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CONTROLLABILITY AND OBSERVABILITY OF DISCRETE SYSTEMS

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A linear dynamic system with controls can be described by the following state equations and observation equations

$$\frac{dx}{dt}=Ax+Bu, \qquad (1)$$
$$y=Cx.$$

Here A is an $n\times n$ matrix, B is an $n\times r$ matrix, C is a $p\times n$ matrix, and $x(t)$, $u(t)$ and $y(t)$ are $n\times 1$, $r\times 1$ and $p\times 1$ vectors respectively. Let (A, B) denote system (1). For a computer control we must chose a proper sampling period T to discrete the system. We assume that $u(t)$ is constant in every interval $(kT, (k+1)T)$, where $k=0, 1, 2, \cdots$ and by fundamental solution matrix, system (1) can be changed into the following discrete system[1]

$$x_{k+1}=\Phi x_k+Gu_k, \qquad (2)$$
$$y_k=Cx_k.$$

Here $\Phi=e^{AT}$, $G=\int_0^T e^{A\tau}d\tau\, B$, and T is the sampling period.

A naturally raised question is: if system (1) has been judged to be completely controllable and observable, is the discretized system (2) still completely controllable and observable? The answer is No. [2] has indicated with examples that after discretization the controllability and observability may not be reserved. Suppose that A has s distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_s$ and the highest degree of elementary divisor $(\lambda-\lambda_i)^l$ of A corresponding to λ_i is e_i and n_i denotes the multiple of λ_i. For system (2) discretized with fundamental solution matrix, [2] has given a sufficient condition of complete controllability and observability, namely we have the following theorem:

Theorem. *If system* (1) *is completely controllable and observable, and A is real, then the discretized system* (2) *is still completely controllable and observable as long as the sampling period T satisfies the following conditions*:

$$\mathrm{Im}(\lambda_l-\lambda_j)\neq\frac{2k\pi}{T},\quad k=\pm 1, \pm 2, \cdots, \text{ when } \mathrm{Re}(\lambda_i-\lambda_j)=0.$$

In this paper, under the assumption of complete controllability and observability

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of system (1), we will discuss the controllability and observability of the discrete system

$$\begin{aligned} x_{k+1} &= f(A)x_k + g(A)Bu_k, \\ y_k &= Cx_k, \end{aligned} \tag{3}$$

here $f(\lambda)$, $g(\lambda)$ and $h(\lambda)$ are single valued analytical functions on $\{\lambda_1, \cdots, \lambda_s\}$. Obviously system (2) is only a special case of system (3).

Theorem 1. *If* (A, B) *is a single input system* (i. e. $r=1$) *which is completely controllable, then the necessary and sufficient condition of complete controllability of discretized system*

$$x_{k+1} = f(A)x_k + g(A)bu_k. \tag{3}$$

$g(\lambda_i) \neq 0$, $f(\lambda_i) \neq f(\lambda_j)$ if $i \neq j$, $f'(\lambda_i) \neq 0$ if $e_i > 1$ $(i, j = 1, 2, \cdots, s)$.

In order to prove Theorem 1, at first we prove two lemmas.

Suppose $n_1, n_2, \cdots, n_s$ to be positive integers and let

$$\varphi(\mu, \lambda_1, \cdots, \lambda_s) = (\mu - f(\lambda_1))^{n_1}, \cdots (\mu - f(\lambda_s))^{n_s} = \mu^n + a_{n-1}\mu^{n-1} + \cdots + a_1\mu + a_0,$$

which is a polynomial of μ, and its coefficients a_i $(\lambda_1, \cdots, \lambda_s)$ $(i = 0, 1, \cdots, n-1)$ are apparently analytic functions of $\lambda_1, \cdots, \lambda_s$. Furthermore we let

$$\Delta^{k_i}_{n_1, n_2, \cdots, n_s} = \frac{d^{k_i} f^n(\lambda_i)}{d\lambda_i^k} + a_{n-1} \frac{d^{k_i} f^{n-1}(\lambda_i)}{d\lambda_i^k} + \cdots + a_0 \frac{d^{k_i} f^0(\lambda_i)}{d\lambda_i^k}, \tag{4}$$

here $f^0(x) = 1$, $n = n_1 + n_2 + \cdots + n_s$. Evidently $\Delta^0_{n_1, n_2, n_s} = 0$ if $k_i = 0$. Now we have $(f_i = f(\lambda_i))$:

Lemma 1

$$\Delta^{k_i}_{n_1, \cdots, n_s} = \begin{cases} 0, & if\ 0 \leqslant k_i < n_i, \\ n_i!\ (f(\lambda_i))^{n_i} \prod\limits_{j=1, j\neq i}^{s} (f_i - f_j)^{n_j}, & if\ k_i = n_i. \end{cases} \tag{5}$$

Proof Without loss of generality we can suppose $i=1$ when $k_1 = 0$, we have

$$\Delta^0_{n_1, \cdots, n_s} = f_1^n + a_{n-1} f_1^{n-1} + \cdots + a_1 f_1 + a_0 f_1^0 = \varphi(f_1, \lambda_1, \cdots, \lambda_s) = 0,$$

here f_i denotes $f(\lambda_i)$. The above equation is an identity of $\lambda_1, \cdots, \lambda_s$. After derivation with respect to λ_1, we obtain

$$\begin{aligned} \Delta^1_{n_1, \cdots, n_s} &= -a'_{n-1} f_1^{n-1} - \cdots - a'_0 f_1^0 = -\frac{\partial}{\partial \lambda_1} \varphi(\mu, \lambda_1, \cdots, \lambda_s)\big|_{\mu = f_1} \\ &= n_1 f'_1 (\mu - f_1)^{n_1 - 1} (\mu - f_2)^{n_2} \cdots (\mu - f_s)^{n_s} \big|_{\mu = f_1} \end{aligned} \tag{6}$$

and we have

$$\Delta^1_{n_1, \cdots, n_s} = \begin{cases} 0, \text{ when } n_1 > k_1 = 1, \\ f_1^1 (f_1 - f_2)^{n_2} \cdots (f_1 - f_s)^{n_s}, \text{ when } k_1 = n_1 = 1. \end{cases}$$

i. e. the conclusion is valid in the case $k_1 = 1$. In case $k_1 = 2$, if $n_1 \geqslant 2$

$$\Delta^1_{n_1, \cdots, n_s} = (f_1^n)' + a_{n-1}(f_1^{n-1})' + \cdots + a_0 (f_1^0)' \equiv 0,$$

which is also an identity of $\lambda_1, \cdots, \lambda_s$. After derivation with respect to λ_1, we obtain

$$\Delta^2_{n_1, \cdots, n_s} = (f_1^n)'' + a_{n-1}(f_1^{n-1})'' + \cdots + a_0 (f_1^0)'' = -a'_{n-1}(f_1^{n-1})' - \cdots - a'_0 (f_1^0)'.$$

But from (6), we have

$$a'_{n-1}f_1^{n-1}+\cdots+a'_1f_1^0\equiv 0.$$

Similarly, after derivation with respect to λ_1, we obtain

$$-a'_{n-1}(f_1^{n-1})'-\cdots-a'_0(f_1^0)'=a''_{n-1}f_1^{n-1}+\cdots+a''_0f_1^0.$$

Therefore when $n_1\geqslant 2$ we have

$$\begin{aligned}\Delta^2_{n_1,\cdots,n_s}&=a''_{n-1}f_1^{n-1}+\cdots+a''_0f_1^0=\frac{\partial^2}{\partial\lambda_1^2}\varphi(\mu,\lambda_1,\cdots,\lambda_s)\big|_{\mu=f(\lambda_1)}\\&=n_1(n_1-1)(f'_1)^2(f_1-f_1)^{n_1-2}(f_1-f_2)^{n_2}\cdots(f_1-f_s)^{n_s}\\&=\begin{cases}0, & \text{if } n_1>2=k_1\\ 2!\,(f'_1)^2(f_1-f_2)^{n_2}\cdots(f_1-f_s)^{n_s}, & \text{if } n_1=2=k_1^{n_s},\end{cases}\end{aligned}$$

in the case $k_1=2$ the Lemma has been proven. By induction we may prove the Lemma for general k_1.

Lemma 2. *Define*

$$D_i=\begin{bmatrix}1 & 0 & \cdots & 0\\ f_i & (f_i)' & \cdots & (f_i)^{(n_i-1)}\\ \cdots & \cdots & \cdots & \cdots\\ f_i^{n-1} & (f_i^{n-1})' & \cdots & (f_i^{n-1})^{(n_i-1)}\end{bmatrix}$$

and determinant $D_{(n_1,n_2,\cdots,n_s)}=\det|D_1, D_2, \cdots, D_s|$ *of degree* n, *then the necessary and sufficient condition of* $D_{n_1,\cdots,n_s}\neq 0$ is

$$f(\lambda_i)\neq f(\lambda_j)\ \textit{if}\ i\neq j,\ f'(\lambda_i)\neq 0\ \textit{if}\ n_i>1\ (i,\ j=1,\ 2,\ \cdots,\ s).$$

Proof To polynomial

$$\prod_{j=1}^{s-1}(\mu-f_j)^{n_j}\cdot(\mu-f(\lambda_s))^{n_s-1}\equiv\mu^{n-1}+C_{n-2}\mu^{n-2}+\cdots+C_1\mu+C_0,$$

we apply Lemma 1 (change n into $n-1$), we have

(i) when $i\neq s$, and $k_i\leqslant n_i-1$, we have $\Delta^{k_i}_{n_1,\cdots,n_s-1}=0$, namely

$$(f_i^{n-1})^{(k_i)}+C_{n-2}(f_i^{n-2})^{(k_i)}+\cdots+C_1(f_i)^{(k_i)}+C_0(f_i^0)^{(k_i)}\equiv 0,\tag{7}$$

(ii) when $i=s$, we have

$$\begin{aligned}(f_s^{n-1})^{(k_s)}+C_{n-2}(f_s^{n-2})^{(k_s)}+\cdots+C_1(f_s)^{(k_s)}+C_0(f_s^0)^{(k_s)}&=\Delta^{k_s}_{n_1,\cdots,n_s-1}\\&=\begin{cases}0 & \text{if } 0\leqslant k_s<n_s-1,\\ (n_s-1)!\,(f'_s)^{n_s-1}\prod_{j=1}^{s-1}(f_s-f_j)^{n_j} & \text{if } k_s=n_s-1.\end{cases}\end{aligned}\tag{8}$$

Multiply the first row, the second row, $\cdots$, the $(n-1)$th row of $D_{n_1,\cdots,n_s}$ with C_0, C_1, C_2, $\cdots$, C_{n-2} respectively, and add all these to the nth row; noticing equation (7) and (8), we obtain

$$D=\left|\begin{array}{c|c} D_{n_1,\cdots,n_{s-1},n_s-1} & \begin{matrix}0\\ f_s^{(n_s-1)}\\ \vdots\\ (f_s^{n-2})^{(n_s-1)}\end{matrix}\\ \hline 0\cdots 0 & a_1\end{array}\right|,$$

where

$$a_1=(n_s-1)!(f_s')^{n_s-1}\prod_{j=1}^{s-1}(f_s-\lambda_j)^{n_j}.$$

By induction we can prove

$$D_{n_1,\cdots,n_s}=\prod_{i=1}^{s}(n_i-1)!\cdots 2!(f_i')^{1+2\cdots+(n_i-1)}\prod_{i=1}^{s}\left(\prod_{j=1}^{i-1}(f_i-f_j)^{n_j}\right)^{n_i}.$$

From this, we can see that the necessary and sufficient condition of $D_{n_1,\cdots,n_s}\neq 0$ is $f(\lambda_i)\neq f(\lambda_j)$ if $i\neq j$, $f'(\lambda_i)\neq 0$ if $n_i>1$, and Lemma 2 is proven.

The proof of Theorem 1:

By assumption of Theorem 1, (A, B) is completely controlable, therefore the rank of matrix $(B, AB, \cdots, A^{n-1}B)$ is n. Since the system of n-dimensional vectors B, $AB, \cdots, A^{n-1}B$ are linearly independent, the minimal zero-polynomial is of degree n and $e_1+\cdots+e_s=n$. But since $e_i\leqslant n_i$ and $n_1+\cdots+n_s=n$, we have $e_i=n_i(i=1, 2, \cdots, s)$.

According to the finite expression theorem of matrix function, for any single valued analytic function $F(\lambda)$ defined on set$\{\lambda_1, \cdots, \lambda_s\}$, we have

$$F(A)=\sum_{i=1}^{s}\sum_{j=0}^{e_i-1}[F(\lambda_i)]^{(j)}Z_i^j(A),$$

where $Z_i^k(\lambda)$ is a polynomial whose degree is less than n. It satisfies the following relation

$$(Z_i^j(\lambda_l))^{(k)}=\begin{cases}1, & \text{if } l=i \text{ and } k=j,\\ 0, & \text{otherwise,}\end{cases}$$

and the representation is unique.

Specially putting $F(\lambda)=\lambda^k(k=0, 1, \cdots, n-1)$, we have

$$A^kB=\sum_{i=1}^{s}\sum_{j=0}^{e_i-1}(\lambda^k)^{(j)}_{\lambda=\lambda_i}Z_i^j(A)B.$$

Since vectors $B, AB, \cdots, A^{n-1}B$ are linearly independent, the n vectors

$$\{Z_i^j(A)B,\ i=1, 2, \cdots, s;\ j=0, 1, \cdots, e_i-1\}$$

are also linearly indepndent, namely it constructs a base of n-dimensional linear space.

Setting $F(\lambda)=f^k(\lambda)$ $(k=0, 1, \cdots, n-1)$, by expression theorem, we have

$$f^k(A)B=\sum_{i=1}^{s}\sum_{j=0}^{e_i-1}[f^k(\lambda_i)]^{(j)}Z_i^j(A)B.$$

From this we can see that $B, f(A)B, \cdots, f^{n-1}(A)B$ are linearly independent if and only if its coefficient determinant $D_{e_1,e_2,\cdots,e_s}\neq 0$ (as for the definition of $D_{e_1,\cdots,e_s}$, see Lemma 2), namely $f(\lambda_i)\neq f(\lambda_j)$ if $i\neq j$, and $f'(\lambda_i)\neq 0$ if $e_i>1$.

On the other hand, the necessary and sufficient condition of complete controllability of discrete system (3) is that the rank of matrix $(g(A)B, f(A)g(A)B, \cdots, f^{n-1}(A)g(A)B)$ is n. Because of $g(A)f^k(A)=f^k(A)g(A)$, the rank of $g(A)(B, f(A)B, \cdots, f^{n-1}(A)B)$ is n, and this is equivalent to that $g(A)$ is full rank and the rank of $(B, f(A)B, \cdots, f^{n-1}(A)B)$ is n. $g(A)$ is full rank if and only if $g(\lambda_i)\neq 0(i=1, 2, \cdots, s)$.

From above we know that the rank of $(B, f(A)B, \cdots, f^{n-1}(A)B)$ is n if and only if $f(\lambda_i) \neq f(\lambda_j)$ $(i \neq j)$ and $f'(\lambda_j) \neq 0$ when $e_i > 1$. So Theorem 1 is proven.

Theorem 2. *If (A, B) is completely controllable and f and g satisfy the same conditions as in Theorem 1, then the discretized system*

$$x_{k+1} = f(A)x_k + g(A)Bu_k$$

is also completely controllable.

Proof The degree of the lowest annihilator polynomial of A is $e = e_1 + \cdots + e_s$. So (A, B) is completely controllable if and only if the rank of $(B, AB, \cdots, A^{e-1}B)$ is n. Block B into the form $B = (b_1, b_2, \cdots, b_r)$, then by the finite expression theorem, we have

$$F(A) = \sum_{k=1}^{s} \sum_{j=1}^{e_k-1} [F(\lambda_k)]^{(j)} Z_k^j(A)$$

or

$$F(A)b_i = \sum_{k=1}^{s} \sum_{j=1}^{e_k-1} [F(\lambda_k)]^{(j)} Z_k^{(j)}(A)b_i.$$

Setting $F(\lambda) = \lambda^l (l = 0, 1, \cdots, e-1)$, we have

$$A^l b_i = \sum_{k=1}^{s} \sum_{j=1}^{e_k-1} (\lambda^l)^{(j)}_{\lambda=\lambda_k} Z_k^{(j)}(A)b_i.$$

Since the rank of $(B, AB, \cdots, A^{e-1}B)$ is n, from above equation we know that the rank of the set of vectors $\{Z_k^j(A)b_i, k = 1, 2, \cdots, s; j = 1, 2, \cdots, e_k - 1; i = 1, 2, \cdots, r\}$ is n.

Setting $F(\lambda) = f^k(\lambda)$, the determinant of coefficient matrix with which $f^k(A)b_i$ $(k = 0, 1, \cdots, e_k - 1, i = 1, 2, \cdots, r)$ is represented by $Z_k^j(A)b_i$, is $D_{e_1, \cdots, e_s}$. Using Lemma 2, we have $\det P_{e_1, \cdots, e_s} \neq 0$ if the conditions in Theorem 2 are satisfied.

But the linear subspace spanned by $b_i, f(A)b_i, \cdots, f^{e-1}(A)$ b_i is just the same as that spanned by $Z_1^0 b_i, \cdots, Z_s^{e_s-1} b_i (i = 1, \cdots, r)$.

Because the latter is an n-dimensional space, the former is also an n-dimensional space, i. e., the rank of matrix $\{B, f(A)B, \cdots, f^{e-1}(A)B\}$ is n. Therefore system (3) is completely controlable.

Using the dual principle, the problem of observability can easily be solved. For system (1), the complete observability of (A, C) is equivalent to the complete controllability of (A^T, C^T). If $f(\lambda)$, $g(\lambda)$ and $h(\lambda)$ are single valued analytic functions defined in D, which includes all the eigenvalues of A, noticing $f(A^T) = f(A)^T$, then by Theorem 1 and Theorem 2, we have

Theorem 3. *If system (A, B, C) is completely controllable and observable,* det $g(\lambda_i) \neq 0$, det $h(\lambda_i) \neq 0$, $f(\lambda_i) \neq f(\lambda_j)$ $(i \neq j)$, *and* $f'(\lambda_i) \neq 0$ *when* $e_i > 1$, *then*

$$x_{k+1} = f(A)x_k + g(A)Bu_k,$$

$$y_k = Ch(A)x_k$$

is also completely controllable and observable. If (A, B, C) is a single input and multi-output system, then the above condition is also necessary.

Setting $f(\lambda)=e^{\lambda T}$, $g(\lambda)=\int_0^T e^{\lambda\tau}d\tau$, and using Theorem 2, we may obtain the results of[3]. So Theorem 3 is an extension of [3].

Reference

[1] 李训经，现代控制理论基础(复旦大学数学系讲义)，1980.
[2] 中科院数学所控制理论研究室，线性控制系统的能控性和能观察性，科学出版社，1975.
[3] Kalman, R. E., Ho, Y. C. and Narendra, K. S., Contribution to Differential Equation, 1 (1963), 189—214.
[4] Гантмахер, 矩阵论(上册).

离散系统的能控性和能观察性

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摘 要

本文考虑有限维空间 E_n 的线性动力系统 $\frac{dx}{dt}=Ax+Bu, u\in E_r$, 与其有关的离散系统 $x_{k+1}=f(A)x_k+g(A)Bu_k$ 的完全能控性和完全能观性的关系. $\lambda_1, \lambda_2, \cdots, \lambda_n$ 表示 A 的 n 个特征根，对应的初等因子最大阶数为 $e_1, e_2, \cdots, e_n$, 则得到：

定理 1 如果 $\frac{dx}{dt}=Ax+Bu$ 是完全能控的单输入系统，那末 $x_{k+1}=f(A)x_k+g(A)Bu_k$ 是完全能控的充要条件是 $g(\lambda_i)\neq 0$, 当 $\lambda_i\neq\lambda_j$ 时 $f(\lambda_i)\neq f(\lambda_j)$, 当 $e_i>1$ 时 $f'(\lambda_i)\neq 0$.

定理 2 在多输入情形，定理 1 的条件是充分的.

应用对偶原理得到对应的完全能观性结果.