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ON A ρ-INCREASING FAMILY OF POINT-TO-SET MAPS

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Dedicated to Professor Su Bu-chin on the Occasion of His 80th Birthday and his 50th Year of Educational Work

1. Introduction.

Since the publication of Zangwill's book "Nonlinear Programming-A Unified Approch" in 1969, the theory of point-to-set maps has come into increasing use during the last ten or more years in papers on optimization, especially, on the convergence theory of the iteration algorithms (Cf. MP Study no. 10, 1979). One of the advantages of the approach introduced by Zangwill is that one can treat in a unified way the problems concerning the convergence properties of algorithms. Zangwill showed that a lot of methods can be viewed as applications of the fixed point method $x_{i+1}\in\Gamma(x_i)$, where Γ is a point-to-set map depending upon the given particular algorithm.

Huard (1975) made a research on Zangwill's results. He designed two models of algorithms (Algorithms 4.2 and 5.2 in his paper), and proved that these algorithms, under some specified conditions, can be viewed as special cases of Zangwill's model. Therefore, Zangwill's results can be applied to these models. The advantage which Huard's models have over the Zangwill's is that it can be used more easily to decide whether a given algorithm belongs or not to these models, and, if it does, the convergence property follows as a consequence.

Denel (1979) introduced a ρ-decreasing family of point-to-set maps and extended Zangwill's approach from another point of view. In his paper (1979), Denel introduced many notions such as monotone decreasing family, uniform regularity, pseudo upper (lower) continuity, etc., and several algorithms. He proved that his algorithms have some convergence properties if certain conditions are satisfied. Denel claimed that his algorithm A_1' leads to the same sequence as those produced by the algorithm 4.2 given by Huard (1975). However, we find that when the sequence $\{\gamma_n\}$ (Cf. Denel (1979)) is given, the sequence produced by A_1' is only a part of those produced by the algorithm 4.2, because the parameter ρ introduced in A_1' has influence on the sequence. Therefore, Huard's algorithm 4.2 can not be said to be a special case of A_1'. However, as pointed

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out by Denel (1979) through two examples, with his algorithm A_1', one can modelize a lot of well-known methods that can not be modelized with the classical approachs, so Denel's approach is, in a sense, a further development of the classical ones.

Huard (1979) made an extension of Zangwill's results, but he did not show what is the use of his result.

In this paper we introduce a ρ-increasing family of point-to-set maps, then define two algorithms and show that under certain simple conditions every accumulation point of the sequence generated by these algorithms belongs to a well-defined set. Then we prove that the results of Zangwill (1969) and Huard (1975) are direct consequences of our results. We prove also that Huard's algorithm (Proposition D (1979)) is convergent if our conditions just mentioned are satisfied. To show that our approach is a real extension of the classical ones, we prove that the convergence of linearized method of centers with partial linearization, which, as Denel showed, could not be modelized with the classical approachs, is a direct consequence of our results.

2. Notations and Definitions.

The following notations are used throughout this paper:

$N=\{1, 2, 3, \cdots\}$;

$E\subset R^n$, a compact set;

$\mathscr{P}(E)$, set of the subsets of E;

$V(x)$, a relative neighbourhood of x, i. e., $V(x)=E\cap U(x)$, where $U(x)$ is a euclidean neighbourhood;

E^0, the interior of E;

$\{F_\rho(x)\,|\,\rho\geqslant 0\}$, an increasing family of point-to-set maps $F_\rho(x): E\to\mathscr{P}(E)$, i. e., the family satisfying

$$F_{\rho_1}(x)\supset F_{\rho_2}(x),\quad \forall\rho_2\geqslant\rho_1\geqslant 0,\quad \forall x\in E;$$

$\Omega\subset R^n$, a given set, $\Omega\supseteq\{x\,|\,x\in F_0(x)\}$;

$P=\{X\in E\,|\,x\in\Omega$ or $F_0(x)=\emptyset\}$.

Definition. *A point-to-set map $F: E\to\mathscr{P}(E)$ is said to be upper continuous* (u. c.) *at $x\in E$, if for any given sequence $\{x_k\in E,\ k\in N\}\to x$ and sequence $\{y_k\in E,\ k\in N\}\to y$ satisfying $y_k\in F(x_k)$, $\forall k\in N$, we have $y\in F(x)$.*

F is said to be upper continuous on E if it is upper continuous at every point of E.

Definition. *A point-to-set map F: $E\to\mathscr{P}(E)$ is said to be lower continuous* (l. c.) *at $x\in E$, if for any sequence $\{x_k\in E,\ k\in N\}\to x$ and any $y\in F(x)$ there exists sequence $\{y_k\in E,\ k\in N\}\to y$ and positive integer k_0 such that $y_k\in E(x_k)$, $\forall k\geqslant k_0$.*

F is said to be lower continuous on E if it is lower continuous at every point of E.

Definition. *F is said to be continuous on E if it is both upper and lower continuous on E.*

3. Assumptions and a Fundamental Theorem.

We make the following assumptions:

H1: If $\rho_\nu \to 0$, $x_\nu \to x_0$, $y_\nu \in F_{\rho_\nu}(x_\nu)$, $y_\nu \to y_0$, then we have $y_0 \in F_0(x_0)$;

H2: There exists a function $h(\cdot)$ continuous on E having the following property: If for some $\rho \geqslant 0$ we have $x' \in F_\rho(x)$ $x \overline{\in} F_\rho(x)$, where $x \overline{\in} P$, then we have $h(x') > h(x)$.

Theorem 1. *Under the assumptions* H1 *and* H2, *there exist, for any given* $x_0 \in E \backslash P$, $V(x_0)$ and $\rho_0 > 0$, *such that*

$$h(x'') > h(x_0), \quad \forall x' \in (E \backslash P) \cap V(x_0), \quad \forall x'' \in F_{\rho_0}(x').$$

Proof Assume that the conclusion is false. Then for any given monotonely decreasing sequence $\{\rho_i\}$ of nonnegative reals converging to zero, there exist sequences $\{x_i'\}$ and $\{x_i''\}$ with $x_i' \in E \backslash P$, $x_i'' \in F_{\rho_i}(x_i')$, $\forall i \in N$, $\{x_i'\} \to x_0$, such that $h(x_i'') \leqslant h(x_0)$, $\forall i$. By assumptions, E is compact, so that there exists $N' \subset N$ such that $\{x_i'', i \in N'\} \to x_0''$. By H1, we have $x_0'' \in F_0(x_0)$. Since h is continuous on E, we have $h(x_0'') \leqslant h(x_0)$. On the other hand, by H2, we have $h(x_0'') > h(x_0)$, since $x_0 \overline{\in} P$. Thus we have a contradiction.

Corollary 1. *Under* H1 *and* H2, *for* $x_0 \in E \backslash P$, *there exist* $V(x_0)$ *and* $\rho_0 > 0$, *such that* $h(x'') > h(x_0)$, $\forall 0 \leqslant \rho \leqslant \rho_0$, $\forall x' \in (E \backslash P) \cap V(x_0)$, $\forall x'' \in F_\rho(x')$.

Proof This is a direct consequence of Theorem 1 and the monotonicity of $\{F_\rho \mid \rho \geqslant 0\}$.

4. Algorithms.

Algorithm A1. Let $x_0 \in E$ be the starting point and $\{\rho_n\}$ be a given monotonely decreasing sequence of nonnegative reals converging to zero.

Step n:

if $x_n \in P$, then stop;

$$\left.\begin{array}{l}\text{if } F_{\rho_n}(x_n) = \emptyset \\ \text{or } x_n \in F_{\rho_n}(x_n)\end{array}\right\}, \text{ then } \begin{cases} x_{n+1} = x_n, \\ \rho_{n+1} \leftarrow \rho_n; \end{cases}$$

$$\left.\begin{array}{l}\text{if } F_{\rho_n}(x_n) \neq \emptyset \\ \text{and } x_n \overline{\in} F_{\rho_n}(x_n),\end{array}\right\} \text{then } \begin{cases} \text{choose } x_{n+1} \in F_{\rho_n}(x_n), \\ \rho_{n+1} \leftarrow \rho_n; \end{cases}$$

End of step n.

5. Convergence property.

Theorem 2. *Let $\{\rho_n\}$ be a given monotonely decreasing sequence converging to zero. Then, under the assumptions* H1 *and* H2, *algorithm* A1 *either constructs an infinite* *sequence having all its accumulation points in P, or terminates at an $x_k \in P$ in a finite number of steps.*

Proof Obviously we can suppose $x_n \overline{\in} P$, $\forall n \in N$. By the definition of $\{x_n\}$ and the assumption H2, it is evident that $\{h(x_n)\}$ is a monotonely increasing sequence, so that it has a limit. We have $F_{\rho_n}(x_n) \neq \emptyset$, $\forall n \in N$, otherwise there exists an n for which $F_{\rho_n}(x_n) = \emptyset$. By the monotonicity of F_ρ, we have $F_0(x_n) = \emptyset$, this means $x_n \in P$ and contradicts our assumption.

Let $\{x_n\}_{N'} \to x^*$. Without loss of generality we assume $x_n \overline{\in} F_{\rho_n}(x_n)$, $\forall n \in N'$. If it is not so, then there are only two possibilities:

(i) From some n_0 onwards we have $x_n \in F_{\rho_n}(x_n)$, $\forall n \in N'$. In this case, by H1, we must have $x^* \in F_0(x^*)$, and the Theorem holds;

(ii) There is an infinite sequence $N'' \subset N'$, for which $x_n \overline{\in} F_{\rho_n}(x_n)$. In this case, we replace N' by N'' without violating the limit of $\{x_n\}_{N'}$.

Now assume that $x^* \overline{\in} P$. By Corollary 1, there exist $V(x^*)$ and $\rho_0 > 0$, such that $h(x'') > h(x^*)$, $\forall 0 \leqslant \rho \leqslant \rho_0$, $\forall x' \in (E \setminus P) \cap V(x^*)$, $\forall x'' \in F_\rho(x')$. Therefore, there exists $n_0 > 0$ such that $h(x_{n+1}) > h(x^*)$, $\forall v \geqslant n_0 + 1$, $n \in N'$ and $x_{n+1} \in F_{\rho_n}(x_n)$. Since $\{h(x_n)\}$ is a monotonely increasing sequence, we have $h(x_n) \geqslant h(x_{n_0+2}) > h(x^*)$, $\forall n \geqslant n_0 + 1$.

By the continuity of $h(x)$, we have $h(x^*) > h(x^*)$, a contradiction. So we must have $x^* \in P$.

6. Applications of Theorem 2.

In this section we will prove that Zangwill's theorem and Huard's theorem (Prop. 4.1, 1975) are special cases of Theorem 2.

Zangwill Theorem *Let $E \subset R$ be compact, $P \subset E$ be a given solution set, F: $E \to \mathscr{P}(E)$ be a point-to-set map, and f: $E \to R$ be a continuous function. Suppose:*

(i) $F(x) \neq \emptyset$, $\forall x \in E \setminus P$;

(ii) *F is upper continuous on $E \setminus P$;*

(iii) $f(y) > f(x)$, $\forall y \in F(x)$, $\forall x \in E \setminus P$.

Let A be the algorithm defined on E:

Starting value: $x_0 \in E$;

Step n: if $x_n \in P$, *then stop;*

if $x_n \overline{\in} P$, *take* $x_{n+1} \in F(x_n)$.

Then A either constructs an infinite sequence having all its accumulation points in P or terminates at an $x_k \in P$ in a finite number of steps.

Corollary of Theorem 2. *Zangwill's theorem holds.*

Proof Let, in Theorem 2, $F_\rho(x)$ be independent of ρ, i. e., $F(x) = F_\rho(x)$, $\forall \rho \geqslant 0$. Assumption (ii) implies that H1 holds. Assumption (iii) implies that H2 holds (in this case, we take $h(x) = f(x)$).

Huard (1975) proved the following Theorem (Prop. 4.1):

Let $T \subset R_+$ be compact;

h: $R^n \to R$ be continuous;

Δ: $E \to \mathscr{P}(E)$ be a point-to-set map such that

(i) Δ is continuous on E;

(ii) $x \in \Delta(x)$, $\forall x \in E$;

g: $E \times E \to R$ be continuous;

$M_\Delta(x, \varepsilon) = \{y \in \Delta(x) \mid g(y, x) \geqslant \max_{z \in \Delta(x)} g(z, x) - \varepsilon\}$.

The function g is supposed such that

$\forall (x, \varepsilon) \in E \times T$, $\forall y \in M_\Delta(x, \varepsilon)$, $g(x, x) < g(y, x) \Rightarrow h(x) < h(y)$.

The algorithm A is defined by:

Starting value: $x_0 \in E$, $\{\varepsilon_n\} \subset T$ be a given non-increasing sequence converging to zero.

Step n: if $x_n \in M_\Delta(x_n, \varepsilon_n)$, then $x_{n+1} = x_n$, $\varepsilon_{n+1} \leftarrow \varepsilon_n$,

if $x_n \overline{\in} M_\Delta(x_n, \varepsilon_n)$, then $x_{n+1} \in M_\Delta(x_n, \varepsilon_n)$, $\varepsilon_{n+1} \leftarrow \varepsilon_n$.

Huard Theorem (Prop. 4.1). *Under the assumptions given above, every accumulation point x^* of the infinite sequence generated by the algorithm A belongs to $M_\Delta(x^*, 0)$.*

Now we are going to prove that the theorem above is a special case of Theorem 2.

Corollary 3. *Huard Theorem* (Prop. 4.1) *holds*.

Proof Taking $F_\rho(x) = M_\Delta(x, \rho)$, $P = \{x \in E \mid F_0(x) = \emptyset$ or $x \in F_0(x)\}$, we only need to prove that under all the assumptions given above, family $\{F_\rho(x) \mid \rho \geqslant 0\}$ satisfies all the requirements of Theorem 2.

It is easy to prove that $F_\rho(x) \neq \emptyset$, $\forall \rho \geqslant 0$, $\forall x \in E$. Namely, $\Delta(x)$ is non-empty and compact, $g(y, x)$ is continuous in y, so for any $x \in E$ and $\rho \geqslant 0$, there exists $y_0 \in \Delta(x)$ such that $g(y_0, x) \geqslant \max_{y \in \Delta(x)} g(y, x) - \rho$.

(i) It is obvious that $\{F_\rho(x) \mid \rho \geqslant 0\}$ is a non-decreasing family of point-to-set maps.

(ii) H1 holds. Let $\rho_\nu \to 0$, $x_\nu \to x_0$, $y_\nu \in M_\Delta(x_\nu \rho_\nu)$, $y_\nu \to y_0$: $g(y_\nu, x_\nu) \geqslant \max_{y \in \Delta(x_\nu)} g(y, x_\nu) - \rho_\nu$. Then, since $\Delta(x)$ is (upper) continuous, we have $y_0 \in \Delta(x_0)$. Now we are going to prove

$$g(y_0, x_0) = \max_{y \in \Delta(x_0)} g(y, x_0).$$

Namely, if $g(x_0, y_0) < \max_{y \in \Delta(x_0)} g(y, x_0)$, there exist $\varepsilon > 0$ and $y_1 \in \Delta(x_0)$ such that $g(y_0, x_0) + \varepsilon < g(y_1, x_0)$. Since $\Delta(x)$ is (lower) continuous, there exist $\bar{y}_\nu \to y_1$, $\bar{y}_\nu \in \Delta(x_\nu)$, $\forall \nu \in N$. From this we have

$$g(y_\nu, x_\nu) \geqslant g(\bar{y}_\nu, x_\nu) - \rho_\nu.$$

Since $g(y, x)$ is continuous, we have

$$g(y_0, x_0) \geqslant g(y_1, x_0),$$

a contradiction.

(iii) H2 holds. This is a direct consequence of the definition of $F_\rho(x)$.

7. Another Algorithm of Huard.

Huard (1979) proved the following

Proposition (Prop. 0). Let $E \subset R^n$ be closed, $Q \subset P \subset E$. Let F_1 and F_0: $(F \backslash Q) \to \mathscr{P}(E)$ be point-to-set maps such that $F_1 \supset F_0$. Suppose that for all $x \in E \backslash Q$ we have

(α_0) $F_0(x) \neq \emptyset$,

(β_0) $x' \in (E \backslash Q) \cap F_1(x) \Rightarrow F_1(x') \subset F_1(x)$, and

(γ_0) if $x \overline{\in} P$, then there exists $V(x)$ such that

$$x' \in (E \backslash Q) \cap V(x),\ x'' \in (E \backslash Q) \cap F_0(x') \Rightarrow x \overline{\in} \overline{F_1(x'')}.$$

Then every accumulation point of the sequence $\{x_k\}$ generated by the following algorithm A3 belongs to P.

Algorithm A3.

$$x_0 \in E;$$

$$x_{i+1} \in F_0(x_i),\ \text{if } x_i \overline{\in} P;$$

$$x_{i+1} \in F_0(x_i) \cup \{x_i\},\ \text{if } x_i \in P \backslash Q;$$

$$x_{i+1} = x_i,\ \text{if } x_i \in Q.$$

Now we are going to prove the following

Theorem 3. *Assume that F_1 is upper continuous on $E \backslash P$, and $P \supset \{x \in E \mid x \in F_1(x)\}$. Then the assumption γ_0 in the proposition mentioned above is a consequence of the other assumptions.*

Proof If the conclusion is false, then there exist $\{x'_\nu\} \to x$, $x'_\nu \in E \backslash Q$, $\forall \nu \in N$, and $x''_\nu \in (E \backslash Q) \cap F_0(x'_\nu)$ such that $x \in \overline{F_1(x''_\nu)}$, $\forall \nu \in N$. Therefore for a fixed ν there exists sequence $\{y_{\nu_j}\}$ with $\lim\limits_{j \to \infty} y_{\nu_j} = x$, and $y_{\nu_j} \in F_1(x''_\nu)$, $\forall\ j$. It is easy to see that there exists sequence $\{y_{n_{j_n}}\}$ with $\lim\limits_{n \to \infty} y_{n_{j_n}} = x$ and $y_{n_{j_n}} \in F_1(x''_n)$. Since $x''_n \in (E \backslash Q) \cap F_0(x'_n)$, we have, by ($\beta_0$), $y_{n_{j_n}} \in F_1(x'_n)$. Since $F_1(x)$ is upper continuous, we have $x \in F_1(x)$, a contradiction. So we must have $x \overline{\in} \overline{F(x)}$.

Remark 1. In the proposition mentioned above, it has been implicitly supposed that $P \supset \{x \in E \mid x \in F_0(x)\}$. Namely, if there exists x_0 such that $x_0 \in F_0(x_0)$, $x_0 \overline{\in} P$, then, taking $x = x' = x'' = x_0$, we have, by (γ_0), $x_0 \overline{\in} \overline{F_1(x_0)}$. Therefore we have $x_0 \overline{\in} F_0(x_0)$, a contradiction.

Remark 2. Since. in Algorithm A3, we have two sets P and Q, Theorem 2 can not be applied to this case. Theorem 3 only shows that, under our assumption, every accumulation point of the sequence $\{x_\nu\}$ generated by A3, according to Huard's (Prop. 0), belongs to P.

8. Algorithm A2.

Now we introduce the following assumptions:

H3: $\{\Delta_\rho(x) \mid \rho \geqslant 0\}$ is a non-increasing (in ρ) family of point-to-set maps;

H4: if $\rho_\nu \to 0$, $x_\nu \to x_0$, $y_\nu \in \Delta_{\rho_\nu}(x_\nu)$, $y_\nu \to y_0$, then $y_0 \in \Delta_0(x_0)$;

H5: if $x_0 \in E$, $\Delta_0(x_0) \neq \emptyset$, then there exist $V(x_0)$ and $\rho_0 > 0$ such that $\Delta_\rho(x) \neq \emptyset$, $\forall 0 \leqslant \rho \leqslant \rho_0$, $\forall x \in V(x_0)$.

Notations:

$$M1 = \{x \in E \mid \Delta_0(x) = \emptyset\},$$
$$M2 = \{x \in E \mid \exists\, z \in \Delta_0(x): F_0(x, z) = \emptyset\},$$
$$M3 = \{x \in E \mid \exists\, z \in \Delta_0(x): x \in F_0(x, z)\},$$
$$M = M1 \cup M2 \cup M3,$$
$$P = M1 \cup M2 \cup \Omega,\ \Omega \supset M3.$$

Algorithm A2. Let $x_0 \in E$, Let $\{\rho_n\}$ and $\{\rho'_n\}$ be two given non-decreasing sequences of non-negative reals converging to zero.

Step n: if $x_n \in P$, stop;

Phase 1: if $\Delta_{\rho_n}(x_n) = \emptyset$, then $x_{n+1} = x_n$, $\rho_{n+1} \leftarrow \rho_n$, $\rho'_{n+1} \leftarrow \rho'_n$;

if $\Delta_{\rho_n}(x_n) \neq \emptyset$, then $z_n \in \Delta_{\rho_n}(x_n)$;

Phase 2: if $F_{\rho'_n}(x_n, z_n) = \emptyset$ or $x_n \in F_{\rho'_n}(x_n, z_n)$, then

$x_{n+1} = x_n$, $\rho_{n+1} \leftarrow \rho_n$, $\rho'_{n+1} \leftarrow \rho'_n$;

if $F_{\rho'_n}(x_n, z_n) \neq \emptyset$ and $x_n \overline{\in} F_{\rho'_n}(x_n, z_n)$, then

$x_{n+1} \in F_{\rho'_n}(x_n, z_n)$; $\rho_{n+1} \leftarrow \rho_n$, $\rho'_{n+1} \leftarrow \rho'_n$.

Theorem 4. *Let $\{F_\rho(x, z) \mid \rho \geqslant 0\}$ be a non-decreasing (in ρ) family of point-to-set maps ($F_\rho(x, z)$ is a map of (x, z)) and satisfy* H1 *and* H2; *let $\Delta_\rho(x)$ be a non-increasing (in ρ) family of point-to-set maps and satisfy* H4 *and* H5. *Algorithm A2 either stops at a step k with $x_k \in M$ or generates a sequence $\{x_k\}$ with all its accumulation points in M.*

Proof It is obvious that we can suppose $x_n \overline{\in} M$, $\forall n \in N$. In this case, we have

(1) $$\Delta_0(x_n) \neq \emptyset,\ \forall n \in N,$$

(2) $$F_0(x_n, z) \neq \emptyset,\ \forall z \in \Delta_0(x_n),\ \forall n \in N.$$

From this we conclude that if $\Delta_{\rho_n}(x_n) \neq \emptyset$, then $F_0(x_n, z) \neq \emptyset\ \forall z \in \Delta_{\rho_n}(x_n)$, and therefore, $F_{\rho'_n}(x_n, z_n) \neq \emptyset$.

Obviously, $\{h(x_n)\}$ is a monotonely increasing sequence. Let $\{x_n\}_{N'} \to x^*$, $N' \subset N$. If $x^* \overline{\in} M$, then $\Delta_0(x^*) \neq \emptyset$. By H5 and (2) there exists n_0 such that $\Delta_{\rho_n}(x_n) \neq \emptyset$, $z_n \in \Delta_{\rho_n}(x_n)$, $F_{\rho'_n}(x_n, z_n) \neq \emptyset$, $\forall n \geqslant n_0$, $n \in N'$. As in the proof of Theorem 2, we can assume $x_n \overline{\in} F_{\rho'_n}(x_n, z_n)$, $\forall n \in N'$. Namely, if it is not so, then there are two possibilities:

(i) From some n onwards, we have $x_n \in F_{\rho'_n}(x_n, z_n)$, $\forall n \in N'$. Since E is compact, by H4, there exists $N''\subset N'$ such that $\{z_n\}_{N''}\to z^*\in \Delta_0(x^*)$; then by H1, we have $x^*\in F_0(x^*, z^*)$. Thus we have $x^*\in M3\subset M$, and the theorem holds.

(ii) There is an infinite sequence $N''\subset N'$ such that $x_n\bar{\in} F_{\rho_n}(x_n, z_n)$, $\forall n\in N''$. In this case, we replace N' by N'' without violating the limit of $\{x_n\}_{N'}$.

Since E is compact, we can assume $\{z_n\}_{N'}\to \bar{z}$. By H4, we have $\bar{z}\in \Delta_0(x^*)$. Since $x^*\bar{\in} M$, by Corollary 1, there exist $V(x^*, \bar{z})$ and $\rho_0>0$ such that $h(x'')>h(x^*)$, $\forall (x', z')\in V(x^*, \bar{z})$, $\forall x''\in F_\rho(x', z')$, $\forall 0\leqslant\rho\leqslant\rho_0$. Thus there exists n_0 such that $h(x_{n+1})>h(x^*)$, $\forall x_{n+1}\in F_{\rho_n}(x_n, z_n)$, $\forall n\geqslant n_0+1$, $n\in N'$. Since $\{h(x_n)\}$ is a non-decreasing sequence, we have $h(x_n)\geqslant h(x_{n_0+2})>h(x^*)$, $\forall n>n_0+1$, $n\in N'$. Since $h(x)$ is continuous, we have $h(x^*)>h(x^*)$, a contradiction. Therefore we must have $x^*\in M$.

9. Applications of Theorem 4.

Huard (1975) proved the following theorem (Prop. 5.1):

Theorem. *Let $A\subset R^n$ be closed and $B\subset R^n$ be convex compact such that $A\cap B\neq\emptyset$. Let $T\subset R_+$ be compact, f: $R^n\to R$ continuous, Δ: $B\to\mathscr{P}(B)$ upper continuous on B, $\Delta(x)\neq\emptyset$, $\forall x\in A\cap B$, g: $B\times B\to R$ continuous*

$$M(x, z, \varepsilon)=\{y\in [x, z]\,|\,g(y, x)\geqslant \max_{w\in[x,z]} g(w, x)-\varepsilon\}.$$

Suppose that

(*) $\quad M(x, z, \varepsilon)\cap A\cap B\neq\emptyset$, $\forall x\in A\cap B$, $\forall z\in \Delta(x)$, $\forall \varepsilon\geqslant 0$, *and that*

$\forall x\in A\cap B$, $\forall z\in B$, $\forall \varepsilon\in T$, $\forall y\in M(x, z, \varepsilon)$, $g(x, x)<g(y, x)\Rightarrow f(x)<f(y)$.

Then the following algorithm either terminates at a finite step k with $x_k\in M$ or generates an infinite sequence having all its accumulation points in M, where

$$M=A\cap B\cap M(x, z, 0).$$

Algorithm A4. $x_0\in A\cap B$ is arbitrarily given.

Step k. We have $x_k\in A\cap B$, and choose: $z_k\in\Delta(x_k)$, $\varepsilon_k\in T$

$$x_{k+1}\in A\cap B\cap M(x_k, z_k, \varepsilon_k), \text{ if } x_k\bar{\in} M(x_k, z_k, \varepsilon_k).$$

$$x_{k+1}=x_k, \text{ if } x_k\in M(x_k, z_k, \varepsilon_k).$$

Now we are going to prove the following

Theorem 5. *Huard's theorem above* (Prop. 5.1) *is a special case of Theorem* 4.

Proof Set

$$E=A\cap B,\ \Delta_\rho(x)=\Delta(x),\ F_\rho(x, z)=A\cap B\cap M(x, z, \rho).$$

Let $\{\rho_\nu\}$ be a non-increasing sequence of non-negative reals converging to zero. Then we want to prove that, under the assumptions given in the above Theorem, all the conditions of Theorem 4 are satisfied, and that Algorithm A4 is a special case of Algorithm A2.

(i) It is obvious that $F_\rho(x, z)$ is a non-decreasing family (in ρ).

(ii) From (*) we have $F_\rho(x, z)\neq\emptyset$, $\forall\rho\geqslant 0$. Thus we have $M1=M2=\emptyset$.

(iii) Obviously $[x, z]$ is a continuous map of (x, z), so the part (ii) in the proof of Corollary 3 is also effective in this case. Therefore H1 holds.

(iv) H2 follows immediately from the definition of $M(x, z, \varepsilon)$.

From the fact that $F_\rho(x, z)\neq\emptyset$ and $\Delta(x)\neq\emptyset$, $\forall x\in A\cap B$, we see immediately that Algorithm A4 is a special case of Algorithm A2.

Denel (1979) has cited two examples to show that his algorithms have more applications than those given by Huard (1975). Now we show that Theorem 4 is also applicable to these examples. As the arguments are quite similar, we only discuss the first one in detail.

Example Linearized method of centers with partial linearization (Huard(1978)).

The problem to be solved is

$$\max f(x)$$
$$\text{s. t. } g_i(x)\geqslant 0,\ i=1, \cdots, m,$$
$$x\in B,$$

where the functions are coucave, continuously differentiable and B is a compact polyhedron.

Denote
$$A=\{x\,|\,g_i(x)\geqslant 0,\ i=1, \cdots, m\}.$$

We suppose that $A^0\cap B\neq\emptyset$.

For a given $\varepsilon>0$, denote

$$d_\varepsilon'(z, x)=\min\{f'(z, x)-f(x),\ g_i'(z, x),\ i\in I_\varepsilon(x)\}, \text{ where}$$
$$f'(z, x)=f(x)+\nabla f(x)\cdot(z-x),$$
$$g_i'(z, x)=g_i(x)+\nabla g_i(x)\cdot(z-x),\ i=1, \cdots, m,$$
$$I_\varepsilon(x)=\{i\in\{1, \cdots, m\}\,|\,g_i(x)<\varepsilon\},$$
$$d(t, f(x))=\min\{f(t)-f(x),\ g_i(t),\ i=1, \cdots, m\}.$$

By "linearized method of centres with partial linearization" we mean the following two-phase algorithm:

(1) For a given x_ν, find z_ν such that
$$d_\varepsilon'(z_\nu, x_\nu)=\max_{z\in B} d_\varepsilon'(z, x_\nu);$$

(2) For given x_ν and z_ν, find $x_{\nu+1}$ such that
$$d(x_{\nu+1}, f(x_\nu))=\max_{t\in[x_\nu, z_\nu]} d(t, f(x_\nu)).$$

For fixed $\rho>0$ we define

$$\Delta_\rho(x)=\{z\in B\,|\,d_\varepsilon'(z; x)\geqslant d_\varepsilon'(t; x),\ \forall t\in B\},$$
$$\Delta_0(x)=\{z\in B\,|\,d_\varepsilon'(z; x)\geqslant d(t, f(x)),\ \forall t\in B\},$$
$$F_\rho(x, z)=\{y\in[x, z]\,|\,d(y, f(x))\geqslant \max_{t\in[x,z]} d(t, f(x))-\rho\},$$
$$F_0(x, z)=\{y\in[x, z]\,|\,d(y, f(x))\geqslant \max_{t\in[x,z]} d(t, f(x))\}.$$

Now we are going to prove that families $\{\Delta_\rho(x)\,|\,\rho\geqslant 0\}$ and $\{F_\rho(x, z)\,|\,\rho\geqslant 0\}$ satisfy all the conditions in Theorem 4:

(i) $\{\Delta_\rho(x) \mid \rho \geqslant 0\}$ is non-increasing in ρ. In fact, for any concave function $\varphi(x)$ we have

$$\varphi(t) \leqslant \varphi(x) + \nabla\varphi(x) \cdot (t-x),$$

and therefore

$$\begin{aligned} d(t, f(x)) &= \min\{f(t)-f(x),\ g_i(t),\ i=1, \cdots, m\} \\ &\leqslant \min\{f'(t, x)-f(x),\ g_i'(t, x),\ i=1, \cdots, m\} \\ &\leqslant \min\{f'(t, x)-f(x),\ g_i'(t, x),\ i \in I_\varepsilon(x)\} \\ &= d_\varepsilon'(t;\ x). \end{aligned}$$

From this we have $\Delta_\rho(x) \subset \Delta_0(x)$.

(ii) $\{\Delta_\rho(x) \mid \rho \geqslant 0\}$ satisfies H4.

Proof Let $x_\nu \to \bar{x}$, $z_\nu \to \bar{z}$ and $d_\varepsilon'(z_\nu, x_\nu) = \max_{t \in B} d_\varepsilon'(t, x_\nu)$. Now we are going to prove that $\bar{z} = \Delta_0(\bar{x})$, or

$$d_\varepsilon'(\bar{z}, \bar{x}) \geqslant \max_{t \in B} d(t, f(\bar{x})).$$

It is known that $d_\varepsilon'(z, x)$ is an upper semicontinuous function of (z, x) and that max $\{d(t, f(x)) \mid t \in B\}$ is a continuous function of x (Cf. Denel (1979), p. 64). From this we have

$$\overline{\lim_{\nu\to\infty}}\, d_\varepsilon'(z_\nu, x_\nu) \leqslant d_\varepsilon'(\bar{z}, \bar{x}).$$

So we have only to show

$$\overline{\lim_{\nu\to\infty}}\, d_\varepsilon'(z_\nu, x_\nu) \geqslant d(t, f(\bar{x})), \quad \forall t \in B.$$

By definition of z_ν, for all $t \in B$ we have

$$\begin{aligned} d_\varepsilon'(z_\nu, x_\nu) &\geqslant d_\varepsilon'(t, x_\nu) \\ &\geqslant \min\{\nabla f(x_\nu) \cdot (t-x_\nu),\ g_i(x_\nu) + \nabla g_i(x_\nu) \cdot (t-x_\nu), i=1, \cdots, m\}. \end{aligned}$$

Since $\{f(x_\nu)\}$ is a non-decreasing family, we have

$f(x_\nu) \leqslant f(\bar{x})$, therefore the above expression

$$\geqslant \min\{f(t)-f(\bar{x}),\ g_i(t),\ i=1, \cdots, m\} = d(t;\ f(\bar{x})).$$

This is what we want to prove.

(iii) $F_\rho(x, z)$ is obviously non-decreasing in ρ.

(iv) H1 is a direct consequence of Theorem A15 in Huard (1975).

(v) H2 follows directly from the definition of $F_\rho(x, z)$.

(vi) Evidently $\Delta_0(x) \neq \emptyset$, $\Delta_\rho(x) \neq \emptyset$, $\forall x \in A \cap B$, so we have H5.

(i)—(vi) indicates that conditions H1—H5 in Theorem 4 all hold. Therefore all the accumulation points of the sequence $\{x_\nu\}$ generated by the algorithm belong to

$$\Gamma = \{x \in A \cap B \mid x \in F_0(x, z),\ z \in \Delta_0(x)\}.$$

Now we are going to prove that, if $x \in \Gamma$, then x is an optimal solution of the given problem (following the way of Huard (1975), p. 325).

Since $x \in F_0(x;\ z)$, we have

$$d(x, f(x)) \geqslant d(t, f(x)), \quad \forall t \in [x, z], \text{ or}$$

$$\min\{0,\ g_i(x),\ i=1,\ \cdots,\ m\} \geqslant \min\{f(t)-f(x),\ g_i(t),\ i=1,\ \cdots,\ m\},\quad \forall t\in[x,\ z].$$

From this we have, $\forall t\in[x, z]$

$$\min\{f(t)-f(x),\ g_i(t),\ i=1,\ \cdots,\ m\}\leqslant 0.$$

Therefore, we have

$$\min\{f(t)-f(x),\ g_i(t),\ i=I_s(x)\}=\min\{f(t)-f(x),\ g_i(t),\ i=1,\ \cdots,\ m\} \leqslant 0,\quad \text{if } t \text{ is sufficiently near to } x. \tag{1}$$

Now we are going to prove that

$$\min\{\nabla f(x)\cdot(z-x),\ \nabla g_i(x)\cdot(z-x)+g_i(x),\ i\in I_s(x)\}\leqslant 0. \tag{2}$$

If, on the contrary, (2) is false, then there exists $b=b(x, z)>0$ such that

$$\nabla f(x)\cdot(z-x)\geqslant b,\ g_i(x)+\nabla g_i(x)\cdot(z-x)\geqslant b,\ \forall i\in I_s(x).$$

Let $t=x+\lambda(z-x)$, $0\leqslant\lambda\leqslant 1$. Since $f(t)$, $g_i(t)$, $i=1, \cdots, m$ are all continuously differentiable, then, by mean-value theorem, there exist $0<\theta$, $\theta_i<1$, such that

$$f(t)-f(x)=\lambda\nabla f(x+\theta\lambda(z-x))\cdot(z-x),$$

$$g_i(t)=g_i(x)+\lambda\nabla g_i(x+\theta_i\lambda(z-x))\cdot(z-x),\ \forall i\in I_s(x).$$

Thus, if λ is small enough, we have

$f(t)-f(x)>0$, $g_i(t)>0$, $\forall i\in I_s(x)$, which contradicts (1).

Since $z\in\Delta_0(x)$, by (2), we have

$$\min\{f(t)-f(x),\ g_i(t),\ i=1,\ \cdots,\ m\} \leqslant\min\{f'(z,\ x)-f(x),\ g_i'(z,\ x),\ i\in I_s(x)\}<0,\ \forall t\in B.$$

From the inequality, it follows immediately that

$$f(t)\leqslant f(x),\ \forall t\in A^0\cap B.$$

Since A is convex, $f(x)$ is continuous, we have

$$f(t)\leqslant f(x),\ \forall t\in A\cap B.$$

This is what we want to prove.

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ρ 单增点到集映象簇

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摘 要

本文引入了一个 ρ 单增点到集映象簇，并定义了两类算法．在相当简单的条件之下，证明了由这些算法所产生的叙列的每一聚点皆属于某一特定的集．然后，证明了诸如 Zangwill, Huard 等人的结果皆是本文结果的直接推理．最后，用例子证明本文结果是上述诸人的结果的一真正扩充．