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单叶函数系数的渐近性质

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一、引 言

设 $f(z)\in S,\ F(\zeta)=\dfrac{1}{f\left(\dfrac{1}{\zeta}\right)}=\zeta+\sum\limits_{n=0}^{\infty}\beta_n\zeta^{-n}\ (|\zeta|>1)$ 记 $\log\dfrac{f(z)}{z}=2\sum\limits_{n=1}^{\infty}r_nz^n$，若

$$\lim_{\rho\to1}\frac{(1-\rho)^2}{\rho}\max_{|z|=\rho}|f(z)|=\lim_{\rho\to1}\frac{(1-\rho)^2}{\rho}|f(\rho e^{i\theta_0})|=\alpha_f.$$

巴西列维奇[1]证明了

$$\sum_{n=1}^{\infty}n\left|r_n-\frac{e^{-in\theta_0}}{n}\right|^2\leqslant\frac{1}{2}\log\frac{1}{\alpha_f}. \tag{1.1}$$

海曼[2]证明了

$$\lim_{n\to\infty}\frac{\left|\left\{\left(\dfrac{f(z)}{z}\right)^{\lambda}\right\}_n\right|}{d_n(2\lambda)}=\alpha_f^{\lambda},\ \lambda>\frac{1}{4}. \tag{1.2}$$

此地 $d_n(2\lambda)$ 为 $\dfrac{1}{(1-x)^{2\lambda}}=\sum\limits_{n=0}^{\infty}d_n(2\lambda)x^n$ 中系数.

又面积定理

$$\sum_{n=1}^{\infty}n|\beta_n|^2<1. \tag{1.3}$$

本文共分三部分. 第一部分改进(1.1)及(1.3)式，并把(1.1)，(1.3)两式结合起来，第二部分给海曼正则定理一个新证明. 第三部分研讨 S 中子族 S^*——星象函数的系数渐近性质. 1956 年[3]已证明，当 $f(z)\in S^*$ 时，海曼正则定理当 $0<\lambda\leqslant\dfrac{1}{4}$ 时亦成立. 当时用的是海曼方法. 在此我们给它两个新的证明(完全不用海曼的方法)，并得到一些新结果.

二、巴西列维奇不等式的改进

对巴西列维奇不等式，任福尧同志曾首先给予改进[6]. 但表示式较复杂. 我们在此给一个较清晰的改进.

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定理一　设 $f(z)\in S$, $F(\zeta)=\dfrac{1}{f\left(\frac{1}{\zeta}\right)}=\zeta+\sum\limits_{n=0}^{\infty}\beta_n\zeta^{-n}\in\Sigma$. $\log\dfrac{f(z)}{z}=2\sum\limits_{n=1}^{\infty}r_nz^n$,

$\lim\limits_{\rho\to1}\dfrac{(1-\rho)^2}{\rho}\max\limits_{|z|=\rho}|f(z)|=\lim\limits_{\rho\to1}\dfrac{(1-\rho)^2}{\rho}|f(\rho e^{i\theta_0})|=\alpha_f$. $\alpha_f\neq0$ 则有

$$\left(1-\sum_{n=1}^{\infty}n|\beta_n|^2\right)\left(\frac{1}{2}\log\frac{1}{\alpha_f}-\sum_{n=1}^{\infty}n\left|r_n-\frac{e^{-in\theta_0}}{n}\right|^2\right)$$
$$\geqslant\left|\overline{r_1-e^{-i\theta_0}}+\sum_{n=1}^{\infty}n\left(r_n-\frac{e^{-in\theta_0}}{u}\right)\bar{\beta}_n\right|^2. \tag{2.1}$$

此式显然是(1.1)和(1.3)两式的改进.

证　设

$$\log\frac{t-\zeta}{F(t)-F(\zeta)}=\sum_{n=1}^{\infty}A_n(\zeta)t^{-n}=-\sum_{m,n=1}^{\infty}d_{m,n}\zeta^{-m}t^{-n},$$

则有

$$F(\zeta)=\beta_0+\zeta-\sum_{m=1}^{\infty}d_{m,1}\zeta^{-m},\ -d_{m,1}=\beta_m. \tag{2.2}$$

$$F(e^{-i\theta_0})=0,\ \zeta=e^{-i\theta_0},\ t=\frac{1}{z},\ |z|<1,$$

$$\log\frac{f(z)}{z}(1-ze^{-i\theta_0})=2\sum_{n=1}^{\infty}r_nz^n-\sum_{n=1}^{\infty}\frac{e^{-in\theta_0}}{n}z^n=\sum_{n=1}^{\infty}A_n(e^{-i\theta_0})z^n,$$

得

$$A_n(e^{-i\theta_0})=2r_n-\frac{e^{-in\theta_0}}{n}. \tag{2.3}$$

记

$$\Phi_1(z_\mu,z_\nu)=\log\left|\frac{f(z_\mu)-f(z_\nu)}{z_\mu-z_\nu}\cdot\frac{z_\mu z_\nu}{f(z_\mu)f(z_\nu)}\cdot\frac{1}{(1-z_\mu\bar{z}_\nu)}\right|,$$

$$\Phi_2(z_\mu,z_\nu)=\frac{1}{2}\sum_{n=1}^{\infty}n\left(A_n(z_\mu)-\frac{z_\mu^n}{n}\right)\left(\overline{A_n(z_\nu)}-\frac{\bar{z}_\nu^n}{n}\right)$$

有[5]

$$\sum_{\mu,\nu=0}^{N}\lambda_\mu\bar{\lambda}_\nu\Phi_2(z_\mu,z_\nu)\leqslant\sum_{\mu,\nu=0}^{N}\lambda_\mu\bar{\lambda}_\nu\{\Phi_1(z_\mu,z_\nu)\}. \tag{2.4}$$

取 $\lambda_0=x_1y_0$, $\lambda_k=x_2y_k$, $k=1,2,\cdots,N$.

$M_{2,2}=\sum\limits_{\mu,\nu=1}^{N}y_\mu\bar{y}_\nu\{\Phi_1(z_\mu,z_\nu)-\Phi_2(z_\mu,z_\nu)\}$, $M_{1,1}=|y_0|^2\{\Phi_1(z_0,z_0)-\Phi_2(z_0,z_0)\}$

$M_{1,2}=y_0\sum\limits_{\nu=1}^{N}\bar{y}_\nu\{\Phi_1(z_0,z_\nu)-\Phi_2(z_0,z_\nu)\}$.

所以(2.4)可以写为

$$\sum_{l,k=1}^{2}M_{l,k}x_l\bar{x}_k\geqslant0. \tag{2.5}$$

由正定矩阵的性质,有

$$M_{2,2}M_{1,1}-M_{1,2}^2\geqslant0. \tag{2.6}$$

取 $y_k=\rho^{-1}\alpha_k$, $z_k=\rho\rho_k$, $N>1$,

$$\sum_{k=0}^{N}\alpha_k\rho_k^n=\begin{cases}1, & n=1,\\ 0, & n=0,2,3,\cdots,N.\end{cases} \tag{2.7}$$

$$\sum_{\mu,\nu=1}^{N} y_\mu \bar{y}_\nu (\Phi_1(z_\mu, z_\nu) - \Phi_2(z_\mu, z_\nu)) = 1 + Rd_{1,1} - \left\{\frac{1}{2}|d_{1,1}+1|^2 + \frac{1}{2}\sum_{u=2}^{\infty} n|d_{n,1}|^2\right\} + O(\rho)$$

$$= \frac{1}{2} - \frac{1}{2}\sum_{n=1}^{\infty} n|d_{n,1}|^2 + O(\rho) = \frac{1}{2}\left(1 - \sum_{n=1}^{\infty} n|\beta_n|^2\right) + O(\rho). \quad (2.8)$$

此由(2.2)式

$$\sum_{\nu=1}^{N} y_\nu \{\Phi_1(e^{i\theta_0}, z_\nu) - \Phi_2(e^{i\theta_0}, z_\nu)\}$$

$$= 2\mathrm{Re}(e^{i\theta_0} - r_1) - \left(e^{i\theta_0} - r_1 - \sum_{n=1}^{\infty} n\left(r_n - \frac{e^{-in\theta_0}}{n}\right)\bar{d}_{n,1}\right) + O(\rho)$$

$$= \overline{e^{i\theta_0} - r_1} - \sum_{n=1}^{\infty} n\left(r_n - \frac{e^{-in\theta_0}}{n}\right)\bar{\beta}_n + O(\rho). \quad (2.9)$$

$$\Phi_1(z_0, z_0) = \log\left|\frac{f'(z_0)z_0^2}{f^2(z_0)(1-|z_0|^2)}\right| \leqslant \log\frac{|z_0|}{|f(z_0)|(1-|z_0|)^2} \leqslant \log\frac{1}{\alpha_f}. \quad (2.10)$$

此因

$$\alpha_f = \lim_{\rho\to 1}\frac{(1-\rho)^2}{\rho}|f(\rho e^{i\theta_0})| \leqslant \frac{(1-|z_0|)^2}{|z_0|}|f(z_0)|,$$

又

$$\lim_{z_0\to e^{i\theta_0}} \Phi_2(z_0, z_0) = 2\sum_{n=1}^{\infty} n\left|r_n - \frac{e^{-in\theta_0}}{n}\right|^2. \quad (2.11)$$

由(2.6),(2.8),(2.9),(2.10)及(2.11),令 $\rho\to 0$ 即得定理.

推论 同上假设,有

$$\alpha_f \leqslant \exp\{-(2-|a_2|)^2/(1+|a_3-a_2^2|)\} \quad (2.12)$$

定理二 同上假设,有

$$\left\{\left(\frac{1}{2}\log\frac{1}{\alpha_f} - \sum_{n=1}^{\infty} n\left|r_n - \frac{e^{-in\theta_0}}{n}\right|^2\right)\left(1 - \sum_{n=1}^{\infty} n|\beta_n|^2\right) - \left|\overline{e^{-i\theta_0} - r_1} - \sum_{n=1}^{\infty} n\left(r_n - \frac{e^{-in\theta_0}}{n}\right)\bar{\beta}_n\right|^2\right\}.$$

$$\left\{\left(|1-\beta_1|^2 + \sum_{n=1}^{\infty} n|\beta_n|^2\right)\left(\sum_{n=1}^{\infty} n\left|r_n - \frac{e^{-in\theta_0}}{n}\right|^2 - \left|e^{-i\theta_0} - r_1 + \sum_{n=1}^{\infty} n\left(r_n - \frac{e^{-in\theta_0}}{n}\right)\bar{\beta}_n\right|^2\right\}$$

$$\leqslant \left\{\frac{1-R\beta_1}{4}\log\frac{1}{\alpha_f} - |\mathrm{Re}(e^{i\theta} - r_1)|^2\right\}^2. \quad (2.13)$$

证 由 Ky Fan[8] 不等式,若矩阵 A, C 为正定的,则

$$|A|\cdot|C| \leqslant \left|\frac{1}{2}(A+C)\right|^2 \quad (2.14)$$

取

$$B_{2,2} = \sum_{\mu,\nu=1}^{N} y_\mu \bar{y}_\nu \Phi_1(z_\mu, z_\nu), \quad B_{1,2} = y_0 \sum_{\nu=1}^{N} \bar{y}_\nu \Phi_2(z_0, z_\nu),$$

$$B_{1,1} = |y_0|^2 \Phi_1(z_0, z_0),$$

$$C_{2,2} = \sum_{\mu,\nu=1}^{N} y_\mu \bar{y}_\nu \Phi_2(z_\mu, z_\nu), \quad C_{1,2} = y_0 \sum_{\nu=1}^{N} \bar{y}_\nu \Phi_\nu(z_0, z_\nu),$$

$$C_{1,1} = |y_0|^2 \Phi_2(z_0, z_0),$$

$$B = (B_{l,k})_{1<l,k<2}, \qquad C = (C_{lk})_{1<l,k<2},$$

$$A = B - C,$$

A, B, C 均为正定矩阵. 用证明定理一的计算结果代入(2.13),即得定理.

若用[5]

$$\frac{1}{\left|\frac{1}{2}(A+C)\right|^2}\leqslant\frac{1}{|A|\cdot|C|}-\left(\frac{1}{\sqrt{|A+B|\cdot|C|}}-\frac{1}{\sqrt{|A|\cdot|B+C|}}\right)^2. \tag{2.15}$$

便可得到比定理二更好的结果.

三、海曼定理的新证明

引理一 设 $f(z)\in S$, $\left\{\frac{f(z)}{z}\right\}^\lambda=1+\sum\limits_{n=1}^{\infty}D_n(\lambda)z^n$, 若

$$\max_{|z|=\rho}|f(z)|=|f(\rho e^{i\theta_0})|,\ \lim_{\rho\to1}\frac{(1-\rho)^2}{\rho}|f(\rho e^{i\theta_0})|=\alpha_f,$$

则

$$\lim_{n\to\infty}\frac{1}{d_n(2\lambda+1)}\sum_{k=0}^{n}\frac{|D_k(\lambda)|^2}{d_k(2\lambda)}=A\leqslant1. \tag{3.1}$$

此地 $d_n(2\lambda)$ 为 $\frac{1}{(1-x)^{2\lambda}}=\sum\limits_{n=0}^{\infty}d_n(2\lambda)x^n$ 中的系数.

证 设

$$\omega(z)=\log\left\{\frac{f(z)}{z}\right\}^\lambda=\sum_{k=1}^{\infty}A_kz^k=2\lambda\sum_{k=1}^{\infty}r_kz^k,$$

$$\varphi(z)=\exp w(z)=\left\{\frac{f(z)}{z}\right\}^\lambda=1+\sum_{k=1}^{\infty}D_k(\lambda)z^k,$$

由米林-列别杰夫定理[7], 若记

$$\frac{1}{d_n(2\lambda+1)}\cdot\sum_{k=0}^{n}\frac{|D_k(\lambda)|^2}{d_k(2\lambda)}=\theta_n(2\lambda)\exp\frac{2\lambda}{d_n(2\lambda+1)}\left\{\sum_{k=1}^{n}d_{n-k}(2\lambda)\,\varDelta_k(\lambda)\right\},$$

$$\varDelta_k(\lambda)=\frac{1}{(2\lambda)^2}\sum_{l=1}^{k}l|A_l|^2-\sum_{l=1}^{k}\frac{1}{l}=\sum_{l=1}^{k}l|r_l|^2-\sum_{l=1}^{k}\frac{1}{l},$$

则 $\theta_n(2\lambda)$ 为 n 的单调减少函数, 所以要证明(3.1)式, 只要证明

$$\lim_{n\to\infty}\frac{2\lambda}{d_n(2\lambda+1)}\sum_{k=1}^{n}d_{n-k}(2\lambda)\varDelta_k(\lambda). \tag{3.2}$$

存在就可以了, 但米林[8]已证. $\lim\limits_{k\to\infty}\varDelta_k(\lambda)$ 存在 $\left(\leqslant\frac{1}{2}\log\alpha_f\right)$, 因此(3.2)成立. 所以(3.1)成立.

引理二 同上假设, 当 $\lambda>\frac{1}{4}$ 时, 有

$$\lim_{\substack{n\to\infty\\ m/n\to1}}\left|\frac{|D_n(\lambda)|-|D_m(\lambda)|}{d_n(2\lambda)}\right|=0, \tag{3.3}$$

$\lambda=\frac{1}{2}$, 1 为[9]中结果.

证 设 $M(\rho, f)=|f(\rho)|$

$$\phi'(z)=\lambda\frac{f^{\lambda-1}(z)}{z^\lambda}f'(z)-\frac{\lambda}{z^{\lambda+1}}f^\lambda(z),$$

又

$$f^{\frac{1}{p}}(z^p)\in S_p,\ f'(z^p)z^{p-1}=f_p'(z)f^{p-1}(z),$$

所以

$$f'(z)=z^{\frac{1}{p}-1}f_p'(z^{\frac{1}{p}})f^{1-\frac{1}{p}}(z), \tag{3.4}$$

$$|nD_n(\lambda)\rho^n-mD_m(\lambda)\rho^m|=\frac{1}{2\pi}\left|\int_0^{2\pi}(e^{-in\theta}-e^{-im\theta})\phi'(\rho e^{i\theta})\rho e^{i\theta}d\theta\right|$$

$$\leqslant\frac{\rho}{2\pi}\int_0^{2\pi}|1-e^{i(n-m)\theta}|\cdot|\phi'|d\theta$$

$$\leqslant A_1\int_0^{2\pi}|e^{-i(n-m)\theta}-1||f|^{\lambda-\frac{1}{p}}|f_p'|d\theta+A_2\int_0^{2\pi}|f|^\lambda d\theta$$

$$=A_1J_1+A_2J_2. \tag{3.5}$$

$$J_1\leqslant 2^{1-\delta}\max_{0\leqslant\theta\leqslant 2\pi}|e^{i\theta}-1|^\delta|f(\rho e^{i\theta})|^\delta\cdot\int_0^{2\pi}\left|\frac{e^{i(n-m)\theta}-1}{e^{i\theta}-1}\right|^\delta|f|^{\lambda-\frac{1}{p}-\delta}|f_p'|d\theta. \tag{3.6}$$

因 $|e^{i\theta}-1||f(\rho^{i\theta})|\leqslant\frac{A}{1-\rho}$, 又由 Hölder 不等式, 得

$$J_1\leqslant\frac{A}{(1-\rho)^\delta}\left(\int_0^{2\pi}\left|\frac{e^{i(n-m)\theta}-1}{e^{i\theta}-1}\right|^2d\theta\right)^{\frac{\delta}{2}}\left(\int_0^{2\pi}|f|^{\left(\lambda-\frac{1}{p}-\delta\right)\frac{2}{1-\delta}}d\theta\right)^{\frac{1}{2}(1-\delta)}\left(\int_0^{2\pi}|f_\rho'|^2d\theta\right)^{\frac{1}{2}}. \tag{3.7}$$

因 $\lambda>\frac{1}{4}$, 取 p 及 δ, 使 $4\left(\lambda-\frac{1}{p}-\delta\right)>1-\delta$, 有

$$\int_0^{2\pi}|f|^{\left(\lambda-\frac{1}{p}-\delta\right)\frac{2}{1-\delta}}d\theta\leqslant\frac{A_1}{(1-\rho)^{\left(\lambda-\frac{1}{p}-\delta\right)\frac{2}{1-\delta}-1}}. \tag{3.8}$$

又

$$\int_0^{2\pi}|f_p'|^2d\theta\leqslant\frac{A_2}{(1-\rho)^{1+\frac{4}{p}}}. \tag{3.9}$$

由(3.6), (3.7), (3.8)及(3.9)得

$$J_1\leqslant\frac{A_3}{(1-\rho)^{2\lambda}}(n-m)^{\frac{\delta}{2}}(1-\rho)^{\frac{\delta}{2}}. \tag{3.10}$$

当 $\lambda>\frac{3}{4}$ 时, 取 δ 使 $4(\lambda-\delta)>2-\delta$,

$$J_2=\int_0^{2\pi}|e^{i(n-m)\theta}-1||f^\lambda(z)|d\theta$$

$$\leqslant 2^{1-\delta}\max_{0\leqslant\theta\leqslant 2\pi}|1-e^{i\theta}|^\delta|f^\delta(\rho e^{i\theta})|\int_0^{2\pi}\left|\frac{e^{i(n-m)\theta}-1}{|e^{i\theta}-1|}\right|^\delta|f|^{\lambda-\delta}d\theta$$

$$\leqslant\frac{A_4}{(1-\rho)^\delta}\left\{\int_0^{2\pi}\left|\frac{e^{i(n-m)\theta}-1}{e^{i\theta}-1}\right|^2d\theta\right\}^{\frac{\delta}{2}}\left\{\int_0^{2\pi}|f|^{(\lambda-\delta)\frac{2}{2-\delta}}d\theta\right\}^{1-\frac{\delta}{2}}$$

$$=\frac{A_5}{(1-\rho)^{2\lambda}}(n-m)^{\frac{\delta}{2}}(1-\rho)^{\frac{\delta}{2}}. \tag{3.11}$$

当 $0<\lambda<\frac{3}{4}$ 时, 有

$$J_2=\frac{1}{2\pi}\int_0^{2\pi}|e^{i(n-m)\theta}-1||f|^\lambda d\theta\leqslant 2\cdot\frac{1}{2\pi}\int_0^{2\pi}|f|^\lambda d\theta\leqslant A_5\left(\int_0^{2\pi}|f|^{\frac{3}{4}}d\theta\right)^{\frac{4\lambda}{3}}$$

$$\leqslant\frac{A_6}{(1-\rho)^{\frac{2}{3}\lambda}}\leqslant\frac{A_6}{(1-\rho)^{\frac{1}{2}}}. \tag{3.12}$$

取 $\rho^{n-m}=\frac{m}{n}$, $m=n-[\epsilon n]$, $\epsilon>0$, $\rho=\sqrt[n-m]{1-\frac{n-m}{n}}=e^{\frac{1}{[\epsilon n]}\log\left(1-\frac{[\epsilon n]}{n}\right)}\sim e^{\frac{1}{n}}$, 所以由(3.10), (3.11), (3.12)及(3.5)得

$$n||D_n(\lambda)|-|D_m(\lambda)||\leqslant Bn^{2\lambda}\varepsilon^{\frac{\delta}{2}}. \tag{3.13}$$

此即所求结果.

引理三 设 $S_n=\frac{1}{d_n(\lambda+1)}\sum_{k=0}^{n}C_k, \alpha>0$, 若 (i) $\lim_{n\to\infty}S_n=A$, (ii) $\lim_{\substack{n\to\infty\\ m/n\to 1}}\frac{|C_n|-|C_m|}{d_n(\alpha)}=0$, 则 $\lim_{n\to\infty}\frac{C_n}{d_n(\alpha)}=A$.

证 由等式

$$C_m(n-m)=S_n\{d_n(\alpha+1)-d_m(\alpha+1)\}+d_m(\alpha+1)(S_n-S_m)+\sum_{k=m+1}^{n}(C_m-C_k)$$
$$=I_1+I_2+I_3.$$

因 $\lim_{k\to\infty}S_k=A$ 所以可取 $m>m_0$, $|S_m-A|\leqslant\varepsilon$, 又取 $\delta=\sqrt{\varepsilon}$ 及 $n=m+[\delta m]$ 得

$$I_2\leqslant 2\varepsilon d_m(\alpha+1). \tag{3.15}$$

由假设

$$I_3=o\left(\sum_{k=m+1}^{n}d_k(\alpha)\right)=o\left(\sum_{k=m+1}^{n}n^{\alpha-1}\right)=o((n-m)n^{\alpha-1}). \tag{3.16}$$

又

$$d_n(\alpha+1)\doteq\frac{n^{\alpha}}{\Gamma(\alpha+1)},$$

$$I_1\approx S_n\frac{n^{\alpha}-m^{\alpha}}{\Gamma(\alpha+1)}\approx\frac{\alpha}{\Gamma(\alpha+1)}n^{\alpha-1}(n-m)S_n. \tag{3.17}$$

由(3.16), (3.17), (3.18)及(3.15)立刻得出所要求的结果.

定理三(海曼定理) 设 $f(z)\in S$, $\varphi(z)=\left\{\frac{f(z)}{z}\right\}^{\lambda}=1+\sum_{n=1}^{\infty}D_n(\lambda)z^n$, 则

$$\lim_{n\to\infty}\frac{|D_n(\lambda)|}{d_n(2\lambda)}=A\leqslant 1. \tag{3.18}$$

证 因已知 $|D_n(\lambda)|=O(d_n(2\lambda))$, $\lambda>\frac{1}{4}$. 所以由引理二.

$$\left|\frac{|D_n^2(\lambda)|}{d_n(2\lambda)}-\frac{|D_m^2(\lambda)|}{d_m(2\lambda)}\right|\leqslant\left|\frac{|D_m^2(\lambda)|-|D_m^2(\lambda)|}{d_n(2\lambda)}\right|+\frac{|D_m^2(\lambda)||d_n(2\lambda)-d_m(2\lambda)|}{d_n(2\lambda)d_m(2\lambda)}$$
$$=O\left(\frac{|D_n(\lambda)|\cdot||D_n(\lambda)|-|D_m(\lambda)||}{d_n(2\lambda)}\right)+O(d_n(2\lambda)-d_m(2\lambda))$$
$$=o(d_n(2\lambda))+O(|n^{2\lambda-1}-m^{2\lambda-1}|)=o(d_n(2\lambda)). \tag{3.19}$$

由引理一及引理三及(3.19)即得定理的证明.

四、星象函数系数的渐近性质

设 $f(z)\in S^*$, 在[4]中已证海曼定理在 $0<\lambda\leqslant\frac{1}{4}$ 亦成立. 当时用的是海曼方法, 在此我们完全不用海曼方法, 给出两个新的证明. 先证

引理四 若 $f(z)\in S^*$, $\varphi(z)=1+\sum_{n=1}^{\infty}D_n(\lambda)z^n$, 则当 $\lambda>0$ 时, 有

$$\lim_{\substack{n\to\infty\\ m/n\to 1}}\frac{||D_n(\lambda)|-|D_m(\lambda)||}{d_n(2\lambda+1)}=0. \tag{4.1}$$

证　因 $f(z)\in S^*$, $|D_n(\lambda)|\leqslant d_n(2\lambda)$, 并且[3]当 $\alpha\geqslant 1$ 时,

$$\int_0^{2\pi}\left|\frac{\phi'(\rho e^{i\theta})}{\phi(\rho e^{i\theta})}\right|^{\alpha}d\theta\leqslant\frac{A}{(1-\rho)^{\alpha-1}},\ 0<\rho_0\leqslant\rho<1, \tag{4.2}$$

设 $\max\limits_{0\leqslant\theta\leqslant 2\pi}|f(\rho e^{i\theta})|=|f(\rho)|$, 有

$$\begin{aligned}|n\rho^n D_n(\lambda)-m\rho^m D_m(\lambda)|&=\frac{1}{2\pi}\left|\int_0^{2\pi}(e^{-in\theta}-e^{-im\theta})\varphi'(\rho e^{i\theta})\rho e^{i\theta}\,d\theta\right|\\&\leqslant\frac{2^{1-\lambda}}{2\pi}\max_{0\leqslant\theta\leqslant 2\pi}|1-e^{i\theta}|^{\lambda}\frac{|f^{\lambda}(\rho e^{i\theta})|}{\rho^{\lambda}}\int_0^{2\pi}\left|\frac{e^{i(n-m)\theta}-1}{e^{i\theta}-1}\right|^{\lambda}\left|\frac{\varphi'(\rho e^{i\theta})}{\varphi(\rho e^{i\theta})}\right|d\theta\\&\leqslant\frac{B}{(1-\rho)^{\lambda}}\left\{\int_0^{2\pi}\left|\frac{e^{i(n-m)\theta}-1}{e^{i\theta}-1}\right|^2 d\theta\right\}^{\frac{\lambda}{2}}\left\{\int_0^{2\pi}\left|\frac{\varphi'(\rho e^{i\theta})}{\varphi(\rho e^{i\theta})}\right|^{\frac{2}{2-\lambda}}d\theta\right\}^{\frac{2-\lambda}{2}}\\&\leqslant\frac{B_1}{(1-\rho)^{2\lambda}}(n-m)^{\frac{\lambda}{2}}(1-\rho)^{\frac{\lambda}{2}}.\end{aligned}\tag{4.3}$$

取 $\rho^{n-m}=\dfrac{m}{n}$, $m=n-[\epsilon n]$ $(\epsilon>0)$ 即得引理的证明. 从定理三的证明方法立刻有:

定理四[3]　设 $f(z)\in S^*$, $\phi(z)=\left\{\dfrac{\phi(z)}{z}\right\}^{\lambda}=1+\sum\limits_{n=1}^{\infty}D_n z^n$, 则

$$\lim\frac{|D_n(\lambda)|}{d_n(2\lambda)}=A<1. \tag{4.4}$$

此定理的证明, 依据米林-列别杰夫的一定理. 但米-列定理本身的证明很长也不直观, 在下面给出(4.4)的另一证明. 即我们不从米-列定理导出(3.1)而代之陶伯型定理导出(3.1)先证一些引理.

引理五[10]　若 $b_n\geqslant 0$, $g(x)=\sum\limits_{n=0}^{\infty}b_n x^n\sim(1-x)^{-\alpha}$, $\alpha>0$ $(x\to 1)$, 则

$$g'(x)\sim\frac{\alpha}{(1-x)^{\alpha+1}}. \tag{4.5}$$

引理六　同引理五的假设, 有

$$\sum_{k=1}^{n}\frac{b_k}{k^{\alpha-1}}\sim\frac{n}{\Gamma(\alpha)}. \tag{4.6}$$

证　由引理五

$$g'(x)=\sum_{n=1}^{\infty}nb_n x^{n-1}\sim\frac{\alpha}{(1-x)^{\alpha+1}},$$

$$\begin{aligned}g_\alpha(x)&=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}g'(t)\,dt\\&=\frac{1}{\Gamma(\alpha)}\sum_{n=1}^{\infty}nb_n x^{n+\alpha}\int_0^1(1-y)^{\alpha-1}y^{n-1}dy=\sum_{n=1}^{\infty}nb_n x^{n+\alpha}\frac{\Gamma(n)}{\Gamma(n+\alpha)}\\&\approx\sum_{n=1}^{\infty}\frac{b_n}{n^{\alpha-1}}x^n.\end{aligned}\tag{4.7}$$

另一方面

$$\begin{aligned}g_\alpha(x)&\approx\frac{\alpha}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}(1-t)^{-\alpha-1}dt=\frac{\alpha}{\Gamma(\alpha)}x^{\alpha}\int_0^1(1-y)^{\alpha-1}(1-xy)^{-\alpha-1}dy\\&=\sum_{n=1}^{\infty}nd_n(\alpha)\frac{\Gamma(n)}{\Gamma(n+\alpha)}x^{n+\alpha-1}\approx\frac{1}{\Gamma(\alpha)}\sum_{n=1}^{\infty}x^{n-1}\approx\frac{1}{\Gamma(\alpha)(1-x)}.\end{aligned}\tag{4.8}$$

所以

$$g_\alpha(x)=\sum_{n=1}^{\infty}\frac{b_n}{n^{\alpha-1}}x^{n-1}\approx\frac{1}{\Gamma(\alpha)(1-x)}. \tag{4.9}$$

由立特伍特-哈特定理即得定理.

引理七 设 $f(z)\in S^*$, $\phi(z)=\left\{\frac{f(z)}{z}\right\}^\lambda=1+\sum_{n=1}^{\infty}D_nz^n$, 记 $\|\phi(x)\|=1+\sum_{n=1}^{\infty}|D_n(\lambda)|x^n$, 则当 $0<x_2\leqslant x_1<1$ 时, 有 $(1-x_2)^{2\lambda}\|\phi(x_2)\|\leqslant(1-x_1)^{2\lambda}\|\phi(x_1)\|$, 即有

$$\lim_{x\to1}(1-x)^{2\lambda}\|\phi(x)\|=A\leqslant1.$$

证 因

$$x\frac{\phi'(x)}{\phi(x)}=\lambda x\frac{f'(x)}{f(x)}-\lambda=\lambda\sum_{n=1}^{\infty}\alpha_nx^n, \tag{4.10}$$

$$x\phi'(x)=\lambda\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}\alpha_{n-k}D_k\right)x^n, \tag{4.11}$$

$$nD_n(\lambda)=\lambda\sum_{k=0}^{n-1}\alpha_{n-k}D_k.$$

但因 $f(z)\in S^*$, $\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\}\geqslant0$, $|\alpha_n|\leqslant2$, 所以

$$n|D_n(\lambda)|\leqslant2\lambda\sum_{k=0}^{n-1}|D_k|. \tag{4.12}$$

因此 $\sum n|D_n(\lambda)|x^{n-1}\leqslant2\lambda\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}|D_k|\right)x^{n-1}=\frac{2\lambda}{1-x}\sum_{k=0}^{\infty}|D_k|x^k$.

即

$$\frac{\sum_{n=1}^{\infty}n|D_n(\lambda)|x^{n-1}}{\sum_{n=0}^{\infty}|D_n|x^n}\leqslant\frac{2\lambda}{1-x}. \tag{4.13}$$

从 x_2 到 x_1 对(4.13)两边积分, 即得定理.

现在来给定理四另一证明. 由引理六、七即得

$$\sum_{k=0}^{n}\frac{|D_n|}{n^{2\lambda-1}}\sim\frac{An}{\Gamma(2\lambda)}. \tag{4.14}$$

由引理四得, 当 $n\to\infty$, $\frac{m}{n}\to1$ 时有

$$\left|\frac{|D_n(\lambda)|}{d_n(2\lambda)}-\frac{|D_m(\lambda)|}{d_m(2\lambda)}\right|\leqslant\frac{||D_n(\lambda)|-|D_m(\lambda)||}{d_n(2\lambda)}+\frac{|D_m(\lambda)|}{d_m(2\lambda)}\cdot\frac{d_n(2\lambda)-d_m(2\lambda)}{d_n(2\lambda)}=o(1). \tag{4.15}$$

由(4.14), (4.15)适合引理三($\alpha=1$)的条件, 因此即刻得出结论. 证毕.

参考文献

[1] Базилевич, И. Е., *Матем. сб.* **74**(1967), 133—146.
[2] Hayman, W. K., *Proc. Lond. Math. Soc.*, **5**: 19, 257—284,
[3] 胡克, 复旦学报, 1(1957), 55—58.
[4] 胡克, 复旦学报, (1956), 77—81.
[5] 胡克, 江西师院学报(自然科学版), 1(1979), 9—10.
[6] 任福尧, 复旦学报, 4(1981), 402—410.
[7] Милин, И. М., Однолистные Функчий и ортонормированные системы, 87—100.
[8] Ky Fan, *Proc. Nat. Acad. Sci. U. S. A.*, **36**(1950), 31—35.
[9] 龚升, 数学学报, **6**(1956), 115—125.
[10] Hardy, G, H., Divergent Series, 170—172.

THE ASYMPTOTIC BEHAVIOUR OF THE COEFFICIENTS OF SCHLICHT FUNCTIONS

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ABSTRACT

Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in S$, $\dfrac{1}{f\left(\frac{1}{\zeta}\right)}=\zeta+\sum_{n=0}^{\infty}\beta_n\dfrac{1}{\zeta^n}(|\zeta|>1)$, $\log\dfrac{f(z)}{z}=2\sum_{n=1}^{\infty}r_nz^n$ and $\lim_{\rho\to1}\dfrac{(1-\rho)^2}{\rho}\max_{|z|=\rho}|f(z)|=\lim_{\rho\to1}\dfrac{(1-\rho)^2}{\rho}|f(\rho e^{i\theta_0})|=\alpha_f$. Bazilivic, I. E. proved the relation

$$\sum_{n=1}^{\infty}n\left|r_n-\frac{1}{n}e^{-in\theta_0}\right|^2\leqslant\frac{1}{2}\log\frac{1}{\alpha_f}. \tag{1}$$

Hayman, W. K. proved that

$$\lim_{n\to\infty}\frac{\left|\left\{\frac{f(z)}{z}\right\}_n^{\lambda}\right|}{d_n(2\lambda)}=\alpha_f^{\lambda},\ \lambda>\frac{1}{4}. \tag{2}$$

where $d_n(t)$ defined by $\dfrac{1}{(1-x)^t}=\sum_{n=1}^{\infty}d_n(t)x^n, t>0, |x|<1$, and principal area is

$$\sum_{n=1}^{\infty}n|\beta_n|^2\leqslant1. \tag{3}$$

In this paper

(i) We prove.

Theorem.

$$\left(1-\sum_{n=1}^{\infty}n|\beta_n|^2\right)\left(\frac{1}{2}\log\frac{1}{\alpha_f}-\sum_{n=1}^{\infty}n\left|r_n-\frac{e^{-in\theta_0}}{n}\right|^2\right)\geqslant\left|\overline{r_1-e^{-i\theta_0}}+\sum_{n=1}^{\infty}n\left(r_n-\frac{e^{-in\theta_0}}{n}\right)\overline{\beta}_n\right|^2. \tag{4}$$

This result improves inequality (1) or (3).

(ii) We give a new proof of Hayman theorem

(iii) If $f(z)\in S^*$, we give two new proofs of Hayman theorem for $0<\lambda\leqslant1$.