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ON A_2^2-POLYHEDRA

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Dedicated to Professor Su Bu-chin on the Occasion of his 80th Birthday and his 50th Year of Educational Work

A finite simply connected 4-dimensional polyhedron is denoted for brevity by A_2^2-polyhedron. To such a polyhedron there is an associated cohomology ring consisting of cohomology groups and Pontrjagin square[1]. A one-one correspondence between the homotopy types of this sort of polyhedra and the properly isomorphic classes of cohomology rings has been well established by Whitehead, J. H. C.[1]. However, the proof seems to be very difficult. The purpose of this note is to introduce A_2^2-homology co-ring and give simple proofs of the following two Theorems:

Theorem 1. *If an A_2^2-homology co-ring H is given, there exists an A_2^2-polyhedron whose homology co-ring is properly isomorphic to H.*

The polyhedron in Theorem 1 is said to realize the given A_2^2-homology co-ring.

Let H and H' be two A_2^2-homology co-rings realized by K and K' respectively. We have

Theorem 2. *If h: $H \to H'$ is a proper homomorphism, there is a continuous map ϕ: $K \to K'$ such that ϕ induces h.*

These two Theorems imply a one-one correspondence between the homotopy types of A_2^2-polyhedra and the proper isomorphism classes of A_2^2-homology co-rings.

§1. Algebraic Preliminaries.

Let $H_i (i=2, 3, 4)$ be abelian groups of finite generators, where H_4 is free. To each integer $m \geqslant 2$, there is a natural projection $\mu_{m,0}$: $H_i \to (H_i)_m = H_i/mH_i (i=2, 3, 4)$. As usual, ${}_mH_i$ denotes a subgroup of H_i such that $\alpha \in {}_mH_i$ if and only if $m\alpha = 0$. Construct a group $H_i(m)$ as a direct sum

$$H_i(m) = \mu_{m,0}H_i + \Delta_m^* H_{i-1}, \tag{1}$$

Δ_m^* being an isomorphism of ${}_mH_{i-1}$ into $H_i(m)$. For brevity, Δ_m^* is written as Δ^*. Let

$$\Delta: H_i(m) \to H_{i-1}$$

be a homomorphism such that

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$$\Delta^{-1}(0)=\mu_{m,0}H_i,\qquad\qquad (2)$$
$$\Delta\Delta^*|_mH_{i-1}=1.$$

Let a homomorphism

$$\mu_{l,m}: H_i(m)\to H_i(l)$$

be defined by

$$\mu_{l,m}(\mu_{m,0}x+\Delta_m^*y)=\mu_{l,0}\Big(\frac{l}{(l,\ m)}\,x\Big)+\Delta_l^*\Big(\frac{m}{(l,\ m)}\,y\Big),\ x\in H_i,\ y\in {}_mH_{i-1}, \qquad (3)$$

where $(l,\ m)$ denotes the greatest common divisor of l and m.

Suppose $H_2=\sum_{\alpha\in A}Z_\alpha(\sigma_\alpha)$, where A is a finite ordered set and $Z_\alpha(\sigma_\alpha)$ is a cyclic group of order σ_α, which may be zero. An abelian group (see Whitehead, J. H. C.[3]), $\Gamma(H_2)$ is so constructed that it consists of the elements of the form

$$\sum_{\alpha\le\beta,\alpha,\beta\subset A} a_{\alpha,\beta}e_{\alpha,\beta},$$

where $e_{\alpha,\beta}$ is a generator of order $(\sigma_\alpha,\ \sigma_\beta)$ and $a_{\alpha,\beta}\in Z_\alpha((\sigma_\alpha,\ \sigma_\beta))$, a cyclic group of order $(\sigma_\alpha,\ \sigma_\beta)$ if $\alpha\ne\beta$, while $e_{\alpha,\alpha}$ a generator of order $(\sigma_\alpha^2,\ 2\sigma_\alpha)$ and $a_{\alpha,\alpha}\in Z_\alpha((\sigma_\alpha^2,\ 2\sigma_\alpha))$, a cyclic group of order $(\sigma_\alpha^2,\ 2\sigma_\alpha)$ if $\alpha\in A$. In case $\alpha\ne\beta$, we assume $e_{\alpha,\beta}=e_{\beta,\alpha}$. The group $\Gamma(H_2)$ is called the homology module of H_2.

Let f: $H_2\to H_2'$ be a homomorphism, where $H_2'=\sum_{\beta\in A'}Z_\beta(\sigma_\beta')$, A' being also a finite ordered set. Then

$$f(g_\alpha)=\sum_{\beta\in A'} b_{\alpha,\beta}g_\beta',$$

in which g_α and g_β' denote generators of H_2 and H_2' respectively, and $b_{\alpha,\beta}$ are integers satisfying

$$\sigma_\alpha b_{\alpha,\beta}\equiv 0,\ \mathrm{mod}\ \sigma_\beta'.$$

Let $\Gamma(H_2')$ be the homology module of H_2'. There is a homomorphism

$$\Gamma(f): \Gamma(H_2)\to\Gamma(H_2')$$

defined, in correspondence with f: $H_2\to H_2'$, by

$$\Gamma(f)\Big(\sum_{\alpha,\beta\subset A}a_{\alpha,\beta}e_{\alpha,\beta}\Big)=\sum_{\alpha,\beta\subset A}a_{\alpha,\beta}\Gamma(f)e_{\alpha,\beta}, \qquad (4)$$

and

$$\Gamma(f)e_{\alpha,\beta}=\sum_{\mu,\nu\subset A'}b_{\alpha,\mu}b_{\beta,\nu}e_{\mu,\nu}'+\sum_{\mu\in A'}b_{\alpha,\mu}b_{\beta,\mu}e_{\mu,\mu}', \qquad (5)$$
$$\Gamma(f)e_{\alpha,\alpha}=\sum_{\beta\in A'}b_{\alpha,\beta}^2e_{\beta,\beta}'+\sum_{\gamma<\delta,\gamma,\delta\subset A'}b_{\alpha,\gamma}b_{\alpha,\delta}e_{\gamma,\delta}',$$

where the occurring coefficients are well defined according to the following Lemma or its similar arguments.

Lemma 1. *If $a_{\alpha,\beta}$ is a reduced* mod $(\sigma_\alpha,\ \sigma_\beta)$ *integer, then $a_{\alpha,\beta}b_{\alpha,\mu}b_{\beta,\nu}$ is a well defined* mod $(\sigma_\mu',\ \sigma_\nu')$ *integer.*

Proof The integer $a_{\alpha,\beta}$ may vary by a multiple of $(\sigma_\alpha,\ \sigma_\beta)$. The integers $b_{\alpha,\mu}$ and $b_{\beta,\nu}$ may be written as $\Big(\frac{r\sigma_\mu'}{(\sigma_\alpha,\ \sigma_\mu')}+l\sigma_\mu'\Big)$ and $\Big(\frac{s\sigma_\nu'}{(\sigma_\beta,\ \sigma_\nu')}+k\sigma_\nu'\Big)$ respectively, where $r,\ s,\ l,\ k$ are integers. Then

$$(a_{\alpha,\beta}+t(\sigma_\alpha,\ \sigma_\beta))\Big(\frac{r\sigma'_\mu}{(\sigma_\alpha,\ \sigma'_\mu)}+l\sigma'_\mu\Big)\Big(\frac{s\sigma'_\nu}{(\sigma_\beta,\ \sigma'_\nu)}+k\sigma'_\nu\Big)$$
$$\equiv(a_{\alpha,\beta}+t(\sigma_\alpha,\ \sigma_\beta))\Big(\frac{r\sigma'_\mu}{(\sigma_\alpha,\ \sigma'_\mu)}\Big)\Big(\frac{s\sigma'_\nu}{(\sigma_\beta,\ \sigma'_\nu)}\Big),\ \mathrm{mod}(\sigma'_\mu,\ \sigma'_\nu).$$

Since $\sigma_\alpha/(\sigma_\alpha,\ \sigma_\beta)$ and $\sigma_\beta/(\sigma_\alpha,\ \sigma_\beta)$ are relatively prime to each other, there are integers p and q such that

$$p\frac{\sigma_\alpha}{(\sigma_\alpha,\ \sigma_\beta)}+q\frac{\sigma_\beta}{(\sigma_\alpha,\ \sigma_\beta)}=1.$$

Therefore

$$t(\sigma_\alpha,\ \sigma_\beta)\Big(\frac{r\sigma'_\mu}{(\sigma_\alpha,\ \sigma'_\mu)}\Big)\Big(\frac{s\sigma'_\nu}{(\sigma_\beta,\ \sigma'_\nu)}\Big)$$
$$=t(\sigma_\alpha,\ \sigma_\beta)\Big(p\frac{\sigma_\alpha}{(\sigma_\alpha,\ \sigma_\beta)}+q\frac{\sigma_\beta}{(\sigma_\alpha,\ \sigma_\beta)}\Big)\Big(\frac{r\sigma'_\mu}{(\sigma_\alpha,\ \sigma'_\mu)}\Big)\Big(\frac{s\sigma'_\nu}{(\sigma_\beta,\ \sigma'_\mu)}\Big)\equiv 0,\ \mathrm{mod}(\sigma'_\mu,\ \sigma'_\nu),$$

bacause $\sigma_\alpha/(\sigma_\alpha,\ \sigma'_\mu)$ and $\sigma_\beta/(\sigma_\beta,\ \sigma'_\mu)$ are integers. Hence $a_{\alpha,\beta}b_{\alpha,\mu}b_{\beta,\nu}$ are determined independently of the different choices of the representatives of $a_{\alpha,\beta}$, $b_{\alpha,\mu}$ and $b_{\beta,\nu}$.

Similar arguments for other cases are omitted.

Though $\Gamma(f)$ is a homomorphism, yet $\Gamma(f+g)\neq\Gamma(f)+\Gamma(g)$. we have

Lemma 2. *$\Gamma(f)$ is a covariant functor.*

This is evident after computation or geometric reasoning by keeping in mind that $e_{\alpha,\beta}(\alpha\neq\beta)$ actually means Whitehead product and $e_{\alpha,\alpha}$ a map of Hopf invariant 1 (See § 2).

Give a set of homomorphisms

$$\gamma_m:\ H_4(m)\to(\Gamma(H_2))_m,\quad m=0,\ 2,\ 3,\ \cdots,$$

so that

$$\gamma_p\mu_{p,q}\,|\,H_4(q)=\bar{\mu}_{p,q}\gamma_q, \tag{6}$$

in which

$$\bar{\mu}_{p,q}:\ (\Gamma(H_2))_q\to(\Gamma(H_2))_p$$

means the projection of $\frac{p}{(p,\ q)}x'$ into $(\Gamma(H_2))_p$, if x' represents an element of $(\Gamma(H_2))_q$. Evidently it is independent of the choice of x' in its class. A homology co-ring[1] contains groups, which are $H_i(i=2,\ 3,\ 4)$, $H_i(m)$ $(m=2,\ 3,\ \cdots)$ and homology module $(\Gamma(H_2))_m$, and homomorphisms among these groups, which are $\mu_{p,q}$: $H_i(q)\to H_i(p)$ $(p=2,\ 3,\ 4,\ \cdots;\ q=0,\ 2,\ 3,\ \cdots)$, $\bar{\mu}_{p,q}$: $(\Gamma(H_2))_q\to(\Gamma(H_2))_p(q=0,\ 2,\ 3,\ \cdots,\ p=2,\ 3,\ \cdots)$, Δ_m: $H_i(m)\to H_{i-1}(m=2,\ 3,\ \cdots)$, and γ_m: $H_4(m)\to(\Gamma(H_2))_m$ $(m=0,\ 2,\ 3,\ \cdots)$, the latter satisfying (6). Hereafter we use $\mu_{p,q}$ to denote both $\mu_{p,q}$ and $\bar{\mu}_{p,q}$.

Let homomorphisms f_i: $H_i\to H'_i$ $(i=2,3,4)$ be given. They induce homomorphisms

$$f_i(m):\ H_i(m)\to H'_i(m)$$

by

$$f_i(m)\,|\,\mu_{m,0}H_i=\mu_{m,0}f_iH_i, \tag{7}$$
$$f_i(m)\,|\,\Delta^*_m({}_mH_{i-1})=\Delta^*_m f_{i-1\ m}H_{i-1} \tag{8}$$

1) Homology co-ring is dual to cohomology ring in [1].

Lemma 3.
$$f_r(p)\mu_{pq}=\mu_{p,q}f_r(q), \tag{9}$$
$$f_{r-1}\Delta=\Delta f_r(p) \tag{10}$$
$$(r=2,\ 3,\ \cdots,\ p=2,\ 3,\ \cdots;\ q=0,\ 2,\ 3,\ \cdots).$$

This Lemma follows directly from (2), (3), (7), and (8).

Let H and H' be two homology co-rings. A set of homomorphisms $\{f_r(m)\}$ ($m=0, 2, 3, \cdots$) between the corresponding homology groups of H and H' with integer coefficients or with integer coefficients reduced mod m ($m=0, 2, 3, \cdots$) is called a (μ, Δ)-homomorphism of H into H' if (9) and (10) are all satisfied. The homomorphism $\Gamma(f_2)$ of $\Gamma(H_2)$ into $\Gamma(H'_2)$ is induced by f_2: $H_2\to H'_2$ according to (4) and (5), which carries ${}_m\Gamma(H_2)$ into ${}_m\Gamma(H'_2)$ and produces a homomorphism
$$\Gamma(f_2)_m: (\Gamma(H_2))_m\to(\Gamma(H_2))_m.$$
If f_2 is an isomorphism onto, f_2^{-1} exists, and by Lemma 2
$$\Gamma(f_2)\Gamma(f_2^{-1})=1_{\Gamma(H'_2)},\quad \Gamma(f_2^{-1})\Gamma(f_2)=1_{\Gamma(H_2)}.$$
Both $\Gamma(f_2)$ and $\Gamma(f_2^{-1})$ are isomorphisms onto. Hence $(\Gamma(f_2))_m$ and $(\Gamma(f_2^{-1}))_m$ are also isomorphisms onto.

If, besides (6), the following diagrams
$$\begin{array}{ccc} H_4(m) & \xrightarrow{\gamma_m} & (\Gamma(H_2))_m \\ \downarrow{\scriptstyle f_4(m)} & & \downarrow{\scriptstyle \Gamma(f_2)_m} \\ H'_4(m) & \xrightarrow{\gamma'_m} & (\Gamma(H'_2))_m, \end{array} \tag{11}$$
$$m=0,\ 2,\ 3,\ 4,\ \cdots,$$
commute, then the (μ, Δ) homomorphism $\{f_i(m)\}$ ($m=0, 2, 3, \cdots, i=2, 3, 4$) and $\Gamma(f_2)$ are said to constitute a proper homomorphism of the homology co-ring H into another H'. No doubt, proper isomorphism is an equivalence relation.

§ 2. Geometric Interpretation of homology Co-ring.

§ 2.1. The three dimensional skeleton of an A_2^3-polyhedron is able to be reduced to
$$K^3=(S_1^2\cup e_1^3(\sigma_1))\vee\cdots\vee(S_{t_2}^2\cup e^3(\sigma_{t_2}))\vee S_{t_2+1}^2\vee S_{t_2+2}^2\vee\cdots\vee S_{t_2+p_2}^2$$
$$\vee S_1^3\vee\cdots\vee S_{p_3+t_3}^3. \tag{12}$$
An A_2^3-polyhedron may be obtained within its homotopy type by attaching four dimensional cells to K^3. Since $\Pi_3(S_1^2\cup e_1^3(\sigma_1))$ is the image of
$$i: \Pi_3(S_1^2)\to\Pi_3(S_1^2\cup e^3(\sigma_1)),$$
it is necessary to compute the kernel of the homomorphism i which is, however, known (See Chang, S. C.[4]). to be generated by Whitehead product, $[S_1^2, \sigma_1 S_1^2]$ and the composition element $\sigma_1\cdot h$, where h denotes the Hopf map, $S^3\to S^2$, and σ_1 is a map of S^2 to S_1^2 of degree σ_1. Hence Π_3 $(S_1^2\cup e_1^3(\sigma_1))$ is a cyclic group of order

$(\sigma_1^2, 2\sigma_1)$. Evidently

$$\Pi_3(K^3)=\sum_{l=1}^{t_2+p_2} Z_l((\sigma_l^2, 2\sigma_l))+\sum_{\alpha<\beta,1}^{t_2+p_2} Z_{\alpha,\beta}((\sigma_\alpha, \sigma_\beta))+\{b_1, \cdots, b_{p_3+t_3}\},$$

where $\sigma_l=0$, if $l=t_2+1, \cdots, t_2+p_2$, and $\{b_1, \cdots, b_{p_3+t_3}\}$ is a free module of p_3+t_3 generators. Now

$$\Pi_3(K^3)=\Gamma(H_2(K))+\{b_1\cdots, b_{p_3+t_3}\},$$

"+" being the direct sum. Let

$$p: \Pi_3(K^3)\to\Gamma(H_2(K))$$

be a projection. Let $f:A_2^2\to A_2'^2$ be a cellular map, then f induces the homomorphisms

$$f_2: H_2(A_2^2)\to H_2(A_2'^2),$$
$$f_*: \Pi_3(K^3)\to\Pi_3(K'^3),$$

where K^3 and K'^3 are the three skeletons of A_2^2 and $A_2'^2$ respectively, so that

$$f_*|\Gamma(H_2(A_2^2)): \Gamma(H_2(A_2^2))\to\Gamma(H_2(A_2^2))$$

coincides with $\Gamma(f_2)$.

The group $i\Pi_3(K^2)(\subset\Pi_3(K^3))$ is determined by $t_2, \sigma_1, \cdots, \sigma_{t_2}$ and p_2, which are known if we know $H_2(A_2^2)$. Hence $i\Pi_3(K^2)$ is preferably written as $\Gamma(H_2(A_2^2))$.

§ 2.2. An A_2^2-polyhedron K may be constructed within its homotopy type by attaching 4-dimensional cells to K^3 given by

$$\begin{aligned}\beta e_l^4&=\bar{\sigma}_l S_l^3+\sum_{\alpha<\gamma} a_{l,\alpha,\gamma}e_{\alpha,\gamma}, \quad l=1, \cdots, t_3,\\ \beta e_{t_3+r}^4&=\sum_{\alpha<\gamma} a_{t_3+r,\alpha,\gamma}e_{\alpha,\gamma}, \quad r=1, \cdots, p_4,\end{aligned} \tag{13}$$

βe_*^4 being the homotopy boundary of e_*^4, which implies

$$\partial e_l^4=\bar{\sigma}_l S_l^3, \quad l=1, \cdots, t_3,$$
$$\partial e_{t_3+r}^4=0, \quad r=1, \cdots, p_4.$$

For convenience, S_i^3 and e_i^4 are also used to denote their homology classes with integers as coefficients or with integers reduced mod $\bar{\sigma}_l$ as coefficients[2]. If $a=\sum_{i=1}^{p_3+t_3} \alpha_i S_i^3\in {}_mH_3$, then $m\alpha_i\equiv0$, mod $\bar{\sigma}_i$, $i=1, \cdots, p_3+t_3$, whence

$$\alpha_i=\frac{r_i\bar{\sigma}_i}{(m, \bar{\sigma}_i)}, \quad i=1, \cdots, p_3+t_3, \tag{14}$$

where $r_i=1, \cdots, (m, \bar{\sigma}_i)$. The group ${}_mH_3$ is generated by $\bar{\sigma}_l S_l^3/(m, \bar{\sigma}_l)$ $(l=1, \cdots, t_3)$ of order $(m, \bar{\sigma}_l)$. Though Δ^* is not unique, yet a special one may be defined by[3]

$$\Delta_m^*\left(\frac{\bar{\sigma}_l S_l^3}{(m, \bar{\sigma}_l)}\right)=\frac{m}{(m, \bar{\sigma}_l)}e_l^4, \tag{15}$$

because

2) Hereafter $\frac{m}{(m, \bar{\sigma}_l)}e_l^4$ also denotes a homology class mod m and $\frac{\bar{\sigma}_l S_l^3}{(m, \bar{\sigma}_l)}$ a homology class mod $(m, \bar{\sigma}_l)$.

3) It is evident that e_l^4 may vary by a linear expression $\sum_{r=1}^{p_4} u_{l,r}e_{t_3+r}^4$, where $u_{l,r}$ are integers if (16) is required to be valid. It tells that Δ^* is not unique. See § 2 of Whitehead, J. H. C. [1] for its cohomology version.

$$\Delta_m\Delta_m^*\left(\frac{\bar\sigma_l S_l^3}{(m,\bar\sigma_l)}\right)=\Delta_m\left(\frac{me_l^4}{(m,\bar\sigma_l)}\right)=\frac{\bar\sigma_l}{(m,\bar\sigma_l)}S_l^3,\quad l=1,\cdots,t_3. \tag{16}$$

From (15) it also leads to

$$\left(\mu_{p,m}\Delta_m^*\left(\frac{\bar\sigma_l S_l^3}{(m,\bar\sigma_l)}\right)=\mu_{p,m}\frac{m}{(m,\bar\sigma_l)}e_l^4=\frac{p}{(p,m)}\cdot\frac{m}{(m,\bar\sigma_l)}e_l^4\right.$$
$$=\Delta_p^*\left(\frac{m}{(p,m)}\cdot\frac{\bar\sigma_l}{(m,\bar\sigma_l)}S_l^3\right), \tag{17}$$

in which I remark that $p\left(\frac{m}{(p,m)}\right)\left(\frac{\bar\sigma_l}{(m,\bar\sigma_l)}\right)\equiv 0$, mod $\bar\sigma_l$. From (17) and

$$\mu_{p,m}\mu_{m,0}H_i=\mu_{p,0}\left(\frac{p}{(p,m)}H_i\right), \tag{18}$$

it is shown that (3) is valid if $i=4$. To complete the verification of (3) it is still necessary to consider the cases that $i=3$ and $i=2$.

A formula similar to (15) is

$$\Delta_m^*\left(\frac{\sigma_l}{(m,\sigma_l)}S_l^2\right)=\frac{m}{(m,\bar\sigma_l)}e^3(\sigma_l), \tag{19}$$

because it is easy to verify $\Delta_m\Delta_m^*=1$ and

$$\mu_{p,m}\Delta_m^*\left(\frac{\sigma_l}{(m,\sigma_l)}S_l^2\right)=\Delta_p^*\left(\left(\frac{m\sigma_l}{(p,m)(m,\sigma_l)}\right)S_l^2\right).$$

If $i=3$. formula (3) is able to be verified in the same way as the case when $i=4$ by use of (19) in place of (15). The case $i=2$ is simple, because

$$H_2(A_2^2,Z_p)=\mu_{p,0}H_2(A_2^2,Z).$$

This finishes the verification[4] of (3).

Lemma 3. *Let a four-dimensional simply connected CW-complex K be given by* (12) *and*

$$\beta e_\lambda^4=\bar\sigma_\lambda S_\lambda^3+\sum_{\alpha<\gamma}a_{\lambda,\alpha,\gamma}e_{\alpha,\gamma},\quad \lambda=1,\cdots,t_3, \tag{13}$$

then $a_{\lambda,\alpha,\gamma}$*'s are determined* mod $\bar\sigma_\lambda$ *within the homotopy type of* K.

proof Let $g\colon S_\lambda^3\to S_\lambda^3\vee S^3$ be a comultiplication. Map S^3 into K by $\sum c_{\alpha,\gamma}e_{\alpha,\gamma}$, $c_{\alpha,\gamma}$ being arbitrary integers. After these two processes $\beta e_\lambda^4=\bar\sigma_\lambda S_\lambda^3+\sum_{\alpha<\gamma}a_{\lambda,\alpha,\gamma}e_{\alpha,\gamma}$ becomes $\bar\sigma_\lambda S_\lambda^3+\sum_{\alpha<\gamma}a_{\lambda,\alpha,\gamma}e_{\alpha,\gamma}+\bar\sigma_\lambda\sum_{\alpha<\gamma}c_{\alpha,\gamma}e_{\alpha,\gamma}$. Construct a new CW-complex L, which contains, besides all the other cells of K^4 except e_λ^4, a new cell $e_\lambda'^4$, that is attached to K^3 by $\bar\sigma_\lambda S_\lambda^3+\sum_{\alpha<\gamma}(a_{\lambda,\alpha,\gamma}e_{\alpha,\gamma}+\bar\sigma_\lambda c_{\alpha,\gamma}e_{\alpha,\gamma})$. Let E^4 be a euclidean four dimensional unit cell containing the points $\{(x_1,x_2,x_3,x_4)\}$, $0\leqslant x_i\leqslant 1$ $(i=1,2,3,4)$. Let $\phi\colon E^4\to\bar e_\lambda^4$ and $\psi\colon E^4\to\bar e_\lambda'^4$ be the characteristic maps of $\bar e_\lambda^4$ and $\bar e_\lambda'^4$ respectively, where identification topology is used. Then $\psi\phi^{-1}$ is a homomorphism in the interior of $\bar e_\lambda^4$. If βe_λ^4 is

4) The verification has been carried out for reduced complexes. But it is true for general A_2^2-polyhedron, because the latter is of the same homotopy type as a reduced. Or we may choose suitable basis for chain groups of an A_2^2-polyhedron, so that (12) and (13) are valid.

considered as a point set, then $\phi^{-1}(\beta\bar{e}_\lambda^4 \cap S_\lambda^3) = \phi^{-1}S_\lambda^3$ is able to be considered as a three dimensional unit cell E^3, containing the points $\{(x_1, x_2, x_3, 0)\}$, $0 \leqslant x_i \leqslant 1$, $(i=1, 2, 3)$. Divide E^3 into $E_l^3 (l=1, \cdots, \bar{\sigma}_\lambda)$ such that $(x_1, x_2, x_3, 0) \in E_l^3$ if $(l-1) \cdot \frac{1}{\bar{\sigma}_\lambda} \leqslant x_3 \leqslant l \cdot \frac{1}{\bar{\sigma}_\lambda}$, $1 \leqslant l \leqslant \bar{\sigma}_\lambda$. Then $\phi | E_l^3: (E_l^3, \dot{E}_l^3) \to (S^3, 0)$ may be supposed to represent $1 \in \Pi_3(S_l^3)$. Divide E_l^3 again into $E_{l,1}^3$ and $E_{l,2}^3$ such that a point $\mathscr{P}(x_1, x_2, x_3, 0)$ of E_l^3 will belong to $E_{l,1}^3$ or $E_{l,2}^3$, if $0 \leqslant x_1 \leqslant \frac{1}{2}$ or $\frac{1}{2} \leqslant x_1 \leqslant 1$. Assume that $\psi | E_{l,1}^3: (E_{l,1}^3, \dot{E}_{l,1}^3) \to (S_\lambda^3, 0)$ represents $1 \in \Pi_3(S_\lambda^3)$ and $\psi | E_{l,2}^3: (E_{l,2}^3, \dot{E}_{l,2}^3) \to (K^2, 0)$ represents $\sum (c_{\alpha,\gamma} e_{\alpha,\gamma})$ in $\Pi_3(K^3, 0)$. Then

$$\begin{aligned}\psi\phi^{-1} | \beta e_\lambda^4 &= \psi\phi^{-1}(\beta e_\lambda^4 \cap (K^3 - S_\lambda^3) + \beta e_\lambda^4 \cap S_\lambda^3) \\ &= 1 | \beta e_\lambda^4 \cap (K^3 - S_\lambda^3) + (1 \vee \sum c_{\alpha,\gamma} e_{\alpha,\gamma}) \circ g | \bar{\sigma}_\lambda S_\lambda^3,\end{aligned}$$

which is single valued. From Lemma 3 in [2], $\psi\phi^{-1}$ is continuous. Define a continuous map

$$\xi: K \to L$$

so that $\xi | K - e_\lambda^4 = 1$, $\xi | \bar{e}_\lambda^4 = \psi\phi^{-1}$. Evidently ξ induces isomorphisms of the homology groups of K with integer coefficients onto the corresponding groups of L. Hence ξ is a homotopy equivalence. However, the homotopy boundaries of the 4-dimensional cells of K and L are equal except βe_λ^4 in K and $\beta e_\lambda'^4$ in L. They differ by $\bar{\sigma}_\lambda \sum_{\alpha < \gamma} c_{\alpha,\gamma} e_{\alpha,\gamma}$. Repeating this procedure at most t_3 times to alternate the homotopy boundaries of $e_\lambda^4 (\lambda = 1, \cdots, t_3)$ succesively finishes the proof of this Lemma.

The elements of $H_4(m)$ are $\mu_{m,0} x + \Delta_m^* \sum_l \frac{r_l \bar{\sigma}_l}{(m, \bar{\sigma}_l)} S_l^3 (r_l = 1, \cdots, (m, \bar{\sigma}_l))$, where $x \in H_4$. From (15) they are actually represented by

$$\mu_{m,0}\left(x + \sum_l \frac{r_l \cdot m}{(m, \bar{\sigma}_l)} e_l^4\right).$$

Here $x (\in H_4)$ has unique representative, which is written as x for brevity. Now

$$\mu_{m,0}\beta\left(x + \sum_l \frac{r_l \cdot m}{(m, \bar{\sigma}_l)} e_l^4\right)$$

is uniquely determined for each element $\mu_{m,0}\left(x + \sum_l \frac{r_l \cdot m}{(m, \bar{\sigma}_l)} e_l^4\right)$ of $H_4(m)$. Define

$$\gamma_m: H_4(m) \to (\Gamma(H_2))_m, \quad m = 0, 2, 3, \cdots,$$

according to Whitehead, J. H. C.[3] that

$$\begin{aligned}\gamma_m \mu_{m,0}\left(x + \sum_l \frac{r_l \cdot m}{(m, \bar{\sigma}_l)} e_l^4\right) &= \mu_{m,0}\beta\left(x + \sum_l \frac{r_l \cdot m}{(m, \bar{\sigma}_l)} e_l^4\right) \\ &= \mu_{m,0} p \beta\left(x + \sum_l \frac{r_l \cdot m}{(m, \bar{\sigma}_l)} e_l^4\right), \qquad (20)\end{aligned}$$

because of (13) and $\frac{m}{(m, \bar{\sigma}_l)} \cdot \bar{\sigma}_l \equiv 0$, mod m. Hence

$$\gamma_p\mu_{p,m}\mu_{m,0}\Big(x+\sum_l \frac{r_l\cdot m}{(m,\bar\sigma_l)}e_l^4\Big)=\gamma_p\Big\{\mu_{p,0}\Big(\frac{p}{(p,m)}x+\sum_l\frac{p\cdot r_l\cdot m}{(p,m)\cdot(m,\bar\sigma_l)}e_l^4\Big)$$
$$=\mu_{p,0}p\beta\Big(\frac{p}{(p,m)}x+\sum_l\frac{p\cdot r_l\cdot m}{(p,m)\cdot(m,\bar\sigma_l)}e_l^4\Big),$$

while

$$\mu_{p,m}\gamma_m\mu_{m,0}\Big(x+\sum_l\frac{r_l\cdot m}{(m,\bar\sigma_l)}e_l^4\Big)=\mu_{p,0}p\beta\Big(\frac{p}{(p,m)}x+\sum_l\frac{p\cdot r_l\cdot m}{(p,m)\cdot(m,\bar\sigma_l)}e_l^4\Big).$$

Consequently the following diagram

$$\begin{array}{ccc} H_4(m) & \xrightarrow{\mu_{p,m}} & H_4(p) \\ \gamma_m\downarrow & & \downarrow\gamma_p \\ (\Gamma(H_2))_m & \xrightarrow{\mu_{p,m}} & (\Gamma(H_2))_p \end{array} \qquad (6)$$

commutes. This verifies (6) in § 1, if γ_m is defined by (20).

Let

$$\Delta_p:\ H_i(A_2^2,\ Z_p)\rightarrow H_{i-1}(A_2^2)$$

be defined by $\frac{1}{p}\partial$, then $\Delta\Delta^*=1$ follows from (15) and (16). We have the groups $H_i(A_2^2)$ $(i=2,\ 3,\ 4)$, $H_i(A_2^2,\ Z_m)$ $(m=2,\ 3,\ \cdots,)$ and $\Gamma(H_2(A_2^2))$ along with homomorphisms $\mu_{p,q}$, Δ and γ_m. They together constitute a homology co-ring of an A_2^2-polyhedron. This explains the geometric meaning of an abstract homology co-ring associated to a given A_2^2-polyhedron.

§ 3. Realizability.

§ 3.1. If $f:\ A_2^2\rightarrow A_2'^2$ is a cellular map, it induces homomorphisms of the homology groups $H_i(A_2^2,\ Z_m)$ into the corresponding groups $H_i(A_2^2,\ m)$, $i=2,\ 3,\ 4$, $m=0,\ 2,\ 3,\ \cdots$. These homomorphisms, denoted by f_*, commute with $\mu_{p,q}$ and Δ, i. e., (9) and (10) are satisfied. If $x\in H_4(A_2^2,\ m)$ and if x' is a mod$m$ cycle belonging to x, then

$$\begin{array}{ccc} x' & \xrightarrow{\gamma_m} & \Gamma(H_2(A_2^2))_m \\ \downarrow f_4(m) & & \downarrow \Gamma(f_2)_m \\ f_4(m)x & \xrightarrow{\gamma_m} & \Gamma(H_2(A_2'^2)) \end{array} \qquad (21)$$

commute if $m=0,\ 2,\ 3,\ \cdots$. Therefore the homomorphism γ_m: $H_4(A_2^2,\ m)\rightarrow\Gamma(H_2(A_2^2))_m$ satisfies

$$(\Gamma(f_2))_m\cdot\gamma_m=\gamma_m\cdot f_4(m), \qquad (22)$$

where $(\Gamma(f_2))_m$ is determined by $f_2: H_2(A_2^2)\rightarrow H_2(A_2'^2)$. It is worth while to remark that γ_m is able to be considered as a homomorphism

$$\gamma_m:\ H_4(A_2^2,\ m)\rightarrow(i\Pi_3(K^2))_m,$$

then γ_m satisfies

$$f_{*}\cdot\gamma_m=\gamma_m\cdot f_4(m), \tag{23}$$

where f_*: $(i\Pi_3(K^2))_m\to(i\Pi_3(K^2))_m$ is induced by $f|K^2$. Of course (22) and (23) are equivalent. All these homomorphisms induced by a cellular map f: $A_2^3\to A_2'^3$ constitute a proper homomorphism of the homology co-ring $H(A_2^3)$ into the homology co-ring $H(A_2'^3)$.

If $H_i (i=2, 3, 4)$ are given abelian groups, where H_4 is free, then $H_i(m)$, $(m=0, 2, 3, \cdots)$, Δ, $\mu_{p,q}$ can be defined by (1), (2), (3), and $\Gamma(H_2)$ can, moreover, be constructed by its definition in § 1. Furthermore, $\gamma_m(m=0, 2, 3, \cdots)$ may be arbitrarily given homomorphisms if (6) is satisfied. Then an abstract homology co-ring is completely determined with a lot of arbitrary factors. Is it able to be realized by an A_2^3-polyhedron? If so, is there a CW-map to realize a proper homomorphism of a homology co-ring into another? The answers are contained in Theorem 1 and Theorem 2.

Before going to prove Theorem 1 and Theorem 2, we need to prove

Lemma 4. *The homomorphisms*

$$\gamma_m\colon H_4(m)\to(\Gamma(H_2))_m, \quad m=0, 2, 3, \cdots$$

are determined if[5]

$$\gamma_{\bar\sigma_l}|\Delta^*_{\bar\sigma_l}S_l^3\colon \Delta^*_{\bar\sigma_l}S_l^3\to(\Gamma(H_2))_{\bar\sigma_l}, \quad l=1, \cdots, t_2 \tag{24}$$

and

$$\gamma|H_4\colon H_4\to\Gamma(H_2) \tag{25}$$

are known.

Proof Each element of $H_4(m)$ is able to be written as $\mu_{m,0}x+\Delta^*_m\sum k_lS_l^3$, where $k_l=\dfrac{r_l\cdot\bar\sigma_l}{(m, \bar\sigma_l)}$, $r_l=1, \cdots, (m, \bar\sigma_l)$. Hence $\gamma_m(H_4(m))$ is determined if $\gamma_m|\mu_{m,0}H_4$ and $\gamma_m\Delta^*_mk_lS_l^3$ are known. By means of (15) we find

$$\begin{aligned}\gamma_m\Delta^*_mk_lS_l^3&=\gamma_m\Big(\frac{k_l\cdot m}{\bar\sigma_l}e_l^4\Big)=\gamma_m\cdot\mu_{m,\bar\sigma_l}\Big(\frac{k_l\cdot(m, \bar\sigma_l)}{\bar\sigma_l}\Big)e_l^4\\&=\gamma_m\mu_{m,\bar\sigma_l}\Delta^*_{\bar\sigma_l}\Big(\frac{k_l\cdot(m, \bar\sigma_l)}{\bar\sigma_l}S_l^3\Big)\\&=\mu_{m,\bar\sigma_l}\gamma_{\bar\sigma_l}\Delta^*_{\bar\sigma_l}(r_lS_l^3).\end{aligned} \tag{26}$$

Besides this we have from (6) that

$$\gamma_m\mu_{m,0}|H_4=\mu_{m,0}(\gamma|H_4).$$

Hence (24) and (25) provide the lemma.

Proof of Theorem 1. Let K be the CW-complex given by (12) and (13). Then the other boundary relations are

$$\begin{aligned}&\partial e_i^3(\sigma_i)=\sigma_iS_2^2 \quad (i=1, \cdots, t_2),\\&\partial S_j^3=0 \quad (j=1, \cdots, p_3+t_3),\\&\partial S_h^2=0 \quad (h=1, \cdots, p_2+t_2).\end{aligned}$$

5) By S_l^3 we mean a generator of H_3 of order $\bar\sigma_l$.

Therefore $H_i(K)\approx H_i$, $i=2, 3, 4$. Let these three isomorphisms be denoted by h. Because

$$H_i(K, m)=\mu_{m,0}H_i(K)+\Delta_m^*({}_mH_{i-1}(K))$$

and

$$H_i(m)=(H_i)_m+\Delta_m^*({}_mH_{i-1}),$$

we may define

$$h: H_i(K, m)\to H_i(m)$$

by (7) and (8), i. e.,

$$h(\mu_{m,0}x+\Delta_m^*y)=\mu_{m,0}(hx)+\Delta_m^*(hy), \tag{27}$$

where $x\in H_i(K)$, $y\in {}_mH_{i-1}(K)$.

Now $H_n(K, Z_m)$ and $H_n(K, Z_p)$ may be computed from the boundary relations in K. From (27) it is evident that

$$\begin{array}{ccc} H_n(K, Z_m) & \xrightarrow{\mu_{p,m}} & H_n(K, Z_p) \\ \downarrow h & & \downarrow h \\ H_n(m) & \xrightarrow{\mu_{p,m}} & H_n(p) \end{array}$$

is commutative, where $\mu_{p,m}$: $H_n(K, Z_m)\to H_n(K, Z_p)$ is defined by

$$\mu_{p,m}[c]=\left[\frac{p}{(p, m)}c\right],$$

c being a cycle mod$m$ belonging to a homology class mod m of $H_n(K, Z_m)$. Moreover, $h\Delta=\Delta h$, where Δ in the right hand side is that in the abstract homology co-ring H, while in the left hand side is that in $H(K)$. From (27) we know that h: $H_n(K, Z_p)\to H_n(p)$ is isomorphism for all n and p. Lemma 2 assures $[\Gamma(h)]_m$ is an isomorphism onto. By the definition of $\mu_{m,p}$ we see that

$$[\Gamma(h)]_m\mu_{m,p}=\mu_{m,p}[\Gamma(h)]_p. \tag{28}$$

Let $\mu_{m,0}x+\Delta_m^*k_lS_l^3$ denote an element of $H_4(\Delta_2^2)$, where $x\in H_4(\Delta_2^2)$, $k_l=\dfrac{r_l\bar{\sigma}_l}{(m, \bar{\sigma}_l)}$, $r_l=1, \cdots, (m, \bar{\sigma}_l)$. Then

$$\begin{aligned}[\Gamma(h)]_m\gamma_m(\mu_{m,0}x+\Delta_m^*k_lS_l^3) &= [\Gamma(h)]_m(\mu_{m,0}\gamma x)+[\Gamma(h)]_m\mu_{m,\bar{\sigma}_l}\gamma_{\bar{\sigma}_l}\Delta_{\bar{\sigma}_l}^*r_lS_l^3 \\ &=\mu_{m,0}[\Gamma(h)]\gamma x+\mu_{m,\bar{\sigma}_l}[\Gamma(h)]_{\bar{\sigma}_l}\gamma_{\bar{\sigma}_l}\Delta_{\bar{\sigma}_l}^*r_lS_l^3, \end{aligned} \tag{29}$$

from (6), (26) and (28). In case

$$\begin{aligned} &\Gamma(h)\cdot\gamma=\gamma\cdot h\,|\,H_4, \\ &[\Gamma(h)]_{\bar{\sigma}_l}\cdot\gamma_{\bar{\sigma}_l}=\gamma_{\bar{\sigma}_l}\cdot h\,|\,\Delta_{\bar{\sigma}_l}^*S_l^3 \quad (l=1, \cdots, t_3) \end{aligned} \tag{P}$$

are satisfied, we find from (29) that

$$\begin{aligned}[\Gamma(h)]_m\gamma_m(\mu_{m,0}x+\Delta_m^*k_lS_l^3) &=\mu_{m,0}\cdot\gamma\cdot hx+\mu_{m,\bar{\sigma}_l}\cdot\gamma_{\bar{\sigma}_l}\cdot h\Delta_{\bar{\sigma}_l}^*\cdot r_lS_l^3 \\ &=\gamma_m(\mu_{m,0}hx)+\gamma_m\cdot\mu_{m,\bar{\sigma}_l}\Delta_{\bar{\sigma}_l}^*r_lhS_l^3 \\ &=\gamma_m(\mu_{m,0}hx)+\gamma_m\Delta_m^*\left(\frac{r_l\cdot\bar{\sigma}_l}{(m, \bar{\sigma}_l)}\right)hS_l^3 \\ &=\gamma_m\cdot h(\mu_{m,0}x+\Delta_m^*k_lS_l^3) \quad (l=1, \cdots, t_3).\end{aligned}$$

Because any element of $H_4(A_2^3, Z_m)$ is a linear expression of $(\mu_{m,0}x+\Delta_m^* k_l S_l^3)$, it is concluded that $(\Gamma(h))_m\cdot\gamma_m=\gamma_m\cdot h$ for all m. Now $\Gamma(h)$ and $(\Gamma(h))_{\bar{\sigma}_l}$ $(l=1, \cdots, t_3)$ are isomorphisms onto. The condition (P) becomes

$$\begin{aligned}\gamma&=[\Gamma(h)]^{-1}\cdot\gamma(H)\cdot h\,|\,H_4(K),\\ \gamma_{\bar{\sigma}_h}&=[\Gamma(h)]_{\bar{\sigma}_l}^{-1}\cdot\gamma_{\bar{\sigma}_l}(H)\cdot h\,|\,\Delta_{\bar{\sigma}_l}^* S_l^3 \quad (l=1, \cdots, t_3),\end{aligned} \tag{P'}$$

which means h: $H(K)\to H$ is a proper isomorphism if the attaching mappings of the cells e_l^4, $(l=1, \cdots, t_3+p_4)$ are

$$\begin{aligned}\beta e_l^4&=\bar{\sigma}_l S_l^3+[\Gamma(h)]_{\bar{\sigma}_l}^{-1}\cdot\gamma(H)\cdot h(e_l^4(\bar{\sigma}_l)), \quad l=1, \cdots, t_3,\\ \beta e_{t_3+u}^4&=[\Gamma(h)]^{-1}\gamma(H)\cdot h(e_{t_3+u}^4), \quad u=1, \cdots, p_4,\end{aligned} \tag{P''}$$

where $e_l^4(\bar{\sigma}_l)$ denotes a mod $\bar{\sigma}_l$ homology class represented by $e_l^4(\bar{\sigma}_l)$, and $e_{t_3+u}^4$ an integral homology class represented by $e_{t_3+u}^4$. The conditions (P'') can be assigned to K. This completes the proof of Theorem 1.

§ 3.2. Let K and K' be two simply connected CW-complexes and let K^r (or K'^r) be the r-dimensional skeleton of K (or K'). By a chain group $C_n(K)$ we mean $\Pi_n(K^n, K^{n-1}, n=2, 3, \cdots$. Let β: $\Pi_n(K^n, K^{n-1})\to\Pi_{n-1}(K^{n-1})$ denote the homotopy boundary operator and j: $\Pi_{n-1}(K^{n-1})\to\Pi_{n-1}(K^{n-1}, K^{n-2})$ the injection. Then $j\beta=\partial$ is a homology boundary operator. A chain map g_n: $C_n(K)\to C_n(K')$, $n=2, 3, \cdots$ induces a (μ, Δ) homomorphism of the homology co-ring $H(K)$ into the homology co-ring $H(K')$. On the other hand, if a (μ, Δ)-homomorphism h: $H(K)\to H(K')$ is given, we have

Lemma 5. *There is a chain map*

$$g_n: C_n(K)\to C_n(K'), \quad n=0, 1, 2, \cdots$$

realizing h.

Let $a\in H_n(K)$ or $H_n(K, Z_m)$. If $a'\in a$ and $j_m ga'=hj_m a'$ for all a, where j_m denotes the injection of a' or ga' into its homology class, then h is said to be realized by g. This Lemma is the dual of Lemma 4 in [1]. We sketch the proof of Lemma 5 as follows: The chain map g: $C_n(K)\to C_n(K')$ realizes h if and only if $hj_{\sigma_l^n}a_l^n=j_{\sigma_l^n}ga_l^n$ $(l=1, \cdots, q_n)$ for all n, where a basis $a_1^n, \cdots, a_{q_n}^n$ for each group $C_n(K)$ is so chosen that $\partial a_i^n=\sigma_i^n a_i^{n-1}$ $(i=1, \cdots, q_n)$, where σ_i^n may be zero, and a_i^{n-1} are basis elements of $C_{n-1}(K)$. Then the required chain map g is able to be constructed by induction.

Proof of Theorem 2. Through the procedure of the proof of Theorem 1, it is shown that a reduced A_2^3-polyhedron exists so that its homology co-ring is properly isomorphic to a given homology co-ring H. Let K and K' be two reduced A_2^3-polyhedra which realize two homology co-rings H and H' respectively. Let g: $C_n(K)\to C_n(K')$ for all n be a chain map realizing a given proper homomorphism h: $H(K)\to H(K')$. To prove Theorem 2 it becomes necessary and sufficient to show the existence of a CW-map ϕ: $K\to K'$ realizing the chain map g: $C_n(K)\to C_n(K')$. Because K^2 and K'^2 are

6) $C_0(K)=Z$, $C_1(K)=\phi$.

bouquets of 2-dimensional spheres, $g: C_2(K) \to C_2(K')$ is easily realized by a CW-map $\phi: K^2 \to K'^2$. Now K^3 takes its fashion as (12). Analogously K'^3 is written as

$$K'^3 = \bigvee_{n=1}^{p_2'+t_2'} (S_n'^2 \cup e_u'^3(\overline{\sigma}_u')) \vee \bigvee_{v=1}^{p_3'+t_3'} S_v'^3, \tag{12'}$$

where $\overline{\sigma}_u'$ is zero if $u > t_2'$. Then

$$gS_l^3 = \sum_{u=1}^{t_2'} a_{l,u} e_t'^3(\overline{\sigma}_u') + \sum_{v=1}^{p_3'+t_3'} b_{l,v} S_v'^3 \quad (l=1, \cdots, p_3+t_3),$$

$$ge_h^3(\sigma_h) = \sum_{u=1}^{t_2'} a_{hu} e_u'^3(\overline{\sigma}_u') + \sum_{v=1}^{p_3'+t_3'} b_{l,v} S_v'^3 \quad (h=1, \cdots, t_2).$$

Because $\partial g = g\partial$, it follows that

$$a_{lu} = 0 \quad (l=1, \cdots, p_3+t_3,\ u=1, \cdots, t_2'),$$

$$a_{hu}\overline{\sigma}_u' = \sigma_h r_{h,u} \quad (h=1, \cdots, t_2,\ u=1, \cdots, t_2'),$$

in ich we have assumed $gS_h^2 = \sum r_{hu} S_u^2$. Since

$$\beta g S_l^3 = \phi = \phi\beta S_l^3,$$

$$\beta g e_h^3(\sigma_h) = \sum_{u=1}^{t_2'} a_{hu}\overline{\sigma}_u' S_u'^2 = \sum_{u=1}^{t_2'} \sigma_h r_{hu} S_u'^2 = \sigma_h \phi S_h^2 = \phi\beta e_h^3(\sigma_h),$$

the map ϕ is sble to be extended to a map of K^3 into K'^3 so that ϕ induces a chain map $\phi_*: \Pi_3(K^3, K^2) \to \Pi_3(K'^3, K'^2)$ satisfying

$$\phi_* e_h^3(\sigma_h) = g e_h^3(\sigma_h),$$

$$\phi_* S_l^3 = g S_l^3,$$

if $e_h^3(\sigma)$ and S_l^3 are considered as elemento of $\Pi_3(K^3, K^2)$.

Let $e_h^4(\overline{\sigma}_h)$ be a cell of K^4. It is evident that $j_{\overline{\sigma}_h} e_h^4(\overline{\sigma}_h)$ is an element of $H_4(K, \overline{\sigma}_h)$ with unique representative, which is $e_h^4(\overline{\sigma}_h)$ itself. Let $\hat{j}$ denote an isomorphism of the group of spherical homology classes of K^3 generated by $S_1^3, \cdots, S_{p_3+t_3}^3$ in (12) onto the direct summand $\{b_1, \cdots, b_{p_3+t_3}\}$ in

$$\Pi_3(K^3) = i\Pi_3(K^2) + \{b_1, \cdots, b_{p_3+t_3}\}.$$

In other words

$$\Pi_3(K^3) = \Gamma(H_2(K^3) + \hat{j}\{S_1^3, \cdots, S_{p_3+t_3}^3\},$$

where $\{S_1^3, \cdots, S_{p_3+t_3}^3\}$ is a free module generated by the 3-spheres $S_i^3 (i=1, \cdots, p_3+t_1)$ attached at a point. Let $i: \Gamma(H_2(K)) \to \Pi_3(K^3)$ be an injection such that $\boldsymbol{pi} = 1_{\Gamma(H_2(K))}$. Then $\Pi_3(K^3)$ becomes

$$\Pi_3(K^3) = \boldsymbol{ip}\Pi_3(K^3) + (1 - \boldsymbol{ip})\Pi_3(K^3). \tag{30}$$

Hence

$$\beta g e_h^4(\sigma_h') = \boldsymbol{ip}\beta g e_h^4(\sigma_h') + (1 - \boldsymbol{ip})\beta g e_h^4(\sigma_h').$$

Because

$$\cdots \xrightarrow{\beta} \Pi_3(K^2) \xrightarrow{i} \Pi_3(K^3) \xrightarrow{j} \Pi_3(K^3, K^2) \xrightarrow{\beta} \cdots$$

is exact, the homomorphism

$$j(1 - \boldsymbol{ip})\beta g: C_4(K, Z_{\overline{\sigma}_h}) \to \{S_1^3, \cdots, S_{p_3+t_3}^3\}$$

becomes $\partial g = g\partial: C_4(K, Z_{\overline{\sigma}_h}) \to \{S_1^3, \cdots, S_{p_3+t_3}^3\}$ since $\boldsymbol{ip}\Pi_3(K^3) = i\Pi_3(K^3)$ and $ji = 0$.

Furthermore, $j|(1-\boldsymbol{ip})\beta gC_4(K, Z_{\bar{\sigma}_h})$ is an isomorphism because the kernel of j is $\boldsymbol{ip}\Pi_3(K^3)$ which is a direct summand in (30). On the other hand

$$\phi\beta e_h^4(\bar{\sigma}_h)=\phi\boldsymbol{ip}\beta e_h^4(\bar{\sigma}_h)+\phi(1-\boldsymbol{ip})\beta e_h^4(\bar{\sigma}_h).$$

Now

$$j\phi(1-\boldsymbol{ip})\beta e_h^4(\bar{\sigma}_h)=\phi j(1-\boldsymbol{ip})\beta e_h^4(\bar{\sigma}_h)=\phi\partial e_h^4(\bar{\sigma}_h)=g\partial e_h^4(\bar{\sigma}_h),$$

because ϕ is cellular and realizes g in K^3. Hence

$$j\phi(1-\boldsymbol{ip})\beta e_h^4(\sigma)=j(1-\boldsymbol{ip})\beta g e_h^4(\bar{\sigma}_h).$$

Because $j|(1-\boldsymbol{ip})\Pi_3(K^3)$ is isomorphism into, we have

$$\phi(1-\boldsymbol{ip})\beta e_h^4(\bar{\sigma}_h)=(1-\boldsymbol{ip})\beta g e_h^4(\bar{\sigma}_h).$$

Consequently

$$\beta g e_h^4(\bar{\sigma}_h)-\phi\beta e_h^4(\bar{\sigma}_h)=\boldsymbol{ip}\beta g e_h^4(\bar{\sigma}_h)-\phi\boldsymbol{ip}\beta e_h^4(\bar{\sigma}_h)=\boldsymbol{i}(\boldsymbol{p}\beta g-\phi\boldsymbol{p}\beta)e_h^4(\bar{\sigma}_h). \qquad (31)$$

By definition

$$\mu_{\bar{\sigma}_h,0}(\boldsymbol{p}\beta g e_h^4(\bar{\sigma}_h)-\phi\boldsymbol{p}\beta e_h^4(\bar{\sigma}_l))=\gamma_{\bar{\sigma}_h}[g e_h^4(\bar{\sigma}_h)]-\Gamma(\phi)\gamma_{\bar{\sigma}_h}[e_h^4(\bar{\sigma}_h)],$$

in which $[g e_h^4(\bar{\sigma}_h)]$ is the same homology class as $h[e_h^4(\bar{\sigma}_h)]$ owing to the realization of h by g, while $\Gamma(\phi)=\Gamma(h)$ owing to the realization of $h|H_2(K)$ by ϕ. It follows from (22) that

$$\mu_{\bar{\sigma}_h,0}(\boldsymbol{p}\beta g e_h^4(\bar{\sigma}_h)-\phi\boldsymbol{p}\beta e_h^4(\bar{\sigma}_h))=0$$

in the homology module $(\Gamma(H_2))_{\bar{\sigma}_h}$, which means that

$$\boldsymbol{p}\beta(g e_h^4(\bar{\sigma}_h))-\phi\boldsymbol{p}\beta e_h^4(\bar{\sigma}_h)=\bar{\sigma}_h(\sum a'_{\mu,\nu}e'_{\mu,\nu}).$$

From (31) we have

$$\beta g e_h^4(\bar{\sigma}_h)-\phi\beta e_h^4(\bar{\sigma}_h)=\boldsymbol{i}\bar{\sigma}_h(\sum a'_{\mu,\nu}e'_{\mu,\nu}), \qquad (32)$$

where $\sum a'_{\mu,\nu}e'_{\mu,\nu}\in\boldsymbol{i}\Pi_3(K'^2)$. We remark

$$\beta e_h^4(\bar{\sigma}_h)=\bar{\sigma}_h S_h^3+\sum a_{\alpha,\gamma}e_{\alpha,\gamma}.$$

Define a CW-map ϕ': $K^3\to K'^3$ such that

$$\phi'|K^3-S_h^3=\phi|K^3-S_h^3,$$

but $\phi'|S_h^3$ is the composition of the following maps:

$$S_h^3\xrightarrow{\omega}S_1^3\vee S_2^3\xrightarrow{\mu\vee\nu}S_h^3\vee K'^3\xrightarrow{\phi\vee 1}K'^3,$$

where ω clutches S_h^3 by its equator to obtain $S_1^3\vee S_2^3$, μ maps S_1^3 onto S_h^3 of degree 1 and ν maps S_2^3 into K'^3 so that ν represents $_+\sum a'_{\mu,\nu}e'_{\mu,\nu}$. The map ϕ' still realizes the chain map g: $K^3\to K'^3$. But (32) becomes

$$\beta g e_h^4(\bar{\sigma}_h)-\phi'\beta e_h^4(\bar{\sigma}_h)=0.$$

It means that ϕ' can be extended over to realize $g e_h^4(\bar{\sigma}_h)$. If $\bar{\sigma}_h=0$, we have

$$\beta g e_{t_s+u}^4=\boldsymbol{p}\beta g e_{t_s+u}^4=\gamma h[e_{t_s+u}^4],$$

$$\phi\beta e_{t_s+u}^4=\Gamma(h)\cdot\gamma\cdot[e_{t_s+u}^4].$$

From (11) with $m=0$ we have $\gamma\cdot h=\Gamma(h)\cdot\gamma$, which means $\beta g e_{t_s+u}^4=\phi\beta e_{t_s+u}^4$. This guarantees the extendability of ϕ to realize $g e_{t_s+u}^4$, $u=1, \cdots, p_4$. In short, the chain map g is able to be realized by a cellular map. Q. E. D.

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关于 A_2^2-多面体

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摘要

本文提出 A_2^2 同调可环，证明它的实现性，即在任给的抽象的同调可环的正则同构类中，必有某一 A_2^2 多面体的同调可环. 又证明同调可环间的正则同态，必为连续映像所实现，于是获得一个函子使 A_2^2 多面体的伦型与同调可环的正则同构类一一对应. 这里提供一个实例，说明这样的函子是代数与几何的桥梁.