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SOME REMARKS ON BURTON'S ASYMPTOTIC STABLE THEOREM

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To simplify our illustration, we use the notation similar to that of Hale's[1].

The theorem of asymptotic stability for the solutions of the retarded functional differential equations

$$\dot{x}=f(t,\ x_t) \tag{1}$$

is usually based on the assumption that f is completely continuous. However, Burton, in [2], has successfully removed it by introducing other restrictions on the Liapunov functional $V(t,\ \phi)$. In this paper we will prove Burton's result by a direct method which is more simple than Burton's indirect method and will obtain an interesting functional inequality which enables us to estimate the rate at which the solutions are approaching zero. Let H be a set of functions defined as

$$H=\{\bar{u}(\cdot)\,|\,\bar{u}{:}\ R^+\to R^+,\text{ continuous nondecreasing }\bar{u}(0)=0\text{ and }\bar{u}(s)>0\text{ for }s>0\}.$$

Set
$$\|\phi\|_\eta=\eta(|\phi(0)|)+\frac{1}{r}\int_{-r}^{0}\eta(|\phi(\theta)|)d\theta,$$

where $\eta{:}\ R^+\to R^+$ is a continuous strictly increasing function with $\eta(0)=0$. Suppose $f{:}\ R\times C\to R^n$ is continuous. Let $u,\ v,\ w\in H$ and $W_1(s)=W(\eta^{-1}(s))$,

$$h(s)=\int_0^s w_1(s)ds,\qquad k(s)=v(s)+\frac{w_1(1)}{2}rs.$$

Theorem 1. *Suppose there is a continuous functional* $V{:}\ R\times C\to R$ *such that*

$$u(|\phi(0)|)\leqslant V(t,\ \phi)\leqslant v(\|\phi\|_\eta), \tag{2}$$
$$\dot{V}(t,\ \phi)\leqslant -w(|\phi(0)|) \tag{3}$$

then there exists another continuous functional $G{:}\ R\times C\to R$ *such that*

$$\dot{G}(t,\ \phi)\leqslant -g(G(t,\ \phi)), \tag{4}$$
$$V(t,\ \phi)\leqslant G(t,\ \phi) \tag{5}$$

for $\eta(|\phi|)<1$, *where* $g{:}\ R^+\to R^+$ *is defined as* $g(s)=h\left(\frac{1}{2}\,k^{-1}(s)\right)$.

Proof Obviously, $h(s)$ is continuous, convex and strictly increasing with $h(0)=0$, and

$$h(s)\leqslant sw_1(s)\leqslant w_1(s) \tag{6}$$
$$h(s)\leqslant w_1(1)\cdot s \tag{7}$$

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for $0\leqslant s<1$. Let $\beta(\theta)=\frac{\theta}{2r}+\frac{1}{2}$, for $-r\leqslant\theta\leqslant 0$ and

$$G(t,\ \phi)=V(t,\ \phi)+\int_{-r}^{0}\beta(\theta)h(\eta(|\phi(\theta)|))d\theta$$

for $\eta(|\phi|)<1$. (5) is obviously satisfied. Differentiating G with respect to t along (1), we have

$$\dot{G}(t,\ \phi)=\dot{V}(t,\ \phi)+\beta(0)h(\eta(|\phi(0)|))-\beta(-r)h(\eta(|\phi(-r)|))$$
$$-\int_{-r}^{0}\beta'(\theta)h(\eta(|\phi(\theta)|))d\theta.$$

By (3) and (6), we have

$$\dot{G}(t,\ \phi)\leqslant -w(|\phi(0)|)+\frac{1}{2}h(\eta(|\phi(0)|))-\frac{1}{2r}\int_{-r}^{0}h(\eta(|\phi(\theta)|))d\theta$$
$$=-w_1(\eta(|\phi(0)|))+\frac{1}{2}h(\eta(|\phi(0)|))-\frac{1}{2r}\int_{-r}^{0}h(\eta(|\phi(\theta)|))d\theta$$
$$\leqslant -\frac{1}{2}h(\eta(|\phi(0)|))-\frac{1}{2r}\int_{-r}^{0}h(\eta(|\phi(\theta)|))d\theta.$$

According to Jessen's inequality, we have

$$\frac{1}{r}\int_{-r}^{0}h(\eta(|\phi(\theta)|))d\theta\geqslant h\left(\frac{1}{r}\int_{-r}^{0}\eta(|\phi(\theta)|)d\theta\right)$$

and

$$\dot{G}\leqslant -h\left(\frac{\eta(|\phi(0)|)+\frac{1}{r}\int_{-r}^{0}\eta(|\phi(\theta)|)d\theta}{2}\right)=-h\left(\frac{\|\phi\|_\eta}{2}\right). \tag{8}$$

Now that $k(s)=v(s)+\frac{w_1(1)}{2}rs$, it is strictly increasing, so is its invers function $k^{-1}(s)$. From (2) and (7), we have

$$G(t,\ \phi)\leqslant v(\|\phi\|_\eta)+\frac{w_1(1)}{2}\int_{-r}^{0}\eta(|\phi(\theta)|)d\theta$$
$$\leqslant k(\|\phi\|_\eta),$$

for $\eta(|\phi|)<1$, hence

$$k^{-1}(G(t,\ \phi))\leqslant\|\phi\|_\eta$$
$$\dot{G}(t,\ \phi)\leqslant -h\left(\frac{1}{2}k^{-1}(G(t,\ \phi))\right),$$

for $\eta(|\phi|)<1$. Since $g(s)=h\left(\frac{1}{4}k^{-1}(s)\right)$, the theorem has been proved.

Theorem 2. *Suppose the conditions of Theorem 1 are all satisfied. Let*

$$F(y)=\int_{1}^{y}\frac{dz}{g(z)},$$

then there is a positive number ε such that $|\phi_0|<\varepsilon$ implies

$$x(t;t_0,\ \phi_0)|\leqslant u^{-1}(F^{-1}(F(G(t_0,\ \phi_0))+t_0-t)) \tag{9}$$

and the solution $x=0$ is uniformly asymptotic stable.

Proof It is a matter of routine to show that $x=0$ is uniformly stable. Since $\eta(s)$ is continuous, we can find $\varepsilon>0$ such that $|\phi_0|<\varepsilon$ implies $\eta(|x(t;t_0,\ \phi_0)|)<1$, and

Theorem 1 is valid. Now that

$$\frac{\dot{G}(t, x_t)}{g(G(t, x_t))} \leqslant -1,$$

integrating from t_0 to t, we have

$$F(G(t, x_t)) - F(G(t_0, \phi_0)) \leqslant t_0 - t.$$

Hence $$G(t, x_t) \leqslant F^{-1}(F(G(t_0, \phi_0)) + t_0 - t).$$

The first part of (2) immediately leads to (9). Obviously $F(y)$ is strictly increasing and finite for positive y. Let $F(\delta) = \xi$ for $\delta > 0$, then $F^{-1}(z) < \delta$ for $z < \xi$. Finally

$$G(t_0, \phi_0) \leqslant k(\|\phi_0\|_\eta) \leqslant k(2\eta(|\phi_0|)),$$

so $x = 0$ is uniformly asymptotic stable.

Note. If we set $\eta(s) = S^2$, it is obvious that for $v_1, v_2 \in H$, there is a function $v \in H$ such that

$$v_1(|\phi(0)|) + v_2\left(\int_{-r}^{0} |\phi(\theta)|^2 d\theta\right) \leqslant v(\|\phi\|_\eta).$$

Conversely, for each $v \in H$, we can find out $v_1, v_2 \in H$ such that

$$v(\|\phi\|_\eta) \leqslant v_1(|\phi(0)|) + v_2\left(\int_{-r}^{0} |\phi(\theta)|^2 d\theta\right), \tag{10}$$

because $$v(\|\phi\|_\eta) \leqslant v(2\eta(|\phi(0)|)) + v\left(\frac{2}{r}\int_{-r}^{0} \eta(|\phi(\theta)|) d\theta\right),$$

and for $\eta \in H$ we can find out $\eta_1 \geqslant \eta$, $\eta_1 \in H$ and η_1 is concave. Thus

$$\frac{1}{r}\int_{-r}^{0} \eta(|\phi(\theta)|) d\theta \leqslant \frac{1}{r}\int_{-r}^{0} \eta_1(|\phi(\theta)|) d\theta \leqslant \eta_1\left(\frac{1}{r}\int_{-r}^{0} |\phi(\theta)| d\theta\right).$$

Using Hölder's inequality we can easily define v_1, v_2 to satisfy (10). Therefore, the summingly general form of (2) is indeed equaivalent to Burton's original condition.

Although we can actually use the formula here to calculate $g(s)$, we always prefer to find out $g(s)$ in each case respectively to get more exact estimate of the asymptotic behavior of the solutions, regarding the above theorems as justification of our process.

Example 1 (*Burton's original example*). Let

$$\dot{x}(t) = -(t+3)x(t) + x(t-1)$$

and $$V(t, x_t) = x^2(t)/2 + \int_{t-1}^{t} x^2(s) ds.$$

Then $$\dot{V}(t, x_t) \leqslant -x^2(t).$$

Let $$G(t, x_t) = V(t, x_t) + \xi\int_{t-1}^{t} (s-t+1)x^2(s) ds, \quad (0 < \xi < 1),$$

then $$\dot{G}(t, x_t) \leqslant -(1-\xi)x^2(t) - \xi\int_{t-1}^{t} x^2(s) ds,$$

$$G(t, x_t) \leqslant \frac{x^2(t)}{2} + (1+\xi)\int_{t-1}^{t} x^2(s) ds.$$

Set $\xi = \frac{\sqrt{17}-1}{4}$, we have

$$\dot{G}(t, x_t) \leqslant -\frac{5-\sqrt{17}}{2} G(t, x_t).$$

Thus $$\frac{x^2(t)}{2} \leqslant V \leqslant G(t, x_t) \leqslant e^{-\frac{5-\sqrt{17}}{2}(t-t_0)} G(t_0, \phi_0).$$

It follows that all solutions approrch to zero exponentially.

Example 2 (Hale). For

$$x(t) = a(t)x^3(t) + b(t)x^3(t-r),$$

where $a(t)$ and $b(t)$ are arbitrary continuous functions with $a(t) \leqslant -\delta < 0$. $|b(t)| < q\delta$ and $0 < q < 1$, one can choose

$$V(\phi) = \frac{\phi^4(0)}{4} + \frac{\delta}{2}\int_{-r}^{0} \phi^6(\theta)d\theta.$$

We can infer that the solution $x=0$ is unifonmly stable[1]. So we only need to consider the case $|\phi| < 1$.

$$\dot{V} = a(t)\phi^6(0) + b(t)\phi^3(0)\phi^3(-r) + \frac{\delta}{2}\phi^6(0) - \frac{\delta}{2}\phi^6(-r)$$
$$\leqslant -\frac{\delta}{2}\phi^6(0) - \frac{1}{2}\phi^6(-r) + q\delta|\phi(-r)|^3|\phi(0)|^3 \leqslant -p\phi^6(0).$$

where $p = \frac{\delta}{2}(1-q)^2$. Set

$$G = V + p\int_{-r}^{0}\left(\frac{\theta}{2r} + \frac{1}{2}\right)\phi^6(\theta)d\theta,$$

we have $$\dot{G} \leqslant -C_1\|\phi\|_6$$

and $$G \leqslant C_2\|\phi\|_4,$$

where C_1, $C_2 > 0$ are constants and

$$\|\phi\|_i = |\phi(0)|^i + \frac{1}{r}\int_{-r}^{0}|\phi(\theta)|^i d\theta, \quad i = 4, 6.$$

Since $\|\phi\|_6^{1.5} \geqslant C_3\|\phi\|_4$, we have

$$\dot{G}(t, \phi) \leqslant -C_4\{G(t, \phi)\}^{1.5},$$

where $C_4 > 0$ is a constant. Solving the above inequality, we know that there exists $\varepsilon > 0$ such that all the solutions with initial conditions $|x_{t_0}| < \varepsilon$ must approach zero at least as fast as $C(t-t_0)^{-\frac{1}{2}}$.

In each example we get more information about the asymptotic behavior of the solutions than before.

References

[1] Hale, J. K., Theory of Functional Differential Equations Chap. 5., Springer-Verlag, New York Inc., (1977).

[2] Burton, T. A., Uniform Asymptotic stability in functional differential equations, *Amer. Math. Soc.*, **68**: 2 (1978).

关于 Burton 渐近稳定性定理的注记

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摘 要

讨论泛函微分方程 $\dot{x}=f(t, x_t)$ 的解的渐近稳定性理论，往往需要假定 f 的某种全连续性．Burton 在他的论文中讨论了 f 是一般 $R\times C\to R^n$ 的连续泛函的情况．本文的目的是改进 Burton 的工作．证明方法采取更简单的直接证法，证明结果不但同样获得有关解的一致渐近稳定性的结论，而且得到一个有趣的不等式，从中能够导出解的收敛于 0 的估计式．

设 f 是 $R\times C\to R^n$ 连续泛函．η: $R^+\to R^+$ 是严格上升的连续函数，$\eta(0)=0$．设 u, v, w 是单调不减的连续函数，$u(0)=v(0)=w(0)=0$，且对 $s>0$ 有 $u(s), v(s), w(s)>0$，又设 $\|\phi\|_\eta=\eta(|\phi(0)|)+\frac{1}{r}\int_{-r}^{0}\eta(|\phi(\theta)|)\,d\theta$, $w_1(s)=w(\eta^{-1}(s))$, $h(s)=\int_0^s w_1(s)\,ds$, $k(s)=v(s)+\frac{w_1(1)}{2}rs$，那么有如下定理：

定理 1 设 V: $R\times C\to R$ 是连续泛函，使得

$$u(|\phi(0)|)\leqq V(t, \phi)\leqslant v(\|\phi\|_\eta),$$
$$\dot{V}(t, \phi)\leqq -w(|\phi(0)|),$$

那么必有另一个连续泛函 G: $R\times C\to R$，使得对 $\eta(|\mu|)<1$ 有

$$\dot{G}(t, \phi)\leqslant -g(G(t, \phi)),\ V(t, \phi)\leqslant G(t, \phi),$$

其中 g: $R^+\to R^+$ 定义为 $g(s)=h\left(\frac{1}{2}k^{-1}(s)\right)$

定理 2 设定理 1 的条件均满足，设 $F(y)=\int_1^y\frac{dz}{g(z)}$，那么存在 $\varepsilon>0$ 使得对于 $|\phi_0|<\varepsilon$ 有

$$|x(t; t_0, \phi_0)|\leqslant u^{-1}(F^{-1}(F(G(t_0, \phi_0))+t_0-t)),$$

且 $x=0$ 一致渐近稳定．

文章最后给出两个实例说明以上定理的应用．