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奇异积分 $\int_L \frac{f(\tau)}{(\tau-t)^{n+1}}d\tau$ 的 Hadamard 主值

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设函数 $f(\tau)$ 定义在复平面的简单光滑曲线 L 上，t 为 L 上不与端点重合的任一点，积分

$$\int_L \frac{f(\tau)}{(\tau-t)^{n+1}}d\tau \tag{1}$$

一般说来是奇异的，其中 n 为非负整数。

作为对 Cauchy 主值的推广，Fox, C.[1] 曾根据 Hadamard, J.[2,3] 从发散积分引出积分的有限部分的思想，定义积分 (1) 的主值(我们将称之为 Hadamard 主值)，并研究了它的一些性质和应用。1977 年路见可[4] 以另一种形式定义积分 (1) 的 Hadamard 主值，在曲线 L 是封闭时给出它的微分公式，同时把它应用于求解奇异积分方程。

本文目的是给出积分(1)在 Hadamard 主值意义下的微分公式，转换公式，合成公式和反转公式。

§1. Hadamard 主值

设复平面上简单光滑曲线 L 的始点为 a，终点为 b，t 为 L 上不与端点重合的任一点，以 t 为中心、δ 为半径的标准弧[5] 记作 λ_δ，其始点为 t_1，终点为 t_2，$L_\delta=L-\lambda_\delta$。

若函数 $f(\tau)$ 的 n 阶导数 $f^{(n)}(\tau)\in H(\alpha)$，(满足 Hölder 条件)，则记 $f(\tau)\in H_n(\alpha)$，这时成立不等式

$$\left|f(\tau)-\sum_{k=0}^{n}\frac{f^{(k)}(t)}{k!}(\tau-t)^k\right|<M|\tau-t|^{n+\alpha}, \tag{1.1}$$

其中 $\tau\in\lambda_\delta$，M 为正常数。

定义 1.1 设

$$H_0(f)=0$$

$$H_n(f)=\sum_{k=0}^{n-1}\frac{f^{(k)}(t)}{k!}\cdot\frac{1}{n-k}\left[\frac{1}{(t_2-t)^{n-k}}-\frac{1}{(t_1-t)^{n-k}}\right]\quad(n\geqslant1),$$

若极限

$$\lim_{\delta\to0}\left[\int_{L_\delta}\frac{f(\tau)}{(\tau-t)^{n+1}}d\tau-H_n(f)\right]$$

存在，则称之为积分(1)的 Hadamard 主值。

当 $n=0$ 时，它就是 Cauchy 主值。又若曲线 L 是封闭的，则认定 $a=b$ 是 L 上不与 t 重合的任一点。本文的高阶奇异积分均理解为 Hadamard 主值。

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定理 1.1 设$f(\tau)\in H_n(\alpha)$，则积分(1)的 Hadamard 主值存在，且有

$$\int_L \frac{f(\tau)}{(\tau-t)^{n+1}}d\tau=\frac{1}{n}\left[\int_L \frac{f'(\tau)}{(\tau-t)^n}d\tau+\frac{1}{(a-t)^n}-\frac{1}{(b-t)^n}\right] \tag{1.2}$$

及

$$\int_L \frac{f(\tau)}{(\tau-t)^{n+1}}d\tau=\frac{1}{n!}\left[\int_L \frac{f^{(n)}(\tau)}{\tau-t}d\tau+\sum_{k=0}^{n}(n-k-1)!\left[\frac{1}{(a-t)^{n-k}}-\frac{1}{(b-t)^{n-k}}\right]\right]. \tag{1.3}$$

证 容易算出

$$H_n(f)=\frac{1}{n}H_{n-1}(f')+\frac{1}{n}\sum_{k=0}^{n-1}\frac{f^{(k)}(t)}{k!}\left[\frac{1}{(t_2-t)^{n-k}}-\frac{1}{(t_1-t)^{n-k}}\right].$$

由此及分部积分公式可得

$$\int_L \frac{f(\tau)}{(\tau-t)^{n+1}}d\tau-H_n(f)=\frac{1}{n}\left[\int_L \frac{f'(\tau)}{(\tau-t)^n}d\tau+\frac{f(a)}{(a-t)^n}-\frac{f(b)}{(b-t)^n}\right]+\frac{1}{n}\sum_{j=1}^{2}\frac{(-1)^j}{(t_j-t)^n}\left[f(t_j)-\sum_{k=0}^{n}\frac{f^{(k)}(t)}{k!}(t_j-t)^k\right].$$

利用(1.1)立刻推出(1.2)，连续使用(1.2)可得(1.3)。

定义 1.1 是由 Fox, C. 给出的，他还在$f(x)$是区间(a, b)上的实函数时利用微分中值定理写出定理 1.1 的证明．路见可[4]则提出以(1.3)作为积分(1)的主值的定义．并对多项式$P(\tau)$定义了积分$\int_L \frac{f(\tau)}{P(\tau)}d\tau$的主值．以下如遇此类积分将采用他的定义．

§2. 微分公式

定理 2.1 若$f(\tau)\in H_{n+1}(\alpha)$，则

$$\frac{d}{dt}\int_L \frac{f(\tau)}{(\tau-t)^{n+1}}d\tau=(n+1)\int_L \frac{f(\tau)}{(\tau-t)^{n+2}}d\tau. \tag{2.1}$$

证 仿照 Гахов[6] 4.4 段的方法可证明

$$\frac{d}{dt}\int_L \frac{f(\tau)}{\tau-t}d\tau=\int_L \frac{f'(\tau)}{\tau-t}d\tau+\frac{f(a)}{a-t}-\frac{f(b)}{b-t}.$$

依(1.3)上式可写成

$$\frac{d}{dt}\int_L \frac{f(\tau)}{\tau-t}d\tau=\int_L \frac{f(\tau)}{(\tau-t)^2}d\tau.$$

利用数学归纳法及(1.2)即可推出(2.1)。

设L_j为复平面上简单光滑曲线，t_j为L_j上不与端点重合的任一点，$\lambda_{j\delta}$表L_j上以t_j为中心δ为半径的标准弧，$L_{j\delta}=L_j-\lambda_{j\delta}$ $(j=1, 2)$．$\varphi(\tau_1, \tau_2)$定义在$L_1\times L_2$上．记

$$\varphi^{(k_1,k_2)}(\tau_1, \tau_2)=\frac{\partial^{k_1+k_2}\varphi(\tau_1, \tau_2)}{\partial\tau_1^{k_1}\partial\tau_2^{k_2}}$$

$$R[\varphi(\tau_1, \tau_2), \Delta\tau_2]=\varphi(\tau_1, \tau_2+\Delta\tau_2)-\varphi(\tau_1, \tau_2)-\varphi^{(0,1)}(\tau_1, \tau_2)\Delta\tau_2.$$

类似于(1.1)，若$\varphi^{(0,1)}(\tau_1, \tau_2)\in H(\alpha_1, \alpha_2)$（见[5]），则当$\tau_2\in\lambda_{2\delta}$时

$$|R[\varphi(\tau_1, \tau_2), \Delta\tau_2]|<K|\Delta\tau_2|^{1+\alpha_2}. \tag{2.2}$$

其中K为与τ_1无关的正常数．由此立刻得到

定理 2.2 若 $\varphi^{(0,1)}(\tau_1, \tau_2)\in H(\alpha_1, \alpha_2)$*, 则

$$\frac{d}{d\tau_2}\int_{L_1}\varphi(\tau_1, \tau_2)\,d\tau_1=\int_{L_1}\frac{\partial\varphi(\tau_1, \tau_2)}{\partial\tau_2}\,d\tau_1. \tag{2.3}$$

定理 2.3 若 $\varphi^{(n,1)}(\tau_1, \tau_2)\in H(\alpha_1, \alpha_2)$, 则

$$\frac{\partial}{\partial\tau_2}\int_{L_1}\frac{\varphi(\tau_1, \tau_2)}{(\tau_1-t_1)^{n+1}}\,d\tau_1=\int_{L_1}\frac{\partial\varphi(\tau_1, \tau_2)}{\partial\tau_2}\cdot\frac{d\tau_1}{(\tau_1-t_1)^{n+1}}. \tag{2.4}$$

证 我们先证 $n=0$ 时定理成立, 即

$$\frac{\partial}{\partial\tau_2}\int_{L_1}\frac{\varphi(\tau_1, \tau_2)}{\tau_1-t_1}\,d\tau_1=\int_{L_1}\frac{\partial\varphi(\tau_1, \tau_2)}{\partial\tau_2}\cdot\frac{d\tau_1}{\tau_1-t_1}. \tag{2.5}$$

令

$$\Delta I=\left(\int_{L_{1\delta}}+\int_{\lambda_{1\delta}}\right)\left[\frac{\varphi(\tau_1, \tau_2+\Delta\tau_2)-\varphi(\tau_1, \tau_2)}{\Delta\tau_2}-\frac{\partial\varphi(\tau_1, \tau_2)}{\partial\tau_2}\right]\cdot\frac{d\tau_1}{\tau_1-t_1}$$
$$=I_1+I_2.$$

依定理 2.2, $\lim\limits_{\Delta\tau_2\to0}I_1=0$. 可写

$$I_2=\frac{1}{\Delta\tau_2}\int_{\lambda_{1\delta}}\{R[\varphi(\tau_1, \tau_2), \Delta\tau_2]-R[\varphi(t_1, \tau_2), \Delta\tau_2]\}\cdot\frac{d\tau_1}{\tau_1-t_1}$$
$$+\frac{1}{\Delta\tau_2}\int_{\lambda_{1\delta}}R[\varphi(t_1, \tau_2), \Delta\tau_2]\cdot\frac{d\tau_1}{\tau_1-t_1}.$$

由(2.2)及 $\varphi^{(0,1)}(\tau_1, \tau_2)\in H(\alpha_1, \alpha_2)$ 知, 对正整数 m_1 和 m_2.

$$|R[\varphi(\tau_1, \tau_2), \Delta\tau_2]-R[\varphi(t_1, \tau_2), \Delta\tau_2]|^{m_1+m_2}<[A_1|\tau_1-t_1|^{\alpha_1}]^{m_1}\cdot[A_2|\Delta\tau_2|^{1+\alpha_2}]^{m_2},$$

其中 A_1 和 A_2 为正常数, 适当选取 m_1 和 m_2 可得

$$|R[\varphi(\tau_1, \tau_2), \Delta\tau_2]-R[\varphi(t_1, \tau_2), \Delta\tau_2]|<A|\tau_1-t_1|^{\beta_1}\cdot|\Delta\tau_2|^{1+\beta_2}, \tag{2.6}$$

其中 A, β_1, β_2 均为正常数. 故

$$|I_2|\leqslant A|\Delta\tau_2|^{\beta_2}\cdot\int_{\lambda_{1\delta}}\frac{|d\tau_1|}{|\tau_1-t_1|^{1-\beta_1}}+K|\Delta\tau_2|^{\alpha_2}\cdot\left|\int_{\lambda_{1\delta}}\frac{d\tau_1}{\tau_1-t_1}\right|.$$

由此立刻得到 $\lim\limits_{\Delta\tau_2\to0}I_2=0$, 所以 $\lim\limits_{\Delta\tau_2\to0}\Delta I=0$, 即(2.5)成立, 使用定理 1.1 和数学归纳法就可证明(2.4).

定理 2.4 设 $\varphi(\tau, t)$ 定义在 $L\times L$ 上, $\varphi^{(1,0)}(\tau, t)\in H(\alpha_1, \alpha_2)$, $\varphi^{(0,1)}(\tau, t)\in H(\alpha_1, \alpha_2)$, 那末

$$\frac{d}{dt}\int_L\frac{\varphi(\tau, t)}{\tau-t}\,d\tau=\int_L\frac{\varphi(\tau, t)}{(\tau-t)^2}\,d\tau+\int_L\frac{\partial\varphi(\tau, t)}{\partial t}\cdot\frac{d\tau}{\tau-t}. \tag{2.7}$$

证 令

$$I=\frac{1}{\Delta t}\int_L\left[\frac{\varphi(\tau, t+\Delta t)}{\tau-t-\Delta t}-\frac{\varphi(\tau, t)}{\tau-t}\right]d\tau$$
$$=\frac{1}{\Delta t}\int_{L_\delta}\left[\frac{\varphi(\tau, t+\Delta t)}{\tau-t-\Delta t}-\frac{\varphi(\tau, t)}{\tau-t}\right]d\tau+\frac{1}{\Delta t}\int_{\lambda_\delta}\left[\frac{\varphi(\tau, t)}{\tau-t-\Delta t}-\frac{\varphi(\tau, t)}{\tau-t}\right]d\tau$$
$$+\frac{1}{\Delta t}\int_{\lambda_\delta}\frac{\varphi(\tau, t+\Delta t)-\varphi(\tau, t)}{\tau-t-\Delta t}\,d\tau=I_1+I_2+I_3.$$

由定理 2.2

$$\lim_{\Delta t\to0}I_1=\int_{L_\delta}\frac{\varphi(\tau, t)}{(\tau-t)^2}\,d\tau+\int_{L_\delta}\frac{\partial\varphi(\tau, t)}{\partial t}\cdot\frac{d\tau}{\tau-t}.$$

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* 此条件可减轻为 $\varphi^{(0,1)}(\tau_1, \tau_2)$ 在 $L_1\times L_2$ 上连续.

由定理 2.1
$$\lim_{\Delta t\to 0} I_2=\int_{\lambda_\delta}\frac{\varphi(\tau,\ t)}{(\tau-t)^2}\,d\tau.$$
可写
$$I_3=\frac{1}{\Delta t}\int_{\lambda_\delta}\frac{R[\varphi(\tau,\ t),\ \Delta t]-R[\varphi(t+\Delta t,\ t),\ \Delta t]}{\tau-t-\Delta t}\,d\tau$$
$$+\frac{1}{\Delta t}\int_{\lambda_\delta}\frac{R[\varphi(t+\Delta t,\ t),\ \Delta t]}{\tau-t-\Delta t}+\int_{\lambda_\delta}\frac{\varphi^{(0,1)}(\tau,\ t)}{\tau-t-\Delta t}\,d\tau.$$
注意到(2.2)和(2.6)，令 $\Delta t\to 0$ 可得
$$\lim_{\Delta t\to 0} I_3=\int_{\lambda_\delta}\frac{\varphi^{(0,1)}(\tau,\ t)}{\tau-t}\,d\tau=\int_{\lambda_\delta}\frac{\partial\varphi(\tau,\ t)}{\partial t}\cdot\frac{d\tau}{\tau-t}.$$
综上所述立刻得到(2.7).

用数学归纳法不难从定理 2.4 和定理 1.1 证明

定理 2.5 设 $\varphi(\tau,\ t)$ 定义在 $L\times L$ 上，且 $\varphi^{(n+1,0)}(\tau,\ t)\in H(\alpha_1,\ \alpha_2)$，$\varphi^{(n,1)}(\tau,\ t)\in H(\alpha_1,\ \alpha_2)$，那末
$$\frac{d}{dt}\int_L\frac{\varphi(\tau,\ t)}{(\tau-t)^{n+1}}\,d\tau=\int_L\frac{\partial\varphi(\tau,\ t)}{\partial t}\cdot\frac{d\tau}{(\tau-t)^{n+1}}+(n+1)\int_L\frac{\varphi(\tau,\ t)}{(\tau-t)^{n+2}}\,d\tau. \qquad (2.8)$$
当 L 是封闭曲线时，(2.8)曾由路见可[4]得到.

可见，奇异积分的 Hadamard 主值允许在积分号下求导数.

§3. 转换公式，合成公式和反转公式

本节假定曲线 L 是封闭的.

引理 3.1 成立等式
$$\frac{1}{\tau-t}\cdot\frac{1}{(\tau_1-\tau)^{n+1}}=\frac{1}{(\tau_1-t)^{n+1}}\cdot\frac{1}{\tau-t}+\sum_{k=0}^{n}\frac{1}{(\tau_1-t)^{k+1}}\cdot\frac{1}{(\tau_1-\tau)^{n-k+1}} \qquad (3.1)$$
及
$$\frac{1}{(\tau-t)^{n+1}}\cdot\frac{1}{(\tau_1-\tau)^{m+1}}=\sum_{l=0}^{n}C_{m+l}^{l}\frac{1}{(\tau_1-t)^{m+l+1}}\cdot\frac{1}{(\tau-t)^{n-l+1}}$$
$$+\sum_{k=0}^{m}C_{n+k}^{k}\frac{1}{(\tau_1-t)^{n+k+1}}\cdot\frac{1}{(\tau_1-\tau)^{m-k+1}}. \qquad (3.2)$$

引理 3.2 设 $\varphi(\tau,\ \tau_1)$ 定义在 $L\times L$ 上，$\varphi^{(l,n-l+1)}(\tau_1,\ \tau_2)\in H(\alpha_1,\ \alpha_2)$，$(l=0,\ 1,\cdots,\ n+1)$，那末
$$\int_L d\tau_1\int_L\frac{\varphi(\tau,\ \tau_1)}{(\tau-t)(\tau_1-\tau)^{n+1}}\,d\tau=\frac{1}{n}\int_L d\tau_1\int_L\frac{\partial\varphi(\tau,\ \tau_1)}{\partial\tau_1}\cdot\frac{d\tau}{(\tau-t)(\tau_1-\tau)^{n}}. \qquad (3.3)$$
证 由定理 1.1 及定理 2.5
$$\int_L d\tau_1\int_L\frac{\varphi(\tau,\ \tau_1)}{(\tau_1-t)^{k+1}(\tau_1-\tau)^{n-k+1}}\,d\tau$$
$$=\frac{1}{k}\int_L\frac{1}{(\tau_1-t)^{k}}\left[\int_L\frac{\partial\varphi(\tau,\ \tau_1)}{\partial\tau_1}\cdot\frac{d\tau}{(\tau_1-\tau)^{n-k+1}}-(n-k+1)\int_L\frac{\varphi(\tau,\ \tau_1)d\tau}{(\tau_1-\tau)^{n-k+2}}\right]d\tau_1.$$
由此得

$$\frac{k}{n}\int_L d\tau_1\int_L \frac{\varphi(\tau,\ \tau_1)d\tau}{(\tau_1-t)^{k+1}(\tau_1-\tau)^{n-k+1}}+\int_L d\tau_1\int_L \frac{\varphi(\tau,\ \tau_1)}{(\tau_1-t)^{k}(\tau_1-\tau)^{n-k+2}}d\tau$$
$$=\frac{1}{n}\int_L d\tau_1\int_L \frac{1}{(\tau_1-t)^{k}(\tau_1-\tau)^{n-k+1}}\cdot\frac{\partial\varphi(\tau,\ \tau_1)}{\partial\tau_1}d\tau$$
$$+\frac{k-1}{n}\int_L d\tau_1\int_L \frac{\varphi(\tau,\ \tau_1)}{(\tau_1-t)^{k}(\tau_1-\tau)^{n-k+2}}d\tau.$$

依次令 $k=1,\ 2,\ \cdots,\ n$，两边作和，经整理可得

$$\int_L d\tau_1\int_L \sum_{k=0}^{n}\frac{\varphi(\tau,\ \tau_1)}{(\tau_1-t)^{k+1}(\tau_1-\tau)^{n-k+1}}d\tau$$
$$=\frac{1}{n}\int_L d\tau_1\int_L \sum_{k=0}^{n-1}\frac{1}{(\tau_1-t)^{k+1}(\tau_1-\tau)^{(n-1)-k+1}}\cdot\frac{\partial\varphi(\tau,\ \tau_1)}{\partial\tau_1}d\tau. \tag{3.4}$$

利用(3.1)，(3.4)并注意到定理1.1和定理2.3．即可推出(3.3)．

定理3.1　设 $\varphi(\tau,\ \tau_1)$ 定义在 $L\times L$ 上，且满足

$$\frac{\partial^{n+m+2}\varphi(\tau,\ \tau_1)}{\partial\tau^l\partial\tau_1^{n+m+2-l}}\in H(\alpha_1,\ \alpha_2),\quad (l=0,\ 1,\ 2,\ \cdots,\ n+m+2),$$

那末

$$\frac{d}{dt}\int_L d\tau_1\int_L \frac{\varphi(\tau,\ \tau_1)}{(\tau-t)^{n+1}(\tau_1-\tau)^{m+1}}d\tau=(n+1)\int_L d\tau_1\int_L \frac{\varphi(\tau,\ \tau_1)}{(\tau-t)^{n}(\tau_1-\tau)^{m+1}}d\tau. \tag{3.5}$$

证　利用(3.2)及定理2.5和定理2.1可得

$$\frac{d}{dt}\int_L d\tau_1\int_L \frac{\varphi(\tau,\ \tau_1)}{(\tau-t)^{n+1}(\tau_1-\tau)^{m+1}}d\tau$$
$$=\sum_{l=0}^{n}C_{m+l}^{l}\cdot(m+l+1)\int_L \frac{d\tau_1}{(\tau_1-t)^{m+l+2}}\int_L \frac{\varphi(\tau,\ \tau_1)}{(\tau-t)^{n-l+1}}d\tau$$
$$+\sum_{l=0}^{n}C_{m+l}^{l}\cdot(n-l+1)\int_L \frac{d\tau_1}{(\tau_1-t)^{m+l+1}}\int_L \frac{\varphi(\tau,\ \tau_1)}{(\tau-t)^{n-l+2}}d\tau$$
$$+\sum_{k=0}^{m}C_{n+k}^{k}\cdot(n+k+1)\int_L \frac{d\tau_1}{(\tau_1-t)^{n+k+2}}\int_L \frac{\varphi(\tau,\ \tau_1)}{(\tau_1-\tau)^{m-k+1}}d\tau.$$

经整理并再次利用引理3.1即可得到(3.5)．

定理3.2　设 $\varphi(\tau,\ \tau_1)$ 定义在 $L\times L$ 上，且满足

$$\frac{\partial^{n+m+1}\varphi(\tau,\ \tau_1)}{\partial\tau^l\partial\tau_1^{n+m+1-l}}\in H,\quad (l=0,\ 1,\ \cdots,\ n+m+1),$$

则有转换公式

$$\int_L \frac{d\tau}{(\tau-t)^{n+1}}\int_L \frac{\varphi(\tau,\ \tau_1)}{(\tau_1-\tau)^{m+1}}d\tau_1=-\frac{\pi^2}{n!m!}\sum_{l=0}^{n}C_n^l\varphi^{(l,\ n+m-l)}(t,\ t)$$
$$+\int_L d\tau_1\int_L \frac{\varphi(\tau,\ \tau_1)}{(\tau-t)^{n+1}(\tau_1-\tau)^{m+1}}d\tau. \tag{3.6}$$

证　由Cauchy主值的Poincare-Bertrand转换公式出发，对 m 使用数学归纳法，依次使用定理1.1，归纳假设和引理3.2就可证明当 $n=0$ 时定理3.2成立．再对 n 使用数学归纳法，依次使用定理2.1，归纳假设和定理3.1就可证明(3.6)．

特别地，若取 $\varphi(\tau,\ \tau_1)\equiv\varphi(\tau_1)$，则得

$$\int_L \frac{d\tau}{(\tau-t)^{n+1}} \int_L \frac{\varphi(\tau_1)}{(\tau_1-\tau)^{m+1}} d\tau_1$$

$$= -\frac{\pi^2}{n!m!} \varphi^{(n+m)}(t) + \int_L d\tau_1 \int_L \frac{\varphi(\tau_1)}{(\tau-t)^{n+1}(\tau_1-\tau)^{m+1}} d\tau. \tag{3.7}$$

定理 1.1. 定理 2.1 及 Cauchy 主值的合成公式得

定理 3.3(合成公式). 若在简单光滑闭曲线 L 上, $\varphi(\tau) \in H_{n+m}(\alpha)$, 那末

$$\int_L \frac{d\tau}{(\tau-t)^{n+1}} \int_L \frac{\varphi(\tau_1)}{(\tau_1-\tau)^{m+1}} d\tau_1 = -\frac{\pi^2}{n!m!} \varphi^{(n+m)}(t). \tag{3.8}$$

由此我们立刻得到

定理 3.4(反转公式). 设 $\varphi(\tau)$ 定义在简单光滑闭曲线 L 上, 且 $\varphi(\tau) \in H_{n+m}(\alpha)$, 若记

$$\psi(t) = \frac{m!}{\pi i} \int_L \frac{\varphi(\tau)}{(\tau-t)^{m+1}} d\tau$$

那末

$$\varphi^{(n+m)}(t) = \frac{n!}{\pi i} \int_L \frac{\psi(\tau)}{(\tau-t)^{n+1}} d\tau. \tag{3.9}$$

(3.8) 与 (3.9) 是等价的. 又若比较 (3.7) 与 (3.8) 可以发现

$$\int_L d\tau_1 \int_L \frac{\varphi(\tau_1)}{(\tau-t)^{n+1}(\tau_1-\tau)^{m+1}} d\tau = 0. \tag{3.10}$$

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HADAMARD'S PRINCIPAL VALUE OF THE SINGULAR INTEGRAL

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ABSTRACT

In this paper we consider the Hadamard's principal value for the singular integral

$$\int_L \frac{f(\tau)}{(\tau-t)^{n+1}} d\tau,$$

where L is a smooth curve on the complex plane, t is any point on L, but it is not the end point of L, n is a non-negative integer. We obtain the following result:

Theorem 1. If the function $\varphi(\tau_1, \tau_2)$ is defined on $L_1 \times L_2$, and $\frac{\partial^{n+1}\varphi(\tau_1, \tau_2)}{\partial\tau_1^n \partial\tau_2} \in H$, then

$$\frac{\partial}{\partial\tau_2}\int_{L_1} \frac{\varphi(\tau_1, \tau_2)}{(\tau_1-t_1)^{n+1}} d\tau_1 = \int_{L_1} \frac{\partial\varphi(\tau_1, \tau_2)}{\partial\tau_2} \cdot \frac{d\tau_1}{(\tau_1-t_1)^{n+1}}.$$

Theorem 2. If the function $\varphi(\tau, t)$ defined on $L \times L$ satisfies the conditions

$$\frac{\partial^{n+1}\varphi(\tau, t)}{\partial\tau^{n+1}} \in H, \quad \frac{\partial^{n+1}\varphi(\tau, t)}{\partial t\, \partial\tau^n} \in H,$$

then $$\frac{d}{dt}\int_L \frac{\varphi(\tau, t)}{(\tau-t)^{n+1}} d\tau = \int_L \frac{\partial\varphi(\tau, t)}{\partial t} \cdot \frac{d\tau}{(\tau-t)^{n+1}} + (n+1)\int_L \frac{\varphi(\tau, t)}{(\tau-t)^{n+2}} d\tau.$$

Theorem 3. (Substitution formula). If L is a smooth simple closed curve, the function $\varphi(\tau, \tau_1)$ defined on the $L \times L$ satisfies the following conditions

$$\frac{\partial^{n+m+1}\varphi(\tau, \tau_1)}{\partial\tau^l \partial\tau_1^{n+m+1-l}} \in H \quad (l=0, 1, \cdots, n+m+1),$$

then

$$\int_L \frac{d\tau}{(\tau-t)^{n+1}} \int_L \frac{\varphi(\tau, \tau_1)}{(\tau_1-\tau)^{m+1}} d\tau_1 = \frac{-\pi^2}{n!m!}\sum_{l=0}^{n} C_n^l \frac{\partial^{n+m}\varphi(\tau, \tau_1)}{\partial\tau^l \partial\tau_1^{n+m-l}}\Bigg|_{\tau=\tau_1=t} + \int_L d\tau_1 \int_L \frac{\varphi(\tau, \tau_1)}{(\tau-t)^{n+1}(\tau_1-\tau)^{m+1}} d\tau.$$

Theorem 4. If the function $\varphi(\tau)$ is defined on the smooth simple closed curve L, and $\varphi^{(n+m)}(\tau) \in H$, then we have

$$\int_L \frac{d\tau}{(\tau-t)^{n+1}} \int_L \frac{\varphi(\tau_1)}{(\tau_1-\tau)^{m+1}} d\tau_1 = -\frac{\pi^2}{n!m!}\varphi^{(n+m)}(t),$$

and if we write

$$\psi(t)=\frac{m!}{\pi i}\int_L \frac{\varphi(\tau)}{(\tau-t)^{m+1}}\,d\tau,$$

then we have the following inversion formula

$$\varphi^{(n+m)}(t)=\frac{n!}{\pi i}\int_L \frac{\psi(\tau)}{(\tau-t)^{n+1}}\,d\tau.$$