Chin. Ann. of Math.
3 (3) 1982

ABSOLUTE STABILITY OF LINEAR FUNCTIONAL REGULATED SYSTEM WITH *M* NONLINEAR REGULATORS

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§ 1. Introduction

In this paper we consider the following linear functional regulated system with m nonlinear regulators

$$\begin{aligned} \frac{dx(t)}{dt} &= L(x_t) + Rf(\sigma(t)), \\ \sigma(t) &= Cx(t), \end{aligned} \tag{1}$$

and

$$\begin{aligned} \frac{dx(t)}{dt} &= L(x_t) + Rf(\sigma(t)), \\ \frac{d\xi(t)}{dt} &= f(\sigma(t)), \ \sigma(t) = Cx(t) + D\xi(t), \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{dx(t)}{dt} &= L(x_t) + R\xi(t), \\ \frac{d\xi(t)}{dt} &= f(\sigma(t)), \ \sigma(t) = Cx(t) + D\xi(t), \end{aligned} \tag{3}$$

where x is n vector, ξ, σ are m vectors, $m \geqslant 1$, $C([-\tau, 0], R^n)$ is the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^n, $\tau > 0$ is a constant, $x_t \in C([-\tau, 0], R^n)$ is defined by $x_t(\theta) = x(t+\theta)$, $-r \leqslant \theta \leqslant 0$. L is a continuous, linear mapping form $C([-\tau, 0], R^n)$ into R^n. D is $m \times m$ matrix. C is $m \times n$ matrix, R is $n \times m$ matrix, elements in C, D, R are real constants, $f(\sigma)$ is m vector, $f(0) = 0$, $L(x_t) = \int_{-\tau}^{0} d_s\eta(s) x(t+s)$, $\eta(s)$ is $n \times n$ matrix function on $[-\tau, 0]$ of bounded variation.

For systems with $L(x_t) = Ax(t) + Bx(t-\tau)$ and any function $f(\sigma)$ of the from $\{f_i(\sigma_i)\}$, Li Xunjin[1] gives conditions which assure an absolute stability by Popov's method. For systems (1), (2), (3) with $m=1$, Halanay[2] gives conditions which assure an absolute stability.

In this paper we give some sufficient conditions which assure a generalized H-absolute stability for (1), (2), (3) with $m \geqslant 1$.

Manuscript received Aug. 26, 1980.

In § 3 of this paper we also consider

$$\frac{dx(t)}{dt}=L(x_t)+R\xi(t),$$
$$\frac{d^2\xi(t)}{dt^2}=f(\sigma(t)),\ \sigma(t)=Cx(t)+B_1\xi(t)+B_2\frac{d\xi(t)}{dt}, \tag{1'}$$

and

$$\frac{dx(t)}{dt}=L(x_t)+Rf(\sigma(t-\tau)),$$
$$\sigma(t)=Cx(t), \tag{2'}$$

where B_1, B_2 are $m\times m$ matrices.

Definetion 1. *A system is said to be a generalized H-absolute stable system, if this system is an absolute stable system for all functions $f(\sigma)$ satisfing*

$$0<\sigma_j f_j(\sigma)<\sigma_j\sum_{k=1}^{n} h_{jk}\sigma_k\quad(\sigma_i\neq 0,\ j=1,\ 2,\ \cdots,\ m), \tag{4}$$

where $H=(h_{ik})$ is $m\times m$ symmetrical positive definite matrix.

We generalize and simplify the methods in papers [1, 2]. Indeed, if $f_j(\sigma)=f_j(\sigma_j)$, $j=1,\ 2,\ \cdots m$ and H is $m\times m$ diagonal matrix with positive elements, then a generalized H-absolute stability implies a H-absoluts stability.

We use the following notations:

(i) $\|A\|=\sum_{i,j=1}^{n}|a_{ij}|,\ A=(a_{ij}),\ i,\ j=1,\ 2,\ \cdots,\ n;$

(ii) $\|x\|=\left(\sum_{k=1}^{n}|x_k|^2\right)^{1/2},\ \bar{x}'=(x_1,\ x_2,\ \cdots,\ x_n);$

(iii) $\langle x,y\rangle=\sum_{i=1}^{n}x_i\bar{y}_i;$

(iv) $x(\cdot)\in C([-\tau,\ 0],\ R^n),\ \|x(\cdot)\|=\max_{-\tau\le\theta\le 0}\|x(\theta)\|;$

(v) $\Gamma(\lambda)=\lambda E-\int_{-\tau}^{0}d\eta(\theta)e^{\lambda\theta}$, E is $n\times n$ unit matrix;

(vi) $G(\lambda)=-C\Gamma^{-1}(\lambda)R$, if $\Gamma^{-1}(\lambda)$ exists.

§ 2. Sufficient Conditions for Generalized H-absolute Stability of System (1), (2), (3)

(A) Some theorems.

Theorem 1. *Suppose that the solution $x=0$ of linear system*

$$\frac{dx(t)}{dt}=L(x_t) \tag{5}$$

is an asymptotically stable. If there exist $m\times m$ diagonal matrices P, Q, with the following properties:

(i) *The elements of matrices P, Q are positive or zero;*

(ii) *For all real ω*

$$W(\omega)=P+\frac{1}{2}\{(P+i\omega Q)HG(i\omega)+[(P+i\omega Q)HG(i\omega)]^*\} \tag{6}$$

is a positive definite matrix. Here matrix A^* is the conjugate of A, if A is complex;

(iii) $S=\lim_{\omega\to\infty} W(\omega)=P-\frac{1}{2}[QHCR+(QHCR)^*)]$

also is a positive definite matrix, then system (1) is a generalized H-absolute stable.

Theorem 2. *Suppose that the solution $x=0$ of linear system (5) is an asymptotically stable and* $\det D\neq 0$. *Suppose that there exist $m\times m$ diagonal matrices P, Q, with the following properties:*

(i) *The elements of matrices P, Q are positive or zero. PHD is the symmetric and negative definete matrix,*

(ii) *For all real ω*

$$\widetilde{W}(\omega)=P+\frac{1}{2}\{(P+i\omega Q)H\widetilde{G}(i\omega)+[(P+i\omega Q)H\widetilde{G}(i\omega)]^*\} \tag{7}$$

is a positive definite matrix, where

$$\widetilde{G}(i\omega)=-G(i\omega)-\frac{D}{i\omega}=-C\Gamma^{-1}(i\omega)R-\frac{D}{i\omega}, \tag{8}$$

$$\widetilde{S}=\lim_{\omega\to\infty}\widetilde{W}(\omega)=P-\{QH(CR+D)+[QH(CR+D)]^*\} \tag{9}$$

is a positive definite matrix.

Then the system (2) is a generalized H-absolute stable.

Theorem 3. *Suppose for the system (3) conditions of Theorem 2 are modified by the following:*

(i) *We take the symmetric and negative definite matrix*

$$PH\left[D+\int_0^{+\infty}CX(\beta)R\,d\beta\right] \tag{10}$$

in place of PHD, where $X(t)$ is the matrix of solutions of system (5) which verifies conditions $X(0)=E$ (united matrix), $X(t)\equiv 0$ for $t<0$;

(ii) *We take*

$$\widetilde{\widetilde{G}}(i\omega)=-\frac{1}{i\omega}(C\Gamma^{-1}(i\omega)R+D) \tag{10$'$}$$

in place of $\widetilde{G}(i\omega)$.

Then the system (3) is a generalized H-absolute stable.

(B) *Some Lemmas:*

Lemma 1. *Let*

$$\frac{dx(t)}{dt}=L(x_t)+f(t), \tag{11}$$

where $f(t)$ is n vector.

Assume that

(i) *The solution $x=0$, of system (5) is an asymptotically stable,*

(ii) *$f(t)$ is L^2 integrable for all finite interval and $\int_0^T \|f(t)\|^2dt\leqslant\beta^2$.*

Then $\|x(t)\| \leqslant M'\beta + M''\|x(\cdot)\|$ *for* $0<t<T$, *where* M', M'', β *are constants.*

Also if we take

$$\int_0^{+\infty} \|f(t)\|^2 dt \leqslant \beta^2 < +\infty$$

in place of $\int_0^T \|f(t)\|^2 dt \leqslant \beta^2$, *then* $\lim\limits_{t\to+\infty} x(t) = 0$.

Proof To prove this Lemma we need

$$\|X(t)\| \leqslant \tilde{L} e^{-\gamma t}, \tag{11}$$

where $\tilde{L} \geqslant 0$, $\gamma > 0$, $X(t)$ is the matrix of solutions of system (5) which verifies conditions $X(0) = E$ (united matrix), $X(t) \equiv 0$ for $t<0$. Also We need

$$x^T(t) = x^T(0) X(t) + \int_{-\tau}^0 x^T(s) d_s \int_0^\tau \eta(s-\alpha) X(t-\alpha) d\alpha + \int_0^t f^T(\alpha) X(t-\alpha) d\alpha, \tag{12}$$

where $x(t)$ is the solution of (11).

Lemma 2. *Let*

$$\sigma(t) = z(t) + \int_0^t K(t-\alpha) f(\sigma(\alpha)) d\alpha, \tag{13}$$

where $\sigma(t)$, $z(t)$, $f(\sigma(\alpha))$ *are* m *vectors,* $K(t-\alpha)$ *is* $m\times m$ *matrix, and we have*

$$\|\sigma(\cdot)\| \leqslant C_1 \|\varphi\|,$$

$$\|z(t)\| + \left\| \frac{dz(t)}{dt} \right\| \leqslant A_1 \|\varphi\| e^{-\gamma t},$$

$$\|K(t)\| + \left\| \frac{dK(t)}{dt} \right\| \leqslant A_2 e^{-\gamma t}, \tag{14}$$

where $C_1>0$, $A_1>0$, $A_2>0$, $\gamma>0$.

Suppose that there exist $m\times m$ *diagonal matrices* P, Q, *with the following properties:*

(i) *The elements of matrices* P, Q, *are positive or zero;*

(ii) *For all real* ω

$$W(\omega) = P - \frac{1}{2}\{(P+i\omega Q) P\tilde{K}(\omega) + [(P+i\omega Q) H\tilde{K}(\omega)]^*\} \tag{15}$$

is a positive definite matrix, where H *is symmetric and positive definite matrix,* $\tilde{K}(\omega)$ *is the Fourier transformation of* $K(t)$;

(iii) $S = \lim\limits_{\omega\to\infty} W(\omega)$ *is a positive definite matrix. Then for continuous* $f(\sigma)$, *which verifies the condition* (L) *we have*

$$\|f(\sigma(t))\|_{L^2} \leqslant M\|\varphi\|, \quad \|\sigma(t)\| \leqslant M'\|\varphi\|,$$
$$\lim_{t\to\infty} \|\sigma(t)\| = 0. \tag{16}$$

Proof 1. Choose $\bar{\sigma}(t)$ satisfing

$$\|\bar{\sigma}(t)\| + \left\| \frac{d\bar{\sigma}(t)}{dt} \right\| \leqslant B e^{-\gamma t} \|\varphi\| \quad (\gamma>0).$$

Let

$$f_T(t)=\begin{cases} f(\sigma(t)), & 0\leqslant t\leqslant T, \\ 0, & t>T. \end{cases}$$

$$\sigma_T(t)=\begin{cases} \sigma(t), & 0\leqslant t\leqslant T, \\ \bar{\sigma}(t), & t>T. \end{cases}$$

$$z_T(t)=\begin{cases} z(t), & 0\leqslant t\leqslant T, \\ \bar{\sigma}(t)-\int_0^t K(t-\alpha)f_T(\alpha)d\alpha, & t>T. \end{cases}$$

$$\xi_T(t)=\begin{cases} \xi(t)=\frac{dz(t)}{dt}, & 0\leqslant t\leqslant T, \\ \bar{\xi}(t)=0, & t>T. \end{cases}$$

Consider

$$I=\int_0^\infty \langle P[H(\sigma_T(t)-z_T(t))-f_T(t)]+QH\Big[\frac{d\sigma_T(t)}{dt}-\frac{dz_T(t)}{dt}\Big],\ f_T(t)\rangle dt. \qquad (17)$$

We prove that the integrand of (17) is a negative definite quadratic form for $f(\sigma(t))$. From conditions of Lemma follows that there exist Fourier transforms of $\sigma_T(t)$, $f_T(t)$, $z_T(t)$. Let $\tilde{\sigma}_T$, $\tilde{f}_T$, $\tilde{z}_T$ be respectively the Fourier transforms of $\sigma_T(t)$, $f_T(t)$, $z_T(t)$.

From

$$\sigma_T(0)-z_T(0)=0, \qquad (18)$$

$$\int_0^\infty \langle f_1(t),\ f_2(t)\rangle dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{Re}\langle \tilde{f}_1(\omega),\ \tilde{f}_2(\omega)\rangle d\omega, \qquad (19)$$

where $\tilde{f}_1(\omega)$, $\tilde{f}_2(\omega)$ are Fourier transforms of $f_1(t)$, $f_2(t)$, follows

$$\begin{aligned} I&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{Re}\langle P[H(\tilde{\sigma}_T-\tilde{z}_T)-\tilde{f}_T]+i\omega QH[\tilde{\sigma}_T-\tilde{z}_T],\ \tilde{f}_T\rangle d\omega \\ &=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{Re}\langle [PH\tilde{K}-P+i\omega QH\tilde{K}]\tilde{f}_T,\ \tilde{f}_T\rangle d\omega \\ &=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\langle \{P-\mathrm{Re}[(P+i\omega Q)H\tilde{K}]\}\tilde{f}_T,\ \tilde{f}_T\rangle d\omega \\ &=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\langle W(\omega)\tilde{f}_T,\ \tilde{f}_T\rangle d\omega. \end{aligned}$$

Here we have noted the fact that

$$\sigma_T(t)-z_T(t)=\int_0^t K(t-\alpha)f_T(\alpha)d\alpha,\quad \tilde{\sigma}_T-\tilde{z}_T=\tilde{K}\tilde{f}_T.$$

By the conditions of Lemma we obtain that $W(\omega)$ and S are positive definite matrices. Hence there exists $\varepsilon_0>0$ such that

$$I\leqslant -\varepsilon_0\int_0^{+\infty}\|f_T(t)\|^2dt.$$

2. First we develop (17) to obtain the estimate of $\|\sigma(t)\|$. Develop (17) to obtain

$$\begin{aligned} &\int_0^{+\infty}\langle P[H\sigma_T(t)-f_T(t)]+QH\frac{d\sigma_T(t)}{dt},\ f_T(t)\rangle dt \\ &\leqslant \int_0^{+\infty}\Big\langle PHz_T(t)+QH\frac{dz_T(t)}{dt},\ f_T(t)\Big\rangle dt-\varepsilon_0\int_0^{+\infty}\|f_T(t)\|^2dt. \end{aligned}$$

Take into account that $f(\sigma)$ satisfies (14), hence

$$\langle PH\sigma, f(\sigma)\rangle \geqslant \langle Pf(\sigma), f(\sigma)\rangle.$$

Therefore

$$\int_0^{+\infty}\langle P[H\sigma_T(t)-f_T(t)], f_T(t)\rangle dt=\int_0^T\langle P(H\sigma(t)-f(\sigma(t)), f(\sigma(t))\rangle dt\geqslant 0.$$

We also obtain

$$\int_0^{+\infty}\left\langle QH\frac{d\sigma_T(t)}{dt}, f_T(t)\right\rangle dt=\int_0^{\sigma(T)}\langle QHd\sigma, f(\sigma)\rangle-\int_0^{\sigma(0)}\langle QHd\sigma, f(\sigma)\rangle$$
$$\geqslant -\left|\int_0^{\sigma(0)}\langle QHd\sigma, f(\sigma)\rangle\right|$$

and

$$\int_0^{+\infty}\left\langle PHz_T(t)+QH\frac{dz_T(t)}{dt}, f_T(t)\right\rangle dt$$
$$\leqslant A_1\left(\int_0^T e^{-2\gamma t}dt\right)^{1/2}\|\varphi\|\left(\int_0^T\|f(\sigma(t))\|^2dt\right)^{1/2}\leqslant \tilde{L}_2\|\varphi\|\cdot\|f_T\|_{L^2[0,\infty)},$$

where $\tilde{L}_2>0$. Therefore

$$\varepsilon_0\|f_T\|^2_{L^2[0,+\infty)}\leqslant\left|\int_0^{\sigma(0)}\langle QHd\sigma, f(\sigma)\rangle\right|+\tilde{L}_2\|\varphi\|\cdot\|f_T\|_{L^2[0,+\infty)}$$

i. e. $\|f_T\|^2_{L^2[0,+\infty)}\leqslant\tilde{L}_1\|\varphi\|^2+2\tilde{L}_2'\|\varphi\|\cdot\|f_T\|_{L^2[0,+\infty)}$ ($\tilde{L}_1>0$, $\tilde{L}_2'>0$).

We have $(\|f_T\|_L-\tilde{L}_2'\|\varphi\|)^2\leqslant\tilde{L}_1\|\varphi\|^2\tilde{L}_2'^2\|\varphi\|^2$, that is $\|f_T\|_{L^2}\leqslant M\|\varphi\|$, where M is a constant independent of T. From (13) we obtain $\|\sigma(t)\|\leqslant M'\|\varphi\|$. We can prove $\lim_{t\to\infty}\|\sigma(t)\|=0$ as Lemma 1.

Lemma 3. *Let $\tilde{X}^T(\omega)$ be Fourier transformation matrix of solution matrix $X^T(t)$ for* (5). *Then $\tilde{X}^T(\omega)=\Gamma^{-1}(i\omega)$ for all real ω.*

Proof of Lemma 3 is obvious.

Lemma 4. *Consider the system*

$$\sigma(t)=z(t)+\int_0^t K(t-\alpha)f(\sigma(\alpha))d\alpha+D\xi(t),$$
$$\frac{d\xi(t)}{dt}=f(\sigma(t)). \tag{20}$$

where D is $m\times m$ matrix, $z(t)$ and $K(t)$ satisfy the following conditions

$$\|K(t)\|+\left\|\frac{dK(t)}{dt}\right\|\leqslant A_2e^{-\beta t},$$
$$\|z(t)\|+\left\|\frac{dz(t)}{dt}\right\|\leqslant A_1\|\varphi\|e^{-\beta t}, \tag{21}$$
$$\|\sigma(\cdot)\|\leqslant C_1\|\varphi\|,$$

where $\beta>0$, $A_1>0$, $A_2>0$, $C_1>0$.

Suppose that there exist $m\times m$ matrices diagonal P, Q with the following properties:

(i) *The elements of matrices P, Q are positive or zero. PHD is a symmetric and negative definite matrix;*

(ii) *For all real ω*

$$W(\omega)=P-QHD-\frac{1}{2}\{(P+i\omega Q)H\widetilde{K}(\omega)+[(P+i\omega Q)H\widetilde{K}(\omega)]^*\} \tag{22}$$

is a positive definite matrix, where $\widetilde{K}(\omega)$ is the Fourier transform of $K(t)$;

(iii) *$S=\lim_{\omega\to\infty} W(\omega)$ is a positive definite matrix.*

Then for continuous $f(\sigma)$ which verifies the condition (4), we have

$$\int_0^{+\infty}\|f(\sigma(t)\|^2dt\leqslant M\|\varphi\|,\quad \|\xi(t)\|\leqslant M_1\|\varphi\|,$$
$$\|\sigma(t)\|\leqslant M_2\|\varphi\|,\ \lim_{t\to+\infty}\|\xi(t)\|=0,\ \lim_{t\to+\infty}\|\sigma(t)\|=0, \tag{23}$$

where $M\geqslant 0$, $M_1\geqslant 0$, $M_2\geqslant 0$.

Proof For all $0<T<+\infty$ let

$$x(t)=[x^T(0)X(t)]^T+\left[\int_{-\tau}^0 x^T(s)d_s\int_0^\tau \eta(s-\alpha)X(t-\alpha)d\alpha\right]^T$$
$$+\int_0^t X^T(t-\alpha)Rf(\sigma(\alpha))d\alpha; \tag{24}$$

$$\sigma(t)=C[x^T(0)X(t)]^T+C\left[\int_{-\tau}^0 x^T(s)d_s\int_0^\tau \eta(s-\alpha)X(t-\alpha)d\alpha\right]^T$$
$$+\int_0^t CX^T(t-\alpha)Rf(\sigma(\alpha))d\alpha. \tag{25}$$

Set

$$z(t)=C[x^T(0)X(t)]^T+C\left[\int_{-\tau}^0 x^T(s)d_s\int_0^\tau \eta(s-\alpha)X(t-\alpha)d\alpha\right]^T,$$
$$K(t)=CX^T(t)R. \tag{26}$$

By using $\widetilde{X}^T(\omega)=\Gamma^{-1}(i\omega)$ and $\widetilde{K}(\omega)=-G(i\omega)$, we obtain that conditions of Lemma 2 are satisfied. Therefore $\|f(\sigma(t))\|_{L^2[0,+\infty)}\leqslant M'\|\varphi\|$, $\|x(t)\|\leqslant M''\|\varphi\|$, $\lim_{t\to+\infty}\|x(t)\|=0$, where $\|\varphi\|=\|x(\cdot)\|$.

Theorem 1 is proved.

(2) Proof of Theorem 2.

If $x(t)$, $z(t)$, $K(t)$ are defined as in the proof of Theorem 1. Let

$$\sigma(t)=C[x^T(0)X(t)]^T+C\left[\int_{-\tau}^0 x^T(s)d_s\int_0^\tau \eta(s-\alpha)X(t-\alpha)d\alpha\right]^T$$
$$+\int_0^t CX^T(t-\alpha)Rf(\sigma(\alpha))d\alpha+D\xi(t). \tag{27}$$

Then

$$\sigma(t)=z(t)+\int_0^t K(t-\alpha)f(\sigma(\alpha))d\alpha+D\xi(t).$$

From the condition (ii) of Theorem 2 follows that

$$W(\omega)=P+\mathrm{Re}\{(P+i\omega Q)H\widetilde{G}(i\omega)\}$$

is a positive definite matrix, where

$$\widetilde{G}(i\omega)=-C\Gamma^{-1}(i\omega)R-\frac{D}{i\omega}.$$

From $QHD=\mathrm{Re}\left\{(P+i\omega Q)H\dfrac{D}{i\omega}\right\}$ follows that $P-QHD-\mathrm{Re}\{(P+i\omega Q)HC\Gamma^{-1}(i\omega)R\}$ is a positive definite matrix. Therefore, conditions of Lemma 4 are satisfied. Then

Theorem 2 is established.

(3) Proof of Theorem 3.

If $X(t)$, $\sigma(t)$ are defined as in the proof of Theorem 2. Let

$$K_0(t) = -CX^T(t)R \,(t\geqslant 0). \tag{28}$$

Then we can prove that

$$\|K_0(t)\| \leqslant l_0 e^{-\lambda_0 t} \,(l_0>0,\ \lambda_0>0),$$

and then

$$\int_0^{+\infty} K_0(\beta)\,d\beta < \infty.$$

Also put

$$K(t) = \int_t^{+\infty} K_0(\beta)\,d\beta.$$

Then we obtain $\|K(t)\| \leqslant l' e^{-\lambda_0 t} \,(l'>0)$. We deduced

$$\int_0^t CX^T(t-\alpha)R\xi(\alpha)\,d\alpha = \int_0^t -K_0(t-\alpha)\xi(\alpha)\,d\alpha = -\int_0^t \frac{dK(t-\alpha)}{d\alpha}\xi(\alpha)\,d\alpha$$
$$= -K(0)\xi(t) + K(t)\xi(0) + \int_0^t K(t-\alpha) f(\sigma(\alpha))\,d\alpha. \tag{29}$$

Putting

$$z(t) = C[x^T(0)X(t)]^T + C\left[\int_{-\tau}^0 x^T(s)\,d_s \int_0^\tau \eta(s-\alpha)X(t-\alpha)\,d\alpha\right]^T + K(t)\xi(0), \tag{30}$$

we obtain

$$\sigma(t) = z(t) + \int_0^t K(t-\alpha) f(\sigma(\alpha))\,d\alpha + D'\xi(t), \tag{31}$$

where

$$D' = D - K(0),\ K(0) = -\int_0^{+\infty} CX^T(\beta)R\,d\beta.$$

Then we obtain that conditions of Lemma 4 are satisfied and our proof is complete.

§ 3. Sufficient conditions for generalized H-absolute stability of systems (1)′, (2)′

Theorem 4. *Assume that for the system (2)′ we only take $G(i\omega)e^{-i\omega t}$ in place of $G(i\omega)$ in conditions of Theorem 1.*

Then the system (2)′ is a generalized H-absolute stable.

We can prove Theorem 4 using the same deduction as in [2, p. 389] and in Theorem 1.

Lemma 5. *Consider the system*

$$\sigma(t) = z(t) + \int_0^t K(t-\gamma) f(\sigma(\gamma))\,d\gamma - C_1\xi(t) - C_2\eta(t)$$
$$\frac{d\xi(t)}{dt} = \eta(t),\quad \frac{d\eta(t)}{dt} = f(\sigma(t)) \tag{32}$$

where K, C_1, C_2 are $m\times m$ matrices, σ, z, f, ξ, η are m vectors. put

$$K_1(t)=\frac{-dK(t)}{dt},\quad r=K(0)-C_2. \tag{33}$$

Suppose that the following conditions are satisfied:

(i) C_1 *is a symmetric and positive definite matrix;*

(ii) $0<\sigma_j f_j(\sigma)\ (\sigma_j\neq 0,\ j=1,\ 2,\ \cdots,\ m)$;

(iii) $\int_0^{\pm\infty} f_j(\sigma)d\sigma_i=\infty\ (j=1,\ 2,\cdots,\ m)$;

$$\|z(t)\|+\left\|\frac{dz(t)}{dt}\right\|\leqslant A_1\|\varphi\|e^{-\alpha t},\quad \|K(t)\|+\left\|\frac{dK(t)}{dt}\right\|\leqslant A_2\|\varphi\|,$$
$$\|\xi(\cdot)\|+\|\eta(\cdot)\|\leqslant b_1\|\varphi\|,\quad \|\sigma(\cdot)\|\leqslant b_2\|\varphi\|, \tag{34}$$

where A_1, A_2, b_1, b_2, α all are positive constants;

(iv) $W(\omega)=r-\frac{1}{2}[\tilde{K}_1+\tilde{K}_1^*]$ *is a negative definite matrix. $\tilde{K}_1$ is the Fourier transformation of $K_1(t)$;*

(v) $S=\lim\limits_{\omega\to\infty} W(\omega)$ *is a negative definite matrix.*

Then

$$\|f(\sigma(t))\|_{L^2[0,+\infty)}\leqslant M(\|\varphi\|),\quad \|\xi(t)\|\leqslant M_1(\|\varphi\|),\quad \|\eta(t)\|\leqslant M_2(\|\varphi\|),$$
$$\|\sigma(t)\|\leqslant M_3(\|\varphi\|),$$
$$\lim_{t\to+\infty}\|\sigma(t)\|=\lim_{t\to+\infty}\|\xi(t)\|=\lim_{t\to+\infty}\|\eta(t)\|=0,$$

where M, M_1, M_2, M_3 are non-negative functions which tend to zero when their arguments tend to zero.

Proof Let us put

$$I(T)=\int_0^T\left\langle\frac{d\sigma(t)}{dt}+C_1\eta(t)-\frac{dz(t)}{dt},f(\sigma(t))\right\rangle dt, \tag{35}$$

$$f_T(t)=\begin{cases} f(\sigma(t)), & 0\leqslant t\leqslant T,\\ 0 & t>T.\end{cases}$$

$$V_T(t)=\begin{cases}\dfrac{d\sigma(t)}{dt}-\dfrac{dz(t)}{dt}-rf(\sigma(t))+C_1\eta(t), & 0\leqslant t\leqslant T,\\ -\displaystyle\int_0^T K_1(t-\theta)f(\sigma(\theta))d\theta, & t>T.\end{cases}$$

From

$$\frac{d\sigma(t)}{dt}=\frac{dz(t)}{dt}-\int_0^t K_1(t-\theta)f(\sigma(\theta))d\theta+rf(\sigma(t))-C_1\eta(t),$$

we can deduce

$$V_T(t)=-\int_0^t K_1(t-\theta)f_T(\theta)d\theta,$$

that is $\tilde{V}_T=-\tilde{K}_1\tilde{f}_T$, where $\tilde{V}_T$, $\tilde{K}_1$ are Fourier transformations of V_T, K_1 respectively.

Therefore

$$I(T)=\int_0^{+\infty}\langle V_T(t)+rf_T(t),\ f_T(t)\rangle dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\langle\left[r-\frac{1}{2}(\tilde{K}_1+\tilde{K}_1^*)\right]\tilde{f}_T,\ \tilde{f}_T\right\rangle d\omega.$$

From (iv), (v) we deduce that there exists $\varepsilon_0>0$ such that

$$I(T)\leqslant-\varepsilon_0\int_0^{+\infty}\|f_T(t)\|^2dt.$$

We can prove the remainder of Lemma 5 by using the same deduction as in [2, p. 182], [2, p. 187] and in Lemma 2, Lemma 4.

Theorem 5. *Consider system* (1)′. *Suppose that the solution* $x=0$ *of system* (5) *is asymptotically stable. If the following conditions are satisfied:*

(i) $\left[B_1+\int_0^{+\infty}CX^T(t)R\,dt\right]$ *is a symmetry and negative definite matrix.* B_2 *is a negative defiinite matrix;*

(ii) $\overline{W}(\omega)=B_2+\mathrm{Re}\left\{\frac{1}{i\omega}\left[\int_0^{+\infty}CX^T(t)R\,dt-\tilde{K}_0(\omega)\right]\right\}$ *is a negative definite matrix, where* $K_0(t)=CX^T(t)R$, $\tilde{K}_0(\omega)$ *is the Fourier transform of* $K_0(t)$;

(iii) $\overline{S}=\lim_{\omega\to\infty}\overline{W}(\omega)$ *is a negative definite matrix;*

(iv) $\sigma_jf_j(\sigma)>0\ (\sigma_j\neq0)$,

$$\int_0^{\pm\infty}f_j(\sigma)d\sigma_j=\infty\quad(j=1,2,\cdots,m).$$

Then the solution $x=0$ *of system* (1)′ *is a generalized H-absolute stable.*

Proof Let us put

$$K_1(t)=\int_t^{+\infty}K_0(\beta)d\beta,\quad K(t)=\int_t^{+\infty}K_1(\beta)d\beta,$$

$$\frac{d\xi(t)}{dt}=\eta(t).$$

Denote the Fourier transform of $K_1(t)$ by $\tilde{K}_1(\omega)$, thus

$$\tilde{K}_1(\omega)=\frac{1}{i\omega}\left[\int_0^{+\infty}CX^T(t)R\,dt-\tilde{K}_0(\omega)\right].$$

We can deduce as previous

$$\begin{aligned}\sigma(t)=&\left\{C[x^T(0)X(t)]^T+C\left[\int_{-\tau}^0x^T(s)d_s\int_0^\tau\eta(s-\alpha)X(t-\alpha)d\alpha\right]^T\right.\\&\left.-K_1(t)\xi(0)+K(t)\frac{d\xi(0)}{dt}\right\}+\int_0^tK(t-\alpha)f(\sigma(\alpha))d\alpha\\&+[B_1+K_1(0)]\xi(t)+[B_2-K(0)]\eta(t),\\z(t)=&C[x^T(t)X(t)]^T+C\left[\int_{-\tau}^0x^T(s)d_s\int_0^\tau\eta(s-\alpha)X(t-\alpha)d\alpha\right]^T\\&-K_1(t)\xi(0)+K(t)\frac{d\xi(0)}{dt},\\C_1=&-B_1-K_1(0)=-\left[B_1+\int_0^{+\infty}CX^T(\beta)R\,d\beta\right],\\C_2=&K(0)-B_2.\end{aligned}$$

From conditions of this theorem we obtain that conditions of Lemma 5 are satisfied.

The proof of our theorem is completed.

Remark. If any solution $x(t)$ of our system satisfies $\|x(t)\| \leqslant M(\|\varphi\|)$ in the existed inteval $[0, T_0]$, then we can prove that $x(t)$ exists in $[0, +\infty)$.

References

[1] Li Xunjin, The Absolute Stability of retarded systems., *Acta. Math Sinica*, **13** (1963), 558—573.

[2] Halanay, A., Differential Equations, Stability, Oscillations, Time Lags, *Academic press.*, New York, (1966).

[3] Hale, J. K., Theory of Functional Differential Equations. Springer-Varlag New York, (1977).

具有 m 个非线性执行机构的线性泛函型调节系统的绝对稳定性

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摘 要

本文中我们考虑下面系统

$$\frac{dx(t)}{dt}=L(x_t)+Rf(\sigma(t)),\quad \sigma(t)=Cx(t) \tag{1}$$

以及

$$\frac{dx(t)}{dt}=L(x_t)+Rf(\sigma(t)),\quad \frac{d\xi(t)}{dt}=f(\sigma(t)),\ \sigma(t)=Cx(t)+D\xi(t), \tag{2}$$

其中 x, f 是 n 维向量, σ、ξ 是 m 维向量, C、D 是 $m\times n$ 矩阵, R 是 $n\times m$ 矩阵, $m>1$.

我们引入了系统的广义 H-绝对稳定性, 并给出了系统(1), (2)的广义 H-绝对稳定性的充分性判据. 本文中我们推广和简化了文[1, 2]中的方法. 对非线性项 $f(\sigma)$ 去掉了 f_j 仅依赖于 σ_j 的限制.