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ON THE UNIQUENESS OF SOLUTIONS OF NONLINEAR DEGENERATE PARABOLIC EQUATIONS

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1. Introduction

In this paper we shall prove a uniqueness theorem for solutions of the boundary value problem

$$u_t=(a(u)u_x)_x+b(u)u_x, \quad (x,\ t)\in R, \tag{1}$$

where $R=\{(x,\ t)\mid -1<x<1,\ 0<t<T\}$,

$$u(x,\ 0)=u_0(x), \quad -1\leqslant x\leqslant 1, \tag{2}$$

$$u(-1,\ t)=\Psi_1(t),\ u(1,\ t)=\Psi_2(t), \quad 0\leqslant t\leqslant T, \tag{3}$$

in which subscripts denote partial differentiation. The functions a and b are both assumed to be defined and continuous on $[0,\ \infty)$, with

$$a(s)>0 \text{ if } s>0 \text{ and } a(0)=0.$$

T is a fixed positive number, and $u_0(x)$, $\Psi_1(t)$, and $\Psi_2(t)$ are non-negative and continous functions satisfying $u_0(-1)=\Psi_1(0)$, and $u_0(1)=\Psi_2(0)$.

Let

$$b(u)=O(a(u)^{\lambda}), \tag{4}$$

$$u^{\mu}=O(a(u)) \tag{5}$$

in the neighborhood of $u=0$, where λ and μ are positive constants. Since Gilding[1] proved uniqueness when $\lambda=\frac{1}{2}$ and the case in which $\lambda>\frac{1}{2}$ can be treated as the special case of $\lambda=\frac{1}{2}$, so that hereafter we shall assume that $\lambda\leqslant\frac{1}{2}$.

The known results on uniqueness are the followings.

If

$$\lambda=\frac{1}{2}, \tag{6}$$

then the solutions of (1), (2), (3) are unique, see Gilding[1].

If

$$\mu<2,\ \lambda>\frac{1}{4} \tag{7}$$

or if

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$$\mu\geqslant 2,\ 2\lambda+\frac{1}{\mu}-1>0, \tag{8}$$

then the solutions of (1), (2), (3) are unique. This is the result of Wu[2].

In this paper we shall prove the following Theorem.

Theorem. *If*

$$\mu<1,\ \lambda\geqslant 0 \tag{9}$$

or if

$$\mu\geqslant 1,\ 2\lambda+\frac{1}{\mu}-1>0. \tag{10}$$

Then the solutions of (1), (2), (3) *are unique.*

For the standard nonlinear degenerate parabolic equation[3]

$$u_t=(u^m)_{xx}+(u^n)_x,$$

(9) corresponds to $1<m<2,\ n\geqslant 1$, and (10) corresponds to $m\geqslant 2,\ 2n>m$.

Equation (1) is nonlinear and degenerate. At points where $u>0$ equation (1) is parabolic, but at points where $u=0$ it is not. Hence the solution exists only in the weak sence. It's definition is as follows. Set

$$A(s)=\int_0^s a(r)\,dr \quad\text{and}\quad B(s)=\int_0^s b(r)\,dr.$$

Definition. *A function $u(x, t)$ defined on $\bar{R}$: $-1<x<1$, $0<t<T$ is said to be a weak solution of* (1), (2), (3) *if:* (i) *u is nonnegative and continous on $\bar{R}$.* (ii) $u(-1, t)=\Psi_1(t)$, $u(1, t)=\Psi_2(t)$ *for $0\leqslant t\leqslant T$.* (iii) *$A(u)$ has a square-integrable generalized derivative with respect to x in R.* (iv) *u satisfies the identity*

$$\iint_R \{\varphi_x[A(u)_x+B(u)]-\varphi_t u\}\,dx\,dt=\int_{-1}^1\varphi(x, 0)\,u_0(x)\,dx \tag{11}$$

for all $\varphi\in C(\bar{R})$ which vanish for $|x|=1$ and for $t=T$, and which have square-integrable generalized first derivatives in R.

2. Proof of the Theorem

Henceforth let $K, K_1, K_2, \cdots$ denote positive constants.

Suppose the solution of (1), (2), (3) is not unique, i. e. there exist two solutions $u_1(x, t)$, $u_2(x, t)$ of (1), (2), (3) such that $U=u_2-u_1$ is not identically equal to zero. Then there exists at least one point $(x_0, t_0)\in R$ such that $|U(x_0, t_0)|>0$. Let

$$a_1=\frac{A(u_2)-A(u_1)}{U}=\int_0^1 a(u_1\tau+u_2(1-\tau))\,d\tau,$$

$$b_1=\frac{B(u_2)-B(u_1)}{U}=\int_0^1 b(u_1\tau+u_2(1-\tau))\,d\tau.$$

From (4), (5) we can prove the following Lemmas, see[2].

Lemma 1. *If* $0\leqslant\lambda\leqslant\frac{1}{2}$, *then*

$$b_1 = O(a_1^\lambda). \tag{12}$$

Proof If $\lambda=0$, (12) holds obviously.

If $0<\lambda\leqslant\frac{1}{2}$

$$|b_1|\leqslant\int_0^1 b(u_1\tau+u_2(1-\tau))d\tau\leqslant\left(\int_0^1 1^{\frac{1}{1-\lambda}}d\tau\right)^{\frac{1-\lambda}{\lambda}}\left(\int_0^1 |b|^{\frac{1}{\lambda}}d\tau\right)^{\lambda}$$
$$=\left\{\int_0^1 |b|^{\frac{1}{\lambda}}d\tau\right\}^{\lambda}\leqslant\left\{\int_0^1 (Ka^\lambda)^{\frac{1}{\lambda}}d\tau\right\}^{\lambda}=K\left\{\int_0^1 a d\tau\right\}^{\lambda}=Ka_1^\lambda.$$

Lemma 2. *If μ is a positive number, then*

$$|U|^\mu=O(a_1). \tag{13}$$

Proof When $u_1\geqslant u_2$, we have

$$|U|^\mu\leqslant u_1^\mu=(\mu+1)\int_0^1(u_1\tau)^\mu d\tau\leqslant(\mu+1)\int_0^1[u_1\tau+u_2(1-\tau)]^\mu d\tau.$$

The symmetric expression

$$|U|^\mu<(\mu+1)\int_0^1[u_1\tau+u_2(1-\tau)]^\mu d\tau$$

is valid for $u_1<u_2$.

From (5) we have

$$u^\mu<Ka(u),$$

combining the above two expressions we get

$$|U|^\mu<(\mu+1)K\int_0^1 a(u_1\tau+u_2(1-\tau))d\tau=(\mu+1)Ka_1.$$

Therefore (13) is proved.

When $\varphi\in C^2$, (11) implies that

$$\iint_R U(a_1\varphi_{xx}-b_1\varphi_x+\varphi_t)dxdt=0. \tag{14}$$

Take a small positive number ε. Let α, β be positive C^∞ approximations of a_1, b_1 such that

$$|\alpha-a_1|<\varepsilon^{\frac{1+\mu}{\mu}},\quad |\beta-b_1|<\varepsilon^{\frac{2+\mu}{2\mu}}. \tag{15}$$

Let $U_1(x, t)$ be a C^∞ function which equals zero outside a small neighborhood of (x_0, t_0) and such that $U_1(x_0, t_0)=U(x_0, t_0)$. Hence

$$\iint_R UU_1\,dx\,dt>0. \tag{16}$$

Solve the following approximate adjoint problem for $w=w_\varepsilon$ in R.

$$\begin{cases} Fw_{xx}-\beta w_x+w_t=U_1, & -1<x<1,\ 0\leqslant t<T, \quad (17)\\ w|_{x=-1}=w|_{x=1}=w|_{t=T}=0, & \quad (18)\end{cases}$$

where

$$F=F(\alpha,\varepsilon)=(\alpha^{\frac{2+\mu}{2\mu}}+\varepsilon^{\frac{2+\mu}{2\mu}})^{\frac{2\mu}{2+\mu}}. \tag{19}$$

Because α, β, $U_1\in C^\infty$ we have $w\in C^\infty$. By applying the maximum principle we know that w is bounded by a constant independent of ε.

By (12), (15), (19) and the inequality $\lambda\leqslant\frac{1}{2}$, we have

$$\frac{\beta^2}{F}\leqslant K\frac{b_1^2+|\beta^2-b_1^2|}{(a_1+\varepsilon)}\leqslant K\frac{b_1^2}{(a_1+\varepsilon)}+K_1\leqslant K_2(a_1+\varepsilon)^{2\lambda-1}+K_1$$
$$\leqslant K_3(a_1+\varepsilon)^{2\lambda-1}\leqslant K_3\varepsilon^{2\lambda-1}. \tag{20}$$

Now we need the following Lemma.

Lemma 3.

$$|U|(F-\alpha)\leqslant K\varepsilon^{\frac{2+\mu}{2\mu}}\sqrt{F}. \tag{21}$$

Proof Since

$$|U|\leqslant Ka_1^{\frac{1}{\mu}},\quad |\alpha-a_1|<\varepsilon^{\frac{1+\mu}{\mu}}<\varepsilon,\ a_1<\alpha+\varepsilon,$$

therefore

$$|U|<Ka_1^{\frac{1}{\mu}}<K(\alpha+\varepsilon)^{\frac{1}{\mu}},$$

$$\frac{|U|(F-\alpha)}{\varepsilon^{\frac{2+\mu}{2\mu}}\sqrt{F}}<K\frac{[(\alpha^{\frac{2+\mu}{2\mu}}+\varepsilon^{\frac{2+\mu}{2\mu}})^{\frac{2\mu}{2+\mu}}-\alpha](\alpha+\varepsilon)^{\frac{1}{\mu}}}{\varepsilon^{\frac{2+\mu}{2\mu}}(\alpha^{\frac{2+\mu}{2\mu}}+\varepsilon^{\frac{2+\mu}{2\mu}})^{\frac{\mu}{2+\mu}}}$$
$$=K\,(1+s^{\frac{2\mu}{2+\mu}})^{\frac{1}{\mu}}(1+s)^{\frac{-\mu}{2+\mu}}[(1+s)^{\frac{2\mu}{2+\mu}}-1]\frac{1}{s},$$

where we set $\left(\frac{\varepsilon}{\alpha}\right)^{\frac{2+\mu}{2\mu}}=s$. It is easy to see that the right hand side of the above expression is uniformly bounded in $0<s<+\infty$. Therefore the Lemma is proved.

Multiplying (17) by U and integrating on R we have

$$\iint_R UU_1\,dx\,dt=\iint_R U(Fw_{xx}-\beta w_x+w_t)\,dx\,dt. \tag{22}$$

Taking $\varphi=w$ in (14) we get

$$0=\iint_R U(a_1w_{xx}-b_1w_x+w_t)\,dx\,dt. \tag{23}$$

Subtracting (23) from (22) we have

$$\iint_R UU_1\,dx\,dt=\iint_R U[(F-a_1)w_{xx}-(\beta-b_1)w_x]\,dx\,dt$$
$$\leqslant K\varepsilon^{\frac{2+\mu}{2\mu}}\iint_R(\sqrt{F}|w_{xx}|+|w_x|)\,dx\,dt. \tag{24}$$

By (17)

$$\sqrt{F}\,w_{xx}+\frac{w_t}{\sqrt{F}}=\frac{U_1+\beta w_x}{\sqrt{F}}.$$

Squaring, integrating on R, and using the fact that

$$\iint_R w_{xx}w_t\,dx\,dt=\frac{1}{2}\int_{-1}^{1}w_x^2\,dx|_{t=0}>0,$$

we get

$$\iint_R F w_{xx}^2 \, dx\, dt + \iint_R \frac{w_t^2}{F} \, dx\, dt < \iint_R \frac{(U_1+\beta w_x)^2}{F} \, dx\, dt. \tag{25}$$

Multiplying (17) by $\frac{w}{F}$, integrating, and combining with (25) and (20), we have

$$\iint_R w_x^2 dx\, dt = \iint_R \frac{w}{F}(-\beta w_x + w_t - U_1)\, dx\, dt$$

$$\leqslant \left\{ \iint_R \frac{w^2}{F} \, dx\, dt \cdot 2 \iint_R \left[\frac{w_t^2}{F} + \frac{(U_1+\beta w_x)^2}{F} \right] dx\, dt \right\}^{\frac{1}{2}}$$

$$\leqslant 2 \left\{ \iint_R \frac{w^2}{F} \, dx\, dt \iint_R \frac{(U_1+\beta w_x)^2}{F} \, dx\, dt \right\}^{\frac{1}{2}}$$

$$\leqslant K \left\{ \varepsilon^{-1} \left[\iint_R \frac{U_1^2}{F} \, dx\, dt + \iint_R \frac{\beta^2}{F} w_x^2 \, dx\, dt \right] \right\}^{\frac{1}{2}}$$

$$\leqslant K \left\{ \varepsilon^{-1} \left[1 + \varepsilon^{2\lambda-1} \iint_R w_x^2 \, dx\, dt \right] \right\}^{\frac{1}{2}} \quad \left(\text{because of } \frac{U_1^2}{F} \leqslant K \right)$$

$$\leqslant K\varepsilon^{-1} + \left[K\varepsilon^{2\lambda-2} \iint_R w_x^2 \, dx\, dt \right]^{\frac{1}{2}} \leqslant K\varepsilon^{-1} + K\varepsilon^{2\lambda-2} + \frac{1}{2} \iint_R w_x^2 \, dx\, dt,$$

hence

$$\iint_R w_x^2 \, dx\, dt \leqslant K\varepsilon^{2\lambda-2}. \tag{26}$$

By (25), (20), and (26) we have

$$\iint_R F w_{xx}^2 \, dx\, dt < K\varepsilon^{4\lambda-3}. \tag{27}$$

From (24), (26), and (27) we have

$$\iint_R U U_1 \, dx\, dt \leqslant K\varepsilon^{\frac{2+\mu}{2\mu}} \left[\iint_R (F w_{xx}^2 + w_x^2) \, dx\, dt \right]^{\frac{1}{2}} \leqslant K\varepsilon^{\frac{2+\mu}{2\mu}} [\varepsilon^{4\lambda-3} + \varepsilon^{2\lambda-2}]^{\frac{1}{2}}.$$

Therefore

$$\iint_R U U_1 \, dx\, dt \leqslant K\varepsilon^{2\lambda+\frac{1}{\mu}-1}. \tag{28}$$

Hence when (9) or (10) holds, the right hand side of (28) tends to zero as $\varepsilon \to 0$. This contridicts (16) and it follows that solutions of (1), (2), (3) are unique.

This completes the proof of the theorem.

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关于非线性蜕化抛物型方程解的唯一性

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摘　要

本文证明,在条件 $a(s)>0\,(s>0)$, $a(0)=0$, $b(s)=O(a(s)^{\lambda})\left(s\geqslant 0,\ 0\leqslant\lambda\leqslant\frac{1}{2}\right)$, $s^{\mu}=O(a(s))\,(s\geqslant 0,\ \mu>0)$ 之下,混合问题

$$u_t=(a(u)u_x)_x+b(u)u_x,\quad (x,\ t)\in R=\{(x,\ t)\,|\,-1<x<1,\ 0<t<T\},$$

$$u(x,\ 0)=u_0(x)\,(\geqslant 0),\quad -1\leqslant x\leqslant 1,$$

$$u(-1,\ 0)=\Psi_1(t)\,(\geqslant 0),\ u(1,\ t)=\Psi_2(t)\,(\geqslant 0),\quad 0\leqslant t\leqslant T,$$

当 $\mu<1$, $\lambda\geqslant 0$ 或 $\mu\geqslant 1$, $2\lambda+\frac{1}{\mu}>1$ 时,解为唯一的,这改善了[1, 2]的结果。