $Q(u)$ uniformly as a consequence of the periodicity of $Q_r(u)$ with period 2π. It follows that $Q(u)$ is periodic, or $Q(\omega t)$ is quasi periodic. Similarly, we have the quasi periodicity for $C(\omega t)$. By the relation

$$\frac{d}{dt} Q_r(\omega^{(r)}t) = C_r(\omega^{(r)}t) Q_r(\omega^{(r)}t) - Q_r(\omega^{(r)}t) A_r(\omega^{(r)}t),$$

we have $Q(\omega t) \in C^{(1)}$. Let us transform (2.1) by $y = Q(\omega t)x$. Then we have

$$\frac{dy}{dt} = C(\omega t) y, \tag{2.4}$$

and the theorm is proved completely.

Theorem 2. 2. *Suppose that the characteristic exponents of* (2.1) *are different from zero. Then* (2.1) *admits an exponential dichotomy.*

Proof We shall only prove that the theorem holds for (2.4). Suppose that

$$C(\omega t)=\begin{pmatrix} c_1(t) & & \\ & \ddots & \\ 0 & & c_n(t)\end{pmatrix}.$$

The limit
$$\lim_{t\to\pm\infty}\frac{1}{t-s}\int_s^t c_j(s)ds=c_j,\ j=1,\ 2,\ \cdots,\ n,$$
exists uniformly. We may assume that

$$\mathrm{Re}(c_1)\leqslant\cdots\leqslant\mathrm{Re}(c_k)\leqslant-\alpha<0<\alpha<\mathrm{Re}(c_{k+1})\leqslant\cdots\leqslant\mathrm{Re}(c_n).$$

Therefore we have a constant $K>0$ such that

$$\left|\exp\left(\int_s^t c_j(s)ds\right|\leqslant K\exp\left(-\alpha(t-s)\right),\quad t\geqslant s,\ j=1,\ 2,\ \cdots,\ k;$$

$$\left|\exp\left(\int_s^t c_j(s)ds\right|\leqslant K\exp(\alpha(t-s)),\quad s\geqslant t,\ j=k+1,\ \cdots,\ n.$$

If (2.4) is of diagonal form, then the theorem holds. In the case of the triangular form for (2.4) we shall transform (2.4) by

$$x=\mathrm{diag}(\eta,\ \eta^2,\ \cdots,\ \eta^n)z,$$

where η is a sufficiently small positive number. Then we have

$$\frac{dz}{dt}=D(t)z,\tag{2.5}$$

where the diagonal elements of $D(t)$ are exactly $c_1(t)$, $c_2(t)$, $\cdots$, $c_n(t)$, and the absolute values of those elements above the diagonal line of $D(t)$ are small. Since the exponential dichotomy is preserved under small perturbation, the theorem holds for (2.5), hence for (2.4).

§ 3. Almost periodic linear system

Definition 3. 1. *We consider the almost periodic linear system*

$$\frac{dx}{dt}=A(t)x,\tag{3.1}$$

where $A(t)$ *is* $n\times n$ *almost periodic matrix. If* $x(t)$ *is the nontrivial solution of* (3.1), *we define the upper and lower characteristic exponent* $\bar\lambda$, $\underline{\lambda}$ *of* $x(t)$ *as follows*

$$\bar\lambda(+)=\overline{\lim_{t\to\infty}}\frac{1}{t}\log\|x(t)\|,\quad \bar\lambda(-)=\overline{\lim_{t\to-\infty}}\frac{1}{t}\log\|x(t)\|,$$

$$\underline{\lambda}(+)=\underline{\lim}_{t\to\infty}\frac{1}{t}\log\|x(t)\|,\quad \underline{\lambda}(-)=\underline{\lim}_{t\to-\infty}\frac{1}{t}\log\|x(t)\|,$$

$$\bar\lambda=\max\{\bar\lambda(+),\ \bar\lambda(-)\},\ \underline{\lambda}=\min\{\underline{\lambda}(+),\ \underline{\lambda}(-)\}.$$

We say that (3.1) *has no zero characteristic exponent in the extensive sense iff there is a fnndamental matrix* $X(t)=(x_1(t),\ x_2(t),\ \cdots,\ x_n(t))$ *of* (3.1) *such that*

$$\bar\lambda(x_j(t)),\ \underline{\lambda}(x_j(t))>0,\ j=1,\ 2,\ \cdots,\ n.$$

On the contrary we say that (3.1) *has the zero characteristic exponent in the extensive sense.*

Definition 3. 2. *If* $x(t)$ *is a nontrivial solution of* (3.1) *with* $\bar{\lambda}(x(t)) = \underline{\lambda}(x(t)) = \lambda(x(t))$, *then we call* $\lambda(x(t))$ *the strong characteristic exponent of* $x(t)$.

Definition 3. 3. *If* (3.3) *has a fundamental matrix* $X(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ *with the strong characteristic exponents* $\lambda(x_j(t))$, $j=1, 2, \cdots, n$, *then we say that* (3.1) *has the strong characteristic exponents.*

Theorem 3. 1. *Suppose that* (3.1) *has no zero characteristic exponent in the extensive sense. Then* (3.1) *admits an exponential dichotomy.*

Before proving Theorem 3.1 we make some preparation as follows:

(A) To approximate almost system by quasi periodic spstem.

By the approximate theorem of almost periodic functions we may use the trigonometric polynomial matrices $A(\omega^{(m)}t) = A(\omega_1^{(m)}t, \omega_2^{(m)}t, \cdots, \omega_{s(m)}^{(m)}t)$ to approximate to the almost periodic matrix $A(t)$. Furthermore, we may assume that the frequencies $\omega_1^{(m)}$, $\omega_2^{(m)}, \cdots, \omega_{s(m)}^{(m)}$ of the quasi periodic matrices $A_m(\omega^{(m)}t)$ submit to the irrationality conditions

$$\left|\sum_{j=1}^{s(m)} k_j \omega_j^{(m)}\right| \geqslant K(\omega^{(m)}) \left(\sum_{j=1}^{s(m)} |k_j|\right)^{-(s(m)+1)}, \tag{3.2}$$

where $K(\omega^{(m)}) > 0$, $k_1, k_2, \cdots, k_m$ are integers but not all zero.

By Theorem 2.1 there is a quasi periodic unitary transformation

$$y = Q_m(\omega^{(m)}t)x, \; Q_m(\omega^{(m)}t) = Q_m(\omega_1^{(m)}t, \omega_2^{(m)}t, \cdots, \omega_{s(m)}^{(m)}t),$$

which transforms the quasi periodic linear system

$$\frac{dx}{dt} = A_m(\omega^{(m)}t)x \tag{3.3}$$

into the triangular linear system

$$\frac{dy}{dt} = C_m(\omega^{(m)}t)\, y, \; C_m(\omega^{(m)}t) = \begin{pmatrix} c_1^{(m)}(t) & & * \\ & \ddots & \\ 0 & & c_n^{(m)}(t) \end{pmatrix}.$$

By the argument of § 2, the sequences $\{Q_m(\omega^{(m)}t), C_m(\omega^{(m)}t)\}$ are equicontinuous and uniformly bounded, so that we may assume that $Q_m(\omega^{(m)}t) \to Q(t)$ and $C_m(\omega^{(m)}t) \to C(t)$ uniformly on any finite interval.

(B) The fundamental matrix of (3.1).

Suppose that (3.1) has no zero characteristic exponent in the extensive sense with each negative upper characteristic exponent $< -\alpha < 0$, and each positive lower characteristic exponent $> \alpha > 0$. Let us take the fundamental matrix $X(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ of (3.1) such that

$$\begin{aligned} &(x_j(t)) = -\alpha_j < -\alpha < 0, \; j=1, 2, \cdots, k, \\ &(x_j(t)) = \beta_j > \alpha > 0, \; j=k+1, \cdots, n, \\ &\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_k < -\alpha < 0 < \alpha < \beta_{k+1} \leqslant \cdots \leqslant \beta_n. \end{aligned} \tag{3.4}$$

There is an orthogonal matrix

$$P(t)=X(t)R(t),$$

where $R(t)$ is an upper triangular matrix. Next we transform (3.1) by $y=P^*(t)x$, where $P^*(t)$ is the transposed matrix of $P(t)$, then we have

$$\frac{dy}{dt}=C_0(t)y,\ C_0(t)=\begin{pmatrix} c_1^0(t) & & * \\ & \ddots & \\ 0 & & c_n^0(t)\end{pmatrix}, \tag{3.1*}$$

and the fundamental matrix

$$Y(t)=(y_1(t),\ y_2(t),\ \cdots,\ y_n(t))=P^*(t)X(t)=R^{-1}(t)$$

of (3.1)* is the upper triangular matrix. On the other hand, (3.1) has the fundamental matrix of the form

$$Y_0(t)=\begin{pmatrix} \exp\left(\int_0^t c_1^0(s)ds\right) & & * \\ & \ddots & \\ 0 & & \exp\left(\int_0^t c_n^0(s)ds\right)\end{pmatrix},\ Y_0(0)=E.$$

so that $$Y(t)=Y_0(t)C_0=(y_1^0(t),\ y_2^0(t),\ \cdots,\ y_n^0(t))C_0,$$

where C_0 is a constant upper triangular matrix. It is not difficult to prove that

$$\begin{aligned}&\bar\lambda(x_j(t))=\bar\lambda(y_j(t)),\ \bar\lambda(y_j^0(t))<-\alpha<0,\ j=1,\ 2,\ \cdots,\ k,\\ &\underline{\lambda}(x_j(t))=\underline{\lambda}(y_j(t)),\ \underline{\lambda}(y_j^0(t))>\alpha>0,\quad j=k+1,\ \cdots,\ n.\end{aligned} \tag{3.5}$$

Proposition. *By above notation, there are*

(i) $\bar\lambda\left(\exp\left(\int_0^t c_j^0(s)\,ds\right)\right)<-\alpha<0,\ j=1,\ 2,\ \cdots,\ k,$

(ii) $\underline{\lambda}\exp\left(\left(\int_0^t c_j^0(s)ds\right)\right)>\alpha>0,\ j=k+1,\ \cdots,\ n.$

Proof We shall prove the part (i) only. Similarly, we have the part (ii). First

$$\bar\lambda\left(\exp\left(\int_0^t c_1^0(s)ds\right)\right)=\bar\lambda\exp(x_1(t))<-\alpha.$$

If Proposition does not hold, we may suppose that

$$\bar\lambda\left(\exp\left(\int_0^t c_j^0(s)ds\right)\right)<-\alpha,\ j=1,\ 2,\ \cdots,\ r-1,$$

and $$\bar\lambda\left(\exp\left(\int_0^t c_r^0(s)\,ds\right)\geqslant\alpha,\ 1<r\leqslant k.\right.$$

It implies that $\bar\lambda(y_r^0(t))\geqslant-\alpha$, which contradicts to the formula (3.5), and the proof of Proposition is complete.

Corollary. *Suppose that* $z_1(t),\ z_2(t),\ \cdots,\ z_n(t)$ *are the raw vectors of* $X^{-1}(t)$. *Then we have*

$\underline{\lambda}(z_j(t))>0,\ j=1,\ 2,\ \cdots,\ k;$ *and* $\bar\lambda(z_j(t))<0,\ j=k+1,\ \cdots,\ n.$

Suppose that

$$X_1(t)=(x_1(t),\ x_2(t),\ \cdots,\ x_k(t),\ 0,\ \cdots,\ 0),\ X_2(t)=X(t)-X_1(t),$$
$$X^{-1}(s)=Z_1(s)+Z_2(s),\ X(t)X^{-1}(s)=X_1(t)Z_1(s)+X_2(s)Z_2(s).$$

For any $\varepsilon>0$, we have

$$X_1(t)Z_1(s)=O\Big(\sum_{j=1}^{k}\exp((\bar\lambda(x_j+\varepsilon)t+\bar\lambda(z_j+\varepsilon)s)),\ t,\ s\geqslant 0;$$
$$=O\Big(\sum_{j=1}^{k}\exp((\underline{\lambda}(x_j-\varepsilon)t+(\underline{\lambda}(z_j-\varepsilon)s)),\ t,\ s\leqslant 0;$$
$$X_2(t)Z_2(s)=O\Big(\sum_{j=k+1}^{n}\exp((\bar\lambda(x_j+\varepsilon)t+(\bar\lambda(z_j+\varepsilon)s)\Big),\ t,\ s\geqslant 0;$$
$$=O\Big(\sum_{j=k+1}^{n}\exp((\underline{\lambda}(x_j-\varepsilon)t+(\underline{\lambda}(z_j-\varepsilon)s)\Big),\ t,\ s\leqslant 0.$$

Then there is a constant $\alpha_0>0$ such that

$$\alpha+2\varepsilon<\alpha_0<|\bar\lambda(x_j)|,\ |\underline{\lambda}(x_j)|,\ |\bar\lambda(z_j)|,\ |\underline{\lambda}(z_j)|,\ j=1,\ 2,\ \cdots,\ n;$$
$$\begin{aligned}\|X_1(t)Z_1(s)\|&\leqslant M(s)\exp(-\alpha_0(t-s)),\quad t\geqslant s\geqslant 0;\\&\leqslant M^*(t)\exp(-\alpha_0(t-s)),\quad s\leqslant t\leqslant 0;\\\|X_2(t)Z_2(s)\|&\leqslant M^*(t)\exp(\alpha_0(t-s)),\quad s\geqslant t\geqslant 0;\\&\leqslant M(s)\exp(\alpha_0(t-s)),\quad t\leqslant s\leqslant 0.\end{aligned}\tag{3.6}$$

C) The fundamental matrix of (3.3).

Let us consider the fundamental matrix

$$Y_m(t)=\begin{pmatrix}\exp\Big(\int_0^t c_1^{(m)}(s)ds\Big) & & *\\ & \ddots & \\ 0 & & \exp\Big(\int_0^t c_n^{(m)}(s)ds\Big)\end{pmatrix}$$
$$=(y_1^{(m)}(t),\ y_2^{(m)}(t),\ \cdots,\ y_n^{(m)}(t))$$

of (3.3)*. Then the matrix

$$X_m(t)=(x_1^{(m)}(t),\ x_2^{(m)}(t),\ \cdots,\ x_n^{(m)}(t))=\bar Q_m^*(\omega^{(m)}t)Y_m(t)$$

is the fundamental matrix of (3.3). Since (3.3) is the quasi periodic linear system, it is easy to prove that

$$\lambda(x_j^{(m)}(t))=\lambda(y_j^{(m)}(t))=\lambda\Big(\exp\Big(\int_0^t c_j^{(m)}(s)\,ds\Big),\ j=1,\ 2,\ \cdots,\ n.$$

Here we suppose that

$$\lambda(x_j^{(m)}(t))\leqslant\lambda(x_{j+1}^{(m)}(t)),\ \text{and}\ |\lambda(x_j^{(m)}(t))|>\alpha>0.$$

(We shall prove that the last inequality holds for m large enough in Lemma 3.4 below). Then we have the positive number $k_m(\alpha)$ such that

$$\begin{aligned}\|X_{m1}(t)Z_{m1}(s)\|&\leqslant k_m(\alpha)\exp(-\alpha(t-s)),\quad t\geqslant s,\\\|X_{m2}(t)Z_{m2}(t)\|&\leqslant k_m(\alpha)\exp(\alpha(t-s)),\quad s\geqslant t,\end{aligned}\tag{3.7}$$

where $X_m(t)=X_{m1}(t)+X_{m2}(t)$, $X_m^{-1}(s)=Z_{m1}(s)+Z_{m2}(s)$ and

$$X_m(t)X_m^{-1}(s)=X_{m1}(t)Z_{m1}(s)+X_{m2}(t)Z_{m2}(s).$$

Lemme 3.1. *The sequence $k_m(\alpha)$ in (3.7) is bounded iff the sequence $k_{mj}(\alpha)$ is bounded, where $k_{mj}(\alpha)$ is the least number to satisfy the following inequality*

$$\Big|\exp\Big(\int_s^t c_j^{(m)}(s)ds\Big)\Big|\leqslant k_{mj}(\alpha)\exp(-\alpha(t-s)),\ t\geqslant s,\ \textit{when}\ x_j^{(m)}(t)\ \textit{is in}\ X_{m1}(t),$$

and

$$\left|\exp\left(\int_s^t c_j^{(m)}(s)\,ds\right)\right| \leqslant k_{mj}(\alpha)\exp(\alpha(t-s)),\ s\geqslant t,\ \textit{when } x_j^{(m)}(t) \textit{ is in } X_{m2}(t).$$

Proof If $k_{mj}(\alpha)$ is bounded, it is evident that we can take $k_m(\alpha)$ to be bounded. Conversely, if $k_m(\alpha)\leqslant K_0$, where K_0 is a constant independent of m, $m=1, 2, \cdots$, we shall take

$$k_{mj}(\alpha)=\sup\left\{\left|\exp\left(\int_s^t c_j^{(m)}(s)ds+\alpha(t-s)\right)\right| \,\middle|\, t\geqslant s\right\}$$
$$\leqslant \sup\{\|X_{m1}(t)Z_{m1}(s)\exp(\alpha(t-s))\| \mid t\geqslant s\}\leqslant k_m(\alpha),$$

when $x_j^{(m)}(t)$ is in $X_{m1}(t)$;

$$k_{mj}(\alpha)=\sup\left\{\exp\left(\int_s^t c_j^{(m)}(s)ds-\alpha(t-s)\right)\,\middle|\, s\geqslant t\right\}$$
$$\leqslant \sup\{\|X_{m2}(t)Z_{m2}(s)\exp(-\alpha(t+s))\| \mid s\geqslant t\}\leqslant k_m(\alpha),$$

when $x_j^{(m)}(t)$ is in $X_{m2}(t)$, so that $k_{mj}(\alpha)$ is bounded.

Lemma 3. 2. *Suppose that $x(t)$, any nontrivial solution of* (3.1), *has* $\bar{\lambda}(x(t))<-\alpha<0$. (*or* $\underline{\lambda}(x(t))>\alpha>0$.) *Then there is a bounded sequence $k_m(\alpha)$ such that*

$$\|X_m(t)X^{-1}(s)\|\leqslant k_m(\alpha)\exp(-\alpha(t-s)),\ t\geqslant s.$$
$$(or \|X_m(t)X_m^{-1}(s)\|\leqslant k_m(\alpha)\exp(\alpha(t-s)),\ t\leqslant s.)$$

Proof Let $X(t)$ is the fundamental matrix of (3.1). Then $X(t)X^{-1}(s)$ can be expressed by the formula

$$X(t)X^{-1}(s)=X_m(t)X_m^{-1}(s)+\int_s^t X_m(t)X_m^{-1}(r)(A(r)-A_m(r))X(r)X^{-1}(s)dr.$$

Suppose that $\|X(t)X^{-1}(s)\|\leqslant M(s)\exp(-(\alpha+\varepsilon)(t-s)),\ t\geqslant s>0.$
Then we have

$$\|X_m(t)X_m^{-1}(s)\|\leqslant M(s)\exp(-(\alpha+\varepsilon)(t-s))$$
$$+g_m M(s)\int_s^t \|X_m(t)X_m^{-1}(r)\exp(-(\alpha+\varepsilon)(t-s))\|dr,$$

where $\|A(r)-A_m(r)\|\leqslant g_m$, $g_n\to 0$, when s fixed, and m is large enough, hence

$$\|X_m(t)X_m^{-1}(s)\|\leqslant 2M(s)\exp(-\alpha(t-s)),\ t\geqslant s^*. \qquad §§$$

For s, m and α fixed, we choose the least positive number $k_m(s, \alpha)$ such that

$$\|X_m(t)X_m^{-1}(s)\|\leqslant k_m(s, \alpha)\exp(-\alpha(t-s)),\ t\geqslant s,$$

i. e., $k_m(s, \alpha)\leqslant M(s)$, and $k_m(0, \alpha)\leqslant M(0)$. Similarly, when t fixed, we have $M^*(t)$ such that

$$\|X(t)X^{-1}(s)\|\leqslant 2M^*(t)\exp(-(\alpha+\varepsilon)(t-s)),\ 0\geqslant t\geqslant s,$$
$$\|X_m(t)X_m^{-1}(s)\|\leqslant 2M^*(t)\exp(-\alpha(t-s)),\quad 0\geqslant t\geqslant s.$$

For t, m and α fixed, we have the least positive number $k^*_{mj}(t, \alpha)$ such that

$$\|X_m(t)X_m^{-1}(s)\|\leqslant k^*_{mj}(t, \alpha)\exp(-\alpha(t-s)),\ t\geqslant s,$$

i. e., $k^*_{mj}(t, \alpha)\leqslant 2M^*(t)$, and $k^*_{mj}(0, \alpha)\leqslant 2M^*(0)$.

In order to prove that $k_m(\alpha)$ is bounded, by Lemma 3.1, we shall show the boundedness of $k_{mj}(\alpha)$ only. Let us denote by $k_{mj}(s, \alpha)$ and $k^*_{mj}(t, \alpha)$ the least positive

* C. F., "Additional proof of ..." in p. 144.

numbers such that

$$\left|\exp\left(\int_s^t c_j^{(m)}(r)dr\right)\right| \leqslant k_{mj}(s,\ \alpha)\exp(-\alpha(t-s)),\ \text{for all } t \geqslant s$$

and

$$\left|\exp\left(\int_s^t c_j^{(m)}(r)dr\right)\right| \leqslant k_{mj}^*(t,\ \alpha)\exp(-\alpha(t-s)),\ \text{for all } s \leqslant t.$$

It is evident that

$$k_{mj}(\alpha) = \sup_s k_{mj}^*(s,\ \alpha) = \sup_t k_{mj}^*(t,\ \alpha).$$

Suppose that $k_{mj}(\alpha) = k_{mj}(s_0,\ \alpha)$ for some s_0, and $s_0 \leqslant 0 < t$. Then we have

$$\left|\exp\left(\int_{s_0}^t c_j^{(m)}(r)dr\right)\right| \leqslant \left|\exp\left(\int_{s_0}^0 c_j^{(m)}(r)dr\right)\exp\left(\int_0^t c_j^{(m)}(r)dr\right)\right|,$$

hence $\quad k_{mj}(\tau) \leqslant k_{mj}^*(0,\ \alpha)k_{mj}(0,\ \alpha) \leqslant 4M^*(0)M(0).$

If $s_0 > 0$, by the almost periodicity of $c_j^{(m)}(t)$, we have the real number $\tau_0 < 0$ such that

(i) $|c_j^{(m)}(t+\tau_0) - c_j^{(m)}(t)| < \varepsilon$, for all t;

(ii) $s_1 = s_0 + \tau_0 < 0$.

It follows that when $t' = t + \tau_0 > 0$, one has

$$\left|\exp\left(\int_{s_0}^t c_j^{(m)}(r)dr\right)\right| \leqslant \left|\exp\int_{s_1}^{t'} (c_j^{(m)}(r) + \varepsilon)dr)\right|,$$

so that $k_{mj}(\alpha) \leqslant k_{mj}^*(0,\ \alpha+\varepsilon)k_{mj}(0,\ \alpha+\varepsilon)$. We note that $\alpha_0 > \alpha + 2\varepsilon$ in (3.6), which implies that

$$\|X(t)X^{-1}(s)\| \leqslant 2M(s)\exp(-(\alpha+\varepsilon)(t-s)),\quad t \geqslant s \geqslant 0;$$
$$\leqslant 2M^*(t)\exp(-(\alpha+\varepsilon)(t-s)),\quad s \leqslant t \leqslant 0,$$

hence $k_{mj}(0,\ \alpha+\varepsilon)k_{mj}^*(0,\ \alpha+\varepsilon) \leqslant 4M(0)M^*(0)$, and $k_{mj}(\alpha)$ is bounded.

Lemma 3.3. *The sequence $k_m(\alpha)$ in (3.7) is bounded.*

Proof Suppose that $X(t)$ is the fundamental matrix of (3.1) above-mentioned, and take the positive number $r > \bar{\lambda}(x_j(t))$, $j = 1, 2, \cdots, n$, where $x_j(t)$ is the column vector of $X(t)$. There is a transformation

$$z = x\exp(-rt),$$

which transforms (3.1) and (3.3) into

$$\frac{dz}{dt} = (A(t) - rE)z, \tag{3.1**}$$

$$\frac{dz}{dt} = (A_m(t) - rE)z, \tag{3.3**}$$

respectively. Then (3.1)** and (3.3)** have the fundamental matrices $X(t)\exp(-rt)$ and $X_m(t)\exp(-rt)$ with negative upper characteristic exponents. By Lemma 3.2, there exists a bounded sequenence k'_m such that

$$\|X_m(t)X_m^{-1}(s)\exp(-r(t-s))\| \leqslant k'_m\exp(-r_0(t-s)),\ t \geqslant s,$$

where $r_0 > 0$. Therefore $k_{mj}(\alpha) \leqslant k'_m$, $j = 1, 2, \cdots, k$. Similarly, we have $k_{mj}(\alpha) \leqslant k'_m$, $j = k+1, \cdots, n$. By Lemma 3.1, $k_m(\alpha)$ is bounded, and the proof of Lemma 3.3 is complete.

Corollary. *If $\bar{\lambda}(X_{m1}(t))<-\alpha<\sigma<\underline{\lambda}(X_{m2}(t)$, and $k_m(\alpha)$ and $k_m(\sigma)$ are the lesat positive numbers such that*

$$\|X_{m1}(t)Z_{m1}(s)\|\leqslant k_m(\alpha)\exp(-\alpha(t-s)),\ t\geqslant s,$$
$$\|X_{m2}(t)Z_{m2}(s)\|\leqslant k_m(\sigma)\exp(\sigma(t-s)),\ s\geqslant t, \tag{3.8}$$

then $k_m(\alpha)$, $k_m(\sigma)\leqslant K_0$, where K_0 is a constant independent of m, $m=1, 2, 3, \cdots$.

Lemma 3. 4. *Suppose that (3.1) has no zero characteristic exponent in the extensive sense, and there is a fundamental matrix of (3.1) satisfying the inequality (3.6). Then there is the fundamental matrix $X_m(t)$ of (3.3) satisfying (3.7) with $k_m(\alpha)$ bounded, and the rank of $X_{m1}(t)$ is k, when m is sufficiently large.*

Proof We may suppose that $X_m(t)=X_{m1}(t)+X_{m2}(t)$ with $\bar{\lambda}(X_{m1}(t))<-\alpha<\sigma<\underline{\lambda}(X_{m2}(t))$, and the rank of $X_{m1}(t)$ is $N(m)$. Then any nontrivial solution $x(t)$ of (3.1) has $\bar{\lambda}(x(t))<-(\alpha+\varepsilon)$, and we can express $x(t)$ by the formula

$$x(t)=X_m(t)x_0+\Big(\int_{-\infty}^{t}X_{m1}(t)Z_{m1}(r)-\int_{t}^{\infty}X_{m2}(t)Z_{m2}(r)\Big)(A(r)-A_m(r))x(r)dr, \tag{3.9}$$

where $\|A(r)-A_m(r)\|\leqslant g_m$, the sequence g_m tends to zero as m tends to infinite. Since

$$\Big\|\int_t^\infty\Big\|=O\Big(\int_t^\infty\exp(\sigma(t-r)-(\alpha+\varepsilon)x)dr\Big)=O(\exp(-\alpha t)),$$
$$\Big\|\int_0^t\Big\|=O\Big(\int_0^t\exp(-\alpha(t-r)-(\alpha+\varepsilon)r)dr\Big)=O(\exp(-\alpha t)),$$

we have $X_m(t)x_0=X_{m1}(t)x_0$.

By Lemma 3.3 and its Corollary, $X_{m1}(t)$ and $X_{m2}(t)$ satisfy (3.8) with $k_m(\alpha)$ and $k_m(\sigma)$ bounded, and it is easy to prove that in (3.9), $X_{m1}(t)x_0$ does not vanish, when $x(t)$ is the nontrivial solution of (3.1). It means that the rank of $X_{m1}(t)\geqslant k$. Similarly we have $\bar{\lambda}(X_{m2}(t))>\alpha$, and the rank of $X_{m2}(t)\geqslant n-k$, so that $N(m)=k$, when m is large enough.

Proof of Theorem 3.1. We may suppose that $X_m(0)\to X(0)$, which implies that $X_m(t)\to X(t)$, which is a fundamental matrix of (3.1). Since

$$\|X_{m1}(t)Z_{m1}(s)\|\leqslant k_m(\alpha)\exp(-\alpha(t-s))\leqslant K_0\exp(-\alpha(t-s)),\ t\geqslant s,$$
$$\|X_{m2}(t)Z_{m2}(s)\|\leqslant k_m(\alpha)\exp(\alpha(t-s))\leqslant K_0\exp(\alpha(t-s)),\quad s\geqslant t,$$

letting m tend to ∞, we obtain

$$\|X_1(t)Z_1(s)\|\leqslant K_0\exp(-\alpha(t-s)),\ t\geqslant s,$$
$$\|X_2(t)Z_2(s)\|\leqslant K_0\exp(\alpha(t-s)),\quad s\geqslant t.$$

Theorem 3.1 is proved completely.

§ 4. The hull $H(A(t))$ of $A(t)$

Definition 4. 1. *We call the set of matrices*

$$H(A(t))=\{B(t)\mid B(t)=\lim_{r\to\infty}A(t+h_r)\ \text{uniformly}\}$$

as the hull of $A(t)$.

Lemma 4. 1. *Suppose that* $\lim_{r\to\infty} A(t+h_r)=B(t)$, $\lim_{r\to\infty} A_m(t+h_r)=B_m(t)$ *and* $\lim_{m\to\infty} A_m(t)=A(t)$ *uniformly. Then* $\lim_{m\to\infty} B_m(t)=B(t)$ *uniformly.*

This result is obvious, so that the proof is omitted.

Theorem 4. 1. *Suppose that* (3.1) *has no zero characteristic exponent in the extensive sense. Then for any* $B(t)\in H(A(t))$, *the linear system*

$$\frac{dx}{dt}=B(t)x \tag{4.1}$$

admits an exponential dichotomy.

Proof Suppose that $\lim_{r\to\infty} A(t+h_r)=B(t)$ uniformly, and $X(t)$ is the fundamental matrix of (3.1) satisfying the inequality (3.6). By lemmas above-mentioned, (3.3) has the fundamental matrix $X_m(t)$ satisfying the inequality (3.7) with $k_m(\alpha)$ bounded, and

$$\|X_{m1}(t+h_r)Z_{m1}(s+h_r)\|\leqslant k_m(\alpha)\exp(-\alpha(t-s)),\ t\geqslant s;$$
$$\|X_{m2}(t+h_r)Z_{m2}(s+h_r)\|\leqslant k_m(\alpha)\exp(\alpha(t-s)),\quad s\geqslant t.$$

We have $\lim_{r\to\infty} X_{mi}(t+h_r)=W_{mi}(t)$, $\lim_{r\to\infty} Z_{mi}(t+h_r)=U_{mi}(t)$, $i=1, 2$,

where $W(t)=W_{m1}(t)+W_{m2}(t)$ is the fundamental matrix of the linear system

$$\frac{dx}{dt}=B_m(t)x, \tag{4.2}$$

and $W_m^{-1}(s)=U_{m1}(s)+U_{m2}(s)$. Then we have

$$\|W_{m1}(t)U_{m1}(s)\|\leqslant k_m(\alpha)\exp(-\alpha(t-s)),\ t\geqslant s,$$
$$\|W_{m2}(t)U_{m2}(s)\|\leqslant k_m(\alpha)\exp(\alpha(t-s)),\quad s\geqslant t$$

with $k_m(\alpha)$ bounded. By the proof of Theorem 3.1, the linear system (4.1) admits an exponential dichotomy. Theorem 4.1 is proved completely.

§ 5. The spectral theory of almost periodic linear systems

Suppose that $A(t)$ is the almost periodic matrix in (3.1), and λ is a real number. If the linear system

$$\frac{dx}{dt}=(A(t)-\lambda E)x \tag{5.1}$$

admits no exponential dichotomy, then we call λ the spectrum of (3.1).

Theorem 5. 1. (*Sacker and Sell*)[8] *If* (3.1) *has the strong characseristic exponent, i. e., there is a fundamental matrix* $X(t)=(x_1(t), x_2(t), \cdots, x_n(t))$ *of* (3.1) *with* $\bar{\lambda}(x_j(t))=\underline{\lambda}(x_j(t))=\lambda(x_j(t))$, $j=1, 2, \cdots, n$, *then the spectrum of* (3.1) *coincides with its characteristic exponent. In general the spectrums of* (3.1) *consist of several closed* *intervals.*

This result follows immediately from Theoren 3.1. The proof is omitted.

§ 6. The problem of Hale

The differential system

$$\begin{aligned}\frac{d\theta}{dt}&=I+H(x, \theta, t, \varepsilon),\\ \frac{dx}{dt}&=A(\theta)+F(x, \theta, t, \varepsilon)\end{aligned} \tag{6.1}$$

has been considered by Hale, J. K., where x is an n-vector, θ is a k-vector, each component of I is 1, $A(\theta)$, $H(x, \theta, t, \varepsilon)$ and $F(x, \theta, t, \varepsilon)$ are the periodic or almost periodic funcitons of the components $\theta_1, \theta_2, \cdots, \theta_k$ of θ, and continuous for all variables, Lipschitzian for x and θ, where the Lipschitz constant of $A(\theta)$ is M_0, the Lipschitz constants of both $H(x, \theta, t, \varepsilon)$ and $F(x, \theta, t, \varepsilon)$ are $\rho(\varepsilon)=o(1)$, ε is the small parameter, and there is a constant M such that

$$\|H(0, \theta, t, \varepsilon)\|+\|F(0, \theta, t, \varepsilon)\|<M\varepsilon.$$

He wished to establish the center integral manifold of that system.

By Theorem 2.1 or Theorem 3.1, it is not difficult to construct the center integral manifold (6.1).

Theorem 6. 1. *If $A(\theta)$ is the periodic matrix of the component $\theta_1, \theta_2, \cdots, \theta_k$ of θ, and each characteristic exponent of the quasi periodic linear system*

$$\frac{dx}{dt}=A(\theta_0+t)x, \text{ for } \theta_0 \text{ fixed}, \tag{6.2}$$

is different from zero for all θ_0, then (6.1) *has the center integral manifold.*

Theorem 6. 2. *If $A(\theta)$ is the almost periodic matrix of the components $\theta_1, \theta_2, \cdots, \theta_k$ of θ, and the linear system* (6.2) *submits to the conditions of Theorem* 3.1, *then* (6.1) *has the integral manifold.*

The proof of Theorem 6.2 is similar to the proof of Theorem 6.1. We give the proof of Theorem 6.1 only.

The proof of Theorem 6.1; Let us write (6.1) in the following form

$$\begin{aligned}\frac{d\theta}{dt}&=I+H(x, \theta, t, \varepsilon),\\ \frac{dx}{dt}&=A(\theta_0+t)x+G(x, \theta, t, \varepsilon),\end{aligned} \tag{6.1)*$$

where $G(x, \theta, t, \varepsilon)=(A(\theta)-A(\theta_0+t))x+F(x, \theta, t, \varepsilon)$. Since the quasi periodic linear system (6.2) has no zero characteristic exponent in the extensive sense, it admits an exponential dichotomy. Suppose that $X(t)$, the fundamental matrix of (6.2), submits to the inequality (1.2). It is well known that the proof of the existence of the center integral manifold of (6.1)* is equivalent to the proof of the existence of the solution of the following integral system

$$q(\tau,\ \theta,\ t)=\theta+I(\tau-t)+\int_t^\tau H(s)ds,$$
$$f(t,\ \theta,\ \varepsilon)=\left(\int_{-\infty}^t X_1(t)Z_1(s)-\int_t^\infty X_2(t)Z_2(s)\right)G(s)ds, \qquad (5.1)^{**}$$

where $H(s)=H(f(s),\ q(s),\ s,\ \varepsilon)$, $f(s)=f(s,\ q(s),\ s,\ \varepsilon)$, $q(s)=q(s,\ \theta,\ t)$, $G(s)=(A(q(s))-A(\theta_0+s))f(s)+F(s)$, $F(s)=F(f(s),\ q(s),\ s,\ \varepsilon)$.

Since $\theta=\theta_0+t$, we have $A(\theta_0+s)=A(\theta+(s-t))$.

By the approximate method, we shall prove the existence of the solution of (6.1)** as follows:

(A) To construct the sequences of vector functions $f_m(t,\ \theta,\ \varepsilon)$ and $q(\tau,\ \theta,\ t)$.

(a) Take $f_0(t,\ \theta,\ \varepsilon)\equiv 0$.

(b) If $f_m(t,\ \theta,\ \varepsilon)$ has been determined, we take $q_{m+1}(\tau,\ \theta,\ t)$ to be the solution of the following differential equation

$$\frac{dz}{d\tau}=I+H(f_m(\tau,\ z,\ \varepsilon),\ z,\ \tau,\ \varepsilon),\ q_{m+1}(t,\ \theta,\ t)=\theta. \qquad (6.3)$$

By the assumptions of (6.1), the function at the right hand side of (6.3) is Lipschitzian with respect to z, so that $q_{m+1}(\tau,\ \theta,\ t)$ is determined uniquely. Next we take

$$f_{m+1}(t,\ \theta,\ \varepsilon)=\left(\int_{-\infty}^t X_1(t)Z_1(s)-\int_t^\infty X_2(t)Z_2(s)\right)G_m(s)ds,$$

where $G_m(s)=(A(q_{m+1}(s))-A(\theta+(s-t)))f_m(s)+F_m(s)$,

$q_{m+1}(s)=q_{m+1}(s,\ \theta,\ t)$, $f_m(s)=f_m(s,\ q_{m+1}(s),\ \varepsilon)$,

$F_m(s)=F(f_m(s),\ q_{m+1}(s),\ s,\ \varepsilon)$.

(B) Two properties of $f_m(t,\ \theta,\ \varepsilon)$.

(a) There is a constant K^* such that

$$\|f_m(t,\ \theta,\ \varepsilon)\|\leqslant K^*\varepsilon,\ m=0,\ 1,\ 2,\ \cdots. \qquad (6.4)$$

Since
$$\|A(q(s))-A(\theta+(s-t))\|\leqslant M_0\left\|\int_t^s H(s)\,ds\right\|$$
$$\leqslant M_0(\varepsilon+\rho(\varepsilon))\,|t-s|\leqslant M^*(\varepsilon+\rho(\varepsilon))\exp\left(\frac{\alpha}{4}|t-s|,\right)$$

where M^* is a constant, we can prove that (6.4) holds by induction. (The detail is omitted.)

(b) There is a positive number $d(\varepsilon)$ such that

$$\|f_m(t,\ \theta,\ \varepsilon)-f_m(t,\ \theta',\ \varepsilon)\|\leqslant d(\varepsilon)\|\theta-\theta'\|, \qquad (6.5)$$
$$\lim d(\varepsilon)=0,\ \text{as } \varepsilon\to 0.$$

Proof Take a positive number $d(\varepsilon)$ satisfying the following inequalities

$$16K^*M^*(1+d(\varepsilon))(\rho(\varepsilon)+\varepsilon)<d(\varepsilon),$$
$$\rho(\varepsilon)(1+d(\varepsilon))<\frac{1}{4}\alpha,\ \lim_{\varepsilon\to\infty} d(\varepsilon)=0.$$

(6.5) holds evidently for $f_0(t,\ \theta,\ \varepsilon)$. Suppose that (6.5) holds for $f_{m-1}(t,\ \theta,\ \varepsilon)$. We shall prove that it holds for $f_{m-1}(t, \theta, \varepsilon)$ too. Since

$$\|A(q_m(\tau, \theta, t))-A(q_m(\tau, \theta', t))\|+\|A(\theta+(s-t))-A(\theta'+(s-t))\|$$
$$\leqslant M_0(\|\theta-\theta'\|+\|q_m(\tau, \theta, t)-q_m(\tau, \theta', t)\|),$$

one has

$$\|q_m(\tau, \theta, t)-q_m(\tau, \theta', t)\|\leqslant\|\theta-\theta'\|+\rho(\varepsilon)(1+d(\varepsilon))\left|\int_t^\tau\|q_m(s, \theta, t)-q_m(s, \theta', t)\|ds\right|,$$

i. e., $\|q_m(\tau, \theta, t)-q_m(\tau, \theta', t)\|\leqslant\|\theta-\theta'\|\exp\left(\frac{\alpha}{4}|t-s|\right)$. Then we have

$$\|f_m(t, \theta, \varepsilon)-f_m(t, \theta', \varepsilon)\|\leqslant\beta\left(\int_{-\infty}^t\exp(-\alpha(t-s))+\int_t^\infty\exp(\alpha(t-s))\right).$$
$$4M^*K^*(\rho(\varepsilon)+\varepsilon)(1+d(\varepsilon))\|\theta-\theta'\|\exp\left(\frac{\alpha}{4}|t-s|\right)<d(\varepsilon)\|\theta-\theta'\|.$$

(C) To prove the convergence of $f_m(t, \theta, \varepsilon)$ and $q_m(\tau, \theta, t)$.

(a) To prove the convergence of $f_m(t, \theta, \varepsilon)$. Put

$$L(m)=\sup\|f_m(t, \theta, \varepsilon)-f_{m-1}(t, \theta, \varepsilon)\|.$$

It is easy to prove that $L(m+1)\leqslant\frac{1}{2}L(m)$, so that the vector functions $f_m(t, \theta, \varepsilon)$ converge to the vector function $f(t, \theta, \varepsilon)$ uniformly on the real axis.

(b) To prove the convergence of $q_m(\tau, \theta, t)$.

Let $q(\tau, \theta, t)$ be the solution of the differential equation

$$\frac{dz}{d\tau}=I+H(f(\tau, z, \varepsilon), z, \tau, \varepsilon), \quad q(t, \theta, t)=\theta.$$

It is easy to prove that

(i) $\lim_{m\to\infty}q_m(\tau, \theta, t)=q(\tau, \theta, t)$.

(ii) $f(t, \theta, \varepsilon)$ and $q(\tau, \theta, t)$ satisfy (6.1)**.

The detail is omitted. Therefore Theorem 6.1 is proved completely.

Additinal proof of the inequality (§§) in §3.

The inequality (§§) in §3 is not so clear, we shall give the proof of that as follows.

By (**), we may assume that

$$\|X(t)X^{-1}(s)\|\leqslant M(s)\exp(-(\alpha+2\varepsilon)(t-s)),\quad t\geqslant s\geqslant 0,$$
$$\lambda(X_j^{(m)}(t))<-(\alpha+2\varepsilon),$$

and take the numbea $-\alpha_m<0$, and the least positive constant $k_m(\alpha_m+\varepsilon)$ such that

$$-(\alpha+2\varepsilon)<-(\alpha_m+\varepsilon)<-(\alpha+\varepsilon)\alpha+\varepsilon-\alpha_m>\eta_0>0,$$
$$\|X_m(t)X_m^{-1}(s)\|\leqslant k_m(\alpha+\varepsilon)\exp(-(\alpha+\varepsilon)(t-s)),\quad t\geqslant s.$$

hen we have

$$\|X_m(t)X_m^{-1}(s)\|\leqslant M(s)\exp(-(\alpha+2\varepsilon)(t-s))+\frac{\eta_0}{2}\int_s^t k_m(d_m+\varepsilon)\exp(-(\alpha_m+\varepsilon)\cdot(t-r)-(\alpha+2\varepsilon)(t-s))dr$$
$$\leqslant(M(s)+1/2\,k_m(\alpha_m+\varepsilon))\exp(-(\alpha_m+\varepsilon)(t-s)),$$

where $\| A(r) - A_m(r) \| \leqslant g_m$, $M(s) g_m < \frac{1}{2} \eta_0$, when s is fixed and m is large enough, hence $k_m(\alpha_m + \varepsilon) \leqslant 2M(s)$, and

$$\| X_m(t) X_m^{-1}(s) \| \leqslant 2M(s) \exp(-(\alpha+\varepsilon)(t-s)) \leqslant 2M(s) \exp(-\alpha(t-s)).$$

References

[1] Sacker, R J., and Sell, G. R., Existence of Dichotomies and Invariant Splittings for Linear Differenial System, *J. Liff. Eqs.*, **15** (1974), 429—458; **22** (1976), 478—496, 497—522.

[2] Lin Zhen Sheng, The Floquet Theory for Quasi Periodic Linear System (abstract). *Acta Math. Sinica*, **20** (1977), 285—286.

[3] Lin Zhen Sheng, The Floquet Theory for Quasi Periodic Linear System, *Acta Universitatis Fuzhouensis*, (1978).

[4] Lin Zhen Sheng, The Existence of Almost Periodic Solution for Quasi Linear System. *Acta Math. Sinica*, **22** (1979), 515—529.

[5] Besicovich, A. S., Almost Periodic Functios, Cambright 1932.

[6] Hale, J. K., Integral Manifold of Perturbed Differential Equations, *Ann. Math.*, **73** (1961), 496—531.

[7] Coppel. W. A., Dichotomies in Stability Theory, *Lectur notes in math.* 629, Springer-Verlag, Beriin, Heidelberg, New York, 1978.

[8] Srcker, R. J. and Sell, G. R., A Spectral Theory of Linear System, *J. Diff. Eqs.*, **27** (1978), 280—357.

[9] Sell, G. R., The Structure of a Flow in the Vicinity of an Almost Periodic Motion, *J. Diff. Eqs.*, **27** (1978), 359—393.

概周期线性系统与指数型二分法

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摘 要

本文讨论概周期线性系统具有指数型二分法与它的特征指数的关系.

考虑线性系统

$$\frac{dx}{dt}=A(t)x. \tag{1}$$

其中 $A(t)$ 是 $n\times n$ 方阵,它在实轴上连续和有界. 如果(1)有基本方阵 $X(t)$,具有如下的分解

$$X(t)=X_1(t)+X_2(t),\quad X^{-1}(s)=Z_1(s)+Z_2(s),$$
$$X(t)X^{-1}(s)=X_1(t)Z_1(s)+X_2(t)Z_2(s).$$

同时有常数 α, $\beta>0$, 使

$$\|X_1(t)Z_1(s)\|\leqslant\beta\exp(-\alpha(t-s)),\ t\geqslant s;$$
$$\|X_2(t)Z_2(s)\|\leqslant\beta\exp(\alpha(t-s)),\quad s\geqslant t.$$

就说(1)具有指数型二分法.

我们所得的结果,可叙述如下:

一、对拟周期线性系统,存在同频率的酉变换,把它化为三角型系统. 从而推出:

若拟周期线性系统的特征指数异于零,则它具有指数型二分法.

二、对概周期线性系统. 定义了广义的零特征指数. 当它不具有广义的零特征指数,则该系统具有指数型二分法.

三、利用一和二的结果,解决了 Hale 所提的关于中心积分流形的存在性问题.