Chin. Ann. of Math.
4 (3) 1982

ON INFINITE GALOIS THEORY FOR DIVISION RINGS

XU YONGHUA
(*Institute of Mathematics, Fudan University*)

Dedicated to Professor Su Bu-chin on the Occasion of his 80th Birthday and his 50th Year of Educational Work

In [1] we have established the finite structure theorem between complete rings of linear transformations and have used it to study the finite Galois theory of division rings. In this paper, we shall generalize the structure theorem to infinite case and then we study the infinite Galois theory of division rings, which is hard to deal with as we know. In § 1 we shall establish the infinite structure theorem between complete rings of linear transformations. In § 3 we shall use it to study the fundamental theorem of infinite Galois theory of division rings. And in § 4 we shall indicate that our theory includes the well known finite Galois theory of division rings.

§ 1.

If G is a group of automorphisms of division ring F, then we write

$$I(G)=\{f\in F\,|\,f^\sigma=f \text{ for all } \sigma\in G\}.$$

If K is a division subring of F, then we write $A(K)=\{\sigma\in G\,|\,K^\sigma=K \text{ for all } k\in K\}$. And we denote the algebra[1)] of G by E' and the complete ring of P-linear transformations of $\mathfrak{M}$ by $\mathscr{L}(\widetilde{P}, \mathfrak{M})$ just as in [1], where $\widetilde{P}$ is any division subring of F.

Let $\mathfrak{M}=\sum_{i\in I} Fu_i$ be a left vector space over a division ring F, $\mathbf{E}$ be the ring of all endomorphisms of the additive group $(\mathfrak{M}, +)$. A subset $\{x_i\}_{i\in I}$ of $\mathfrak{M}$ is called having ν-order if the cardinal number of $\{x_i\}_{i\in I}$ is $\aleph_\nu$. We denote it by Card. $\{x_i\}_I=\aleph_\nu$. Let D_ν be the class of all sets with order $\leqslant\aleph_\nu$ in $\mathfrak{M}$. Then we can define a mapping $[\quad]$ of D_ν into $\mathfrak{M}$ as follows: for any element $\{x_i\}_{i\in I_1}$ of D_ν with Card. $\{x_i\}_{i\in I_1}\leqslant\aleph_\nu$, it corresponds an element $[\{x_i\}_{i\in I_1}]$ of $\mathfrak{M}$, i. e.,

Manuscript received January 1, 1981.

1) Let E be a ring with an identity, Φ its center and let G be a group of automorphisms in E. Then we call the subalgebra E' over Φ generated by the (regular) elements c such that $I_c\in G$ the algebra of the group G.

$$\{x_i\}_{i\in I_1} \xrightarrow{[\ ]} [\{x_i\}_{i\in I_1}]$$

such that [ ] satisfies the following conditions:

(i) If all $x_i=0$ for $\{x_i\}_{i\in I_1}$, then $[\{x_i\}_{i\in I_1}]=0$,

(ii) $(f[\{x_i\}_{i\in I_1}])\sigma=[\{(fx_i)\sigma\}_{i\in I_1}]$ for $f\in F$, $\sigma\in \mathbf{E}$,

(iii) Let $\{a_i^{(\alpha)}\}_{\substack{i\in I\\ \alpha\in I'}}$ be a set of F, Card. $I\leqslant \aleph_\nu$, Card. $I'\leqslant\aleph_\nu$ and let $\{y_\alpha\}_{\alpha\in I'}$ be a set of F-linearly independent elements of $\mathfrak{M}$. If the system of equations

$$[\{a_i^{(\alpha)}x_i\}_{i\in I}]=y_\alpha \quad \text{for } \alpha\in I' \tag{1.1}$$

is solvable for $\{x_i\}_{i\in I}$ in $\mathfrak{M}$, then it follows that

(1) Card. $I'\leqslant$Card. I,

(2) there exists a subset $\{a_i^{(\alpha)}\}_{i\in I'}$ of $\{a_i^{(\alpha)}\}_{i\in I}$ such that for any element $\{Y_\alpha\}_{\alpha\in I'}$ of D_ν the following system of equations

$$[\{a_i^{(\alpha)}X_i\}_{i\in I'}]=Y_\alpha, \quad \alpha\in I' \tag{1.2}$$

has one and only one solution for $\{X_i\}_{i\in I'}$, that is, if two subsets $\{x_i\}_{i\in I'}$ and $\{x_i^0\}_{i\in I'}$ satisfy the equation (1.2), then this forces $x_i=x_i^0$ for $i\in I'$.

Definition 1. 1. *The above mapping [ ] of $\mathfrak{M}$ is called a ν-soluable function. A vector space is called a ν-soluable space if it has a ν-soluable function. If the function [ ] is defined in the class of all subsets of $\mathfrak{M}$, then we call [ ] simply a soluable function and $\mathfrak{M}$ a soluable space.*

Since $\mathfrak{M}=x\mathbf{E}$ for any non-zero element x of $\mathfrak{M}$, we can extend the ν-soluable function to $\mathbf{E}$. Similarly, a subset $\{\varepsilon_i\}_{i\in I}$ of $\mathbf{E}$ is called a set with ν-order if Card. $I=\aleph_\nu$. Denote the class of all subsets with orders $\leqslant\aleph_\nu$ by D_ν^*, then we define a function of D_ν^* into $\mathbf{E}$ which is still denoted by [ ], if it is not to be confused with the preceding one. Furthermore, these two functions have the following relations: for any element x of $\mathfrak{M}$ we define $x[\{\varepsilon_i\}_{i\in I}]=[\{x\varepsilon_i\}_{i\in I}]$, where Card. $I\leqslant\aleph_\nu$. It is easy to see that the element $[\{x_i\}_{i\in I}]$ of $\mathfrak{M}$ can be written as $[\{x_i\}_{i\in I}]=[\{x\varepsilon_i\}_{i\in I}]$, where $\{\varepsilon_i\}_{i\in I}\subset\mathbf{E}$. It is also easy to see that $[\{\varepsilon_i\}_{i\in I}]\sigma=[\{\varepsilon_i\sigma\}_{i\in I}]$ for any element σ of $\mathbf{E}$.

Let $\mathfrak{M}=\sum_{i\in\Gamma}Fu_i$ be a space, G a group of automorphisms of F, and let $\mathbf{E}$ be the complete ring of endomorphisms of $(\mathfrak{M},+)$. Let T and $\mathscr{L}$ be subsets of $\mathbf{E}$, and I be a set. Then $M(T,\mathscr{L};I)$ of $\mathbf{E}$ is defined as follows

$$M(T,\mathscr{L};I)=\{[\{t_j\sigma_j\}_{j\in I_1}]\,|\,t_j\in T,\ \sigma_j\in\mathscr{L};\ I_1\sqsubseteq I\}, \tag{1.3}$$

where $t_j\in T$, $\sigma_j\in\mathscr{L}$ and $t_j\sigma_j\in\mathbf{E}$ for $j\in I_1$.

Definition 1. 2. *Let $\mathfrak{M}=\sum_j Fu_j$ be a space. A set Q of $\mathbf{E}$ is called $\aleph_\nu$-transitive if and only if for any subset $\{x_i\}_{i\in I}$ of F-linearly independent elements of $\mathfrak{M}$ and any subset $\{y_i\}_{i\in I}$ of $\mathfrak{M}$, where Card. $I\leqslant\aleph_\nu$, there exists an element $\sigma\in Q$ such that $x_i\sigma=y_i$ for $i\in I$.*

Definition 1. 3. *Let* $\mathfrak{M}=\sum_{i\in\Gamma} Fu_i$, $P'=C_F(E')=\{f\in F \mid fe'=e'f$ *for all* $e'\in E\}$[2]. *We say that* $\mathfrak{M}$ *is a* $(\aleph_\nu, 1)$*-type space over* P' *if and only if* $\mathfrak{M}$ *is a soluable space and there exists a set* $M(E'_L, \mathscr{L}(F, \mathfrak{M}); I)$ *which is* $\aleph_\nu$*-transitive in* $\mathfrak{M}$ *as the vector space over* P' *where Card.* $I=\aleph_\nu$.

Definition 1. 4. $M(T, \mathscr{L}; I)$ *is called free if and only if that if* $[\{t_j\sigma_j\}_{j\in I_1}]=0$ *for any element* $[\{t_j\sigma_j\}_{j\in I_1}]$ *of* $M(T, \mathscr{L}; I)$, *then all* $t_j\sigma_j=0$ *for* $j\in I_1$. *In this case we write* $M(T, \mathscr{L}; I)=\oplus M(T, \mathscr{L}; I)$,

we are now ready to establish the following theorem:

Theorem 1. 1. *Let* $\mathfrak{M}=\sum_{i\in\Gamma} Fu_i$ *be a space, G a group of automorphisms of F, and let E′ be the algebra of G,* $P'=C_F(E')$. *Denote the dimension of the left vector space F over a subring P′ by* $[F: P']_L$ *and let I be an index set with Card.* $I=[F: P']_L=\aleph_\nu$. *If* $\mathfrak{M}$ *is a* $(\aleph_\nu, 1)$*-type space over P′, then there exists a subset* $\{r_j\}_{j\in I}$ *of E′ such that*

$$\mathscr{L}(P', \mathfrak{M})=\oplus M(\{r_{jL}\}_{j\in I}, \mathscr{L}(F, \mathfrak{M}); I).$$

Conversely, if $\mathscr{L}(P', \mathfrak{M})=\oplus M(\{r_{jL}\}_{j\in I^*}, \mathscr{L}(F, \mathfrak{M}); I^*)$, *where* $\{r_{jL}\}_{j\in I^*}\subseteq E'_L$, *then there exists a subset* $\{r_{jL}\}_{j\in I'}$ *of* $\{r_{jL}\}_{j\in I^*}$ *such that*

$$\mathscr{L}(P', \mathfrak{M})=\oplus M(\{r_{jL}\}_{j\in I'}, \mathscr{L}(F, \mathfrak{M}); I')$$

and Card. $I'=[F: P']_L$

Proof. we prove the first assertion. Since P' is a division subring of F, there exists a P'-basis $\{f^{(\alpha)}\}_{\alpha\in I}$ such that $F=\sum_{\alpha\in I} P'f^{(\alpha)}$, where Card. $I=[F: P']_L=\aleph_\nu$. Hence $\mathfrak{M}=\sum_{i\in\Gamma} Fu_i=\sum_{i\in\Gamma,\alpha\in I} P'f^{(\alpha)}u_i=\sum_{i\in\Gamma,\alpha\in I} P'v_i^{(\alpha)}$, where $v_i^{(\alpha)}=f^{(\alpha)}u_i$. Clearly $\{v_i^{(\alpha)}\}_{\substack{i\in\Gamma\\ \alpha\in I}}$ is also a P'-basis of $\mathfrak{M}$. Let $\{y_\alpha\}_{\alpha\in I}$ be an arbitrary set of F-linear independent elements of $\mathfrak{M}$. By assumption $\mathfrak{M}$ is a $(\aleph_\nu, 1)$-type space over P'. In view of Definition 1.3 and (1.3) there exists an element $\sigma=[\{r_{jL}\sigma'_j\}_{j\in I_1}]\in M(E'_L, \mathscr{L}(F, \mathfrak{M}); I)$ for $r_{jL}\in E'_L$, $\sigma'_j\in\mathscr{L}(F, \mathfrak{M})$, $I_1\subseteq I$ such that σ satisfies $v_i^{(\alpha)}\sigma=y_\alpha$, $\alpha\in I$. Hence

$$y_\alpha=v_i^{(\alpha)}\sigma=[\{(f^{(\alpha)}u_i)(r_{jL}\sigma'_j)\}_{j\in I_1}]=[\{r_jf^{(\alpha)}(u_i\sigma'_j)\}_{j\in I_1}], \quad \alpha\in I. \tag{1.4}$$

Let $a_{\alpha j}=r_jf^{(\alpha)}$, $x_j=u_i\sigma'_j$ for $\alpha\in I$, $j\in I$, then (1.4) becomes the following formula

$$[\{a_{\alpha j}x_j\}_{j\in I_1}]=y_\alpha, \quad \alpha\in I. \tag{1.5}$$

From the porperty of soluable function [ ] it follows that Card. $I\leqslant$ Card. I_1, furthermore there exists a subset $\{a_{\alpha j}\}_{j\in I_1}$ of $\{a_{\alpha j}\}_{j\in I}$ such that the following system of functional equations

$$[\{a_{\alpha j}X_j\}_{j\in I}]=Y_\alpha, \quad \alpha\in I \tag{1.6}$$

is soluable for any set $[Y_\alpha]_{\alpha\in I}$ of $\mathfrak{M}$ and it has a unique solution. Now we want to show that $\mathscr{L}(P', \mathfrak{M})\subseteq M(\{r_{jL}\}_{j\in I}, \mathscr{L}(F, \mathfrak{M}); I)$.

In fact, let $\sigma^*\in\mathscr{L}(P', \mathfrak{M})$, $v_i^{(\alpha)}\sigma^*=Y_\alpha(i)$, $\alpha\in I$, $i\in\Gamma$. Consider the system of functional equations (1.6). Then from the soluable property of (1.6) for arbitrary

2) E' is the algebra of the group G, where G is a group of automorphisms in F.

subset $\{Y_\alpha\}_{\alpha\in I}$ of $\mathfrak{M}$ it follows that for any element $i\in\Gamma$ the following system of equations

$$[\{a_{\alpha j}X_j(i)\}_{j\in I}]=Y_\alpha(i),\quad \alpha\in I \tag{1.7}$$

has a solution. Since $\mathfrak{M}$ is a soluable space, (1.7) has a unique solution $X_j(i)=x_j(i)\in\mathfrak{M}$ for $j\in I$. On the other hand, since $\mathfrak{M}=\sum_{i\in\Gamma}Fu_i$, there exists an element $\sigma'_j\in\mathscr{L}(F,\mathfrak{M})$ such that

$$u_i\sigma'_j=x_j(i)\ \text{for}\ j\in I,\ i\in\Gamma. \tag{1.8}$$

Let $\bar\sigma=[\{r_{jL}\sigma'_j\}_{j\in I}]$, then we have

$$v_i^{(\alpha)}\bar\sigma=[(f^{(\alpha)}u_i)\{r_{jL}\sigma'_j\}_{j\in I}]=[\{r_jf^{(\alpha)}(u_i\sigma'_j)\}_{j\in I}]=[\{a_{\alpha j}x_j(i)\}_{j\in I}]=v_i^{(\alpha)}\sigma^*.$$

Since $\{v_i^{(\alpha)}\}_{\substack{\alpha\in I\\ i\in\Gamma}}$ is P'-basis of $\mathfrak{M}$, it follows that $\sigma^*=\bar\sigma\in M(\{r_{jL}\}_{j\in I},\mathscr{L}(F,\mathfrak{M});I)$. But σ^* is an arbitrary element of $\mathscr{L}(P',\mathfrak{M})$, we have

$$\mathscr{L}(P',\mathfrak{M})\subset M(\{r_{jL}\}_{j\in I},\mathscr{L}(F,\mathfrak{M});I).$$

Let $p'\in P'$, $v\in\mathfrak{M}$ and $\sigma=[\{r_{jL}\sigma'_j\}_{j\in I^*}]$ is arbitrary element of $M(\{r_{jL}\}_{j\in I},\mathscr{L}(F,\mathfrak{M});I)$, where $\sigma'_j\in\mathscr{L}(F,\mathfrak{M})$, $I^*\subset I_1$, then

$$(p'v)[\{r_{jL}\sigma'_j\}_{j\in I^*}]=[\{r_jp'v\sigma'_j\}_{j\in I^*}]=p'(v[\{r_{jL}\sigma'_j\}_{j\in I^*}]),$$

this is true because $p'\in P'=C_F(E')$ and $r_j\in E'$. Since p' and v are arbitrary, it follows that $[\{r_{jL}\sigma'_j\}_{j\in I^*}]\in\mathscr{L}(P',\mathfrak{M})$. Hence $M(\{r_{jL}\}_{j\in I},\mathscr{L}(F,\mathfrak{M});I)\subseteq\mathscr{L}(P',\mathfrak{M})$.

We need to prove that $M(\{r_{jL}\}_{j\in I},\mathscr{L}(F,\mathfrak{M});I)=\oplus M(\{r_{jL}\}_{j\in I},\mathscr{L}(F,\mathfrak{M});I)$. If $M(\{r_{jL}\}_{j\in I},\mathscr{L}(F,\mathfrak{M});I)$ has an element $[\{r_{jL}\sigma_j\}_{j\in I'}]=0$, where $I'\subseteq I$, then

$$(f^{(\alpha)}u_i)[\{r_{jL}\sigma_j\}_{j\in I'}]=[\{a_{\alpha j}(u_i\sigma_j)\}_{j\in I'}]=0,\quad \alpha\in I.$$

From the property of soluable space of $\mathfrak{M}$ it follows that there exists a subset $\{a_{\alpha j}\}_{j\in I}$ of $\{a_{\alpha j}\}_{j\in I'}$, where $I\subseteq I'$ such that $[\{a_{\alpha j}(u_i\sigma_j)\}_{j\in I}]=0$ for $\alpha\in I$. From the property of unique solution it follows that $u_i\sigma_j=0$ for all $i\in\Gamma$. Hence $\sigma_j=0$ for $j\in I=I'$. This proves $r_{jL}\sigma_j=0$ for $j\in I'$.

Now we prove the second assertion. If $\mathscr{L}(P',\mathfrak{M})=\oplus M(\{r_{jL}\}_{j\in I^*},\mathscr{L}(F,\mathfrak{M});I^*)$, where $\{r_{jL}\}_{j\in I^*}\subset E'_L$, then we want to prove $[F:P']_L\leqslant\text{Card.}\,I^*$. In fact, if we write $F=\sum_{\alpha\in I}P'f^{(\alpha)}$, Card. $I=[F:P']_L$, then $\mathfrak{M}=\sum_\Gamma Fu_i=\sum_{i,\alpha}P'v_i^{(\alpha)}$, $v_i^{(\alpha)}=f^{(\alpha)}u_i$. Since $\{u_i\}_{i\in\Gamma}$ is a basis of $\mathfrak{M}$, it is easy to see that $\{v_i^{(\alpha)}\}_{\substack{i\in\Gamma\\ \alpha\in I}}$ is a P'-basis of $\mathfrak{M}$. Let $\{y_\alpha\}_{\alpha\in I}$ be a given set of F-linearly independent elements of $\mathfrak{M}$, then there exists an element $\sigma\in\mathscr{L}(P',\mathfrak{M})$ such that $v_i^{(\alpha)}\sigma=y_\alpha$, $\alpha\in I$. By assumption $\sigma=[\{r_{jL}\omega_j\}_{j\in I'}]$, Card. $I'\leqslant$ Card. I^*, $\omega_j\in\mathscr{L}(F,\mathfrak{M})$. Hence we have

$$y_\alpha=[\{r_jf^{(\alpha)}(u_i\omega_j)\}_{j\in I'}]=[\{a_{\alpha j}x_j\}_{j\in I'}],\quad \alpha\in I, \tag{1.9}$$

where $a_{\alpha j}=r_jf^{(\alpha)}$, $x_j=u_i\omega_j$. Therefore from the property of soluable function [ ] it follows that Card. $I\leqslant$ Card. I'. But Card. $I'\leqslant$ Card. I^*, hence $[F:P']_L\leqslant$ Card. I^*.

On the other hand, in view of (1.9) and the property of soluable function we can choose a subset $\{a_{\alpha j}\}_{j\in I}$ from $\{a_{\alpha j}\}_{j\in I'}$ such that the system of functional equations

$$[\{a_{\alpha j}X_j\}_{j\in I}]=Y_\alpha, \quad \alpha\in I$$

has a unique solution for any subset $\{Y_\alpha\}_{\alpha\in I}$. Repeating the proof of the first assertion we can immediatly obtain $\mathscr{L}(P', \mathfrak{M})=\oplus M(\{r_{jL}\}_{j\in I}, \mathscr{L}(F, \mathfrak{M}); I)$ and Card. $I=[F: P']_L$, $I\subseteq I^*$. This completes the proof.

Definition 1. 5. *We say that* $\mathscr{L}(P', \mathfrak{M})=\oplus M(\{r_{jL}\}_{j\in I}, \mathscr{L}(F, \mathfrak{M}); I)$ *is an expression of* $\mathscr{L}(P', \mathfrak{M})$ *about* E' *and* $\mathscr{L}(F, \mathfrak{M})$, *where* $\{r_j\}_{j\in I}\subset E'$. *A expression* $\oplus M(\{r_{jL}\}_{j\in I}, \mathscr{L}(F,\mathfrak{M}); I)$ *is called minimal if and only if any expression* $\oplus M(\{r^*_{jL}\}_{j\in I^*}, \mathscr{L}(F, \mathfrak{M}); I^*)$ *of* $\mathscr{L}(P', \mathfrak{M})$ *about* E' *and* $\mathscr{L}(F, \mathfrak{M})$ *has the property Card.* $I\leqslant$ *Card.* I^*. *In this case we call Card. I the cardinal number of expression of* $\mathscr{L}(P', \mathfrak{M})$.

From the last part of Theorem 1.1 and Definition 1.5 follows the following theorem:

Theorem 1. 2. *Under the above hypotheses,* $\mathscr{L}(P', \mathfrak{M})$ *has a minimal expression about* E' *and* $\mathscr{L}(F, \mathfrak{M})$. *And the cardinal number of its expression is equal to* $[F:P']_L$.

Let G be a group of automorphisms of F. It is easy to see that for any element ψ of G there exists an F-semi-linear isomorphism S, which we denote by (S, ψ) if we need to emphasize ψ. Let $\Theta=\{S=(S, \psi)\,|\,\psi\in G\}$. As (1.3) we write

$$M(\Theta, \mathscr{L}(P',\mathfrak{M}); I)=\{[\{S_j\sigma'_j\}_{j\in I_1}]\,|\,S_j\in\Theta, \ \sigma'_j\in\mathscr{L}(P', \mathfrak{M}); \ I_1\subseteq I\}.$$

Definition 1. 6. *Let* $\mathfrak{M}=\sum_{i\in I}Fu_i$, $P=I(G)=\{f\in F\,|\,f^\sigma=f \text{ for all } \sigma\in G\}$. $\mathfrak{M}$ *is called a* $(\aleph_\nu, 2)$*-type space over* P *if* $\mathfrak{M}$ *is a soluable space and there exists a set* $M(\Theta, \mathscr{L}(P', \mathfrak{M}); I)$, *which is* $\aleph_\nu$*-transitive in space* $\mathfrak{M}$ *over* P.

Theorem 1. 3. *Let* $\mathfrak{M}=\sum_{i\in I}Fu_i$, $P'=C_F(E')$, $I(G)=P$, $[P': P]_L=\aleph_\nu$. *Suppose that* $\mathfrak{M}$ *is a* $(\aleph_\nu, 2)$*-type space over* P, *then there exists in* Θ *a subset* $\{v_j\}_{j\in I}$ *with cardinal number* $\aleph_\nu$ *such that* $\mathscr{L}(P, \mathfrak{M})=\oplus M(\{S_j)_{j\in I}, \mathscr{L}(P', \mathfrak{M}); I)$. *Conversely, if*

$$\mathscr{L}(P, \mathfrak{M})=\oplus M(\{S_j\}_{j\in I}, \mathscr{L}(P', \mathfrak{M}); I),$$

then there exists in $\{S_j\}_{j\in I}$ *a subset* $\{S_j\}_{j\in I'}$ *such that* $\mathscr{L}(P, \mathfrak{M})=\oplus M(\{S_j\}_{j\in I'}, \mathscr{L}(P', \mathfrak{M}); I')$ *and Card,* $I'=[P': P]_L$.

Proof Now we prove the first assertion. Since E' is the algebra of G, $P'=C_F(E')$, by [1] $E'^\psi=E'$, $P'^\psi=P'$ for any element ψ of G. Let

$$\mathfrak{M}=\sum_{j\in I'}Pw_j=\sum_{j\in I',\alpha\in I'}Pv_j^{(\alpha)}$$

be P'-and P-spaces respectively, where $P'=\sum_{\alpha\in I'}Pf'^{(\alpha)}$, $v_j^{(\alpha)}=f'^{(\alpha)}w_j$. It is easy to see that $\{v_j^{(\alpha)}\}_{j\in I', \alpha\in I'}$ is a P-basis of $\mathfrak{M}$. Let $\{y_\alpha\}_{\alpha\in I'}$ be a set of F-linearly independent elements of $\mathfrak{M}$. By assumptions there exists an element $\sigma=[\{S_j\sigma'_j\}_{j\in I_1}]\in M(\Theta, \mathscr{L}(P', \mathfrak{M}); I)$, $I_1\subseteq I$ such that

$$y_\alpha=v_i^{(\alpha)}\sigma=(f'^{(\alpha)}w_j)[\{S_j\sigma'_j\}_{j\in I_1}]=[(f'^{(\alpha)}w_j)\{S_j\sigma'_j\}_{j\in I_1}]$$
$$=[\{f'^{(\alpha)\psi_j}(w_jS_j\sigma'_j)\}_{j\in I_1}]=[\{a_{\alpha j}x_j(i)\}_{j\in I_1}], \ \alpha\in I',$$

where $a_{\alpha j}=f'^{(\alpha)\psi_j}$, $f'^{(\alpha)}\in P'$, $x_j(i)=w_j(S_j\sigma'_j)$. Then from the property of soluable function [ ] it follows that Card. $I' \leqslant$ Card. I_1, and there exists a subset $\{a_{\alpha j}\}_{j\in I'}$ of $\{a_{\alpha j}\}_{j\in I_1}$ such that the following system of functional equations

$$[\{a_{\alpha j}X_j(i)\}_{j\in I'}] = Y_\alpha(i), \quad \alpha\in I' \tag{1.10}$$

has a unique solution in $\mathfrak{M}$ for any subset $\{Y_\alpha(i)\}_{\alpha\in I'}$.

Now we let $\sigma^*\in \mathscr{L}(P, \mathfrak{M})$, $v_i^{(\alpha)}\sigma^* = Y_\alpha(i)$ for $\alpha\in I'$. Since $S_j = (S_j, \psi_j)$ is an F-semi-linear isomorphism, we know that $\{w_i S_j\}_{i\in I'}$ is also a P'-basis of $\mathfrak{M}$. Therefore for $j\in I'$ there exists an element $\sigma_j''\in \mathscr{L}(P', \mathfrak{M})$ such that $w_i S_j \sigma_j'' = X_j(i)$ for $i\in I'$. Let $\bar{\sigma} = [\{S_j\sigma_j''\}_{j\in I'}]$, then

$$v_i^{(\alpha)}\bar{\sigma} = [\{f'^{(\alpha)\psi_j}(w_i S_j \sigma_j'')\}_{j\in I'}] = [\{a_{\alpha j}X_j(i)\}_{j\in I'}] = Y_\alpha(i) = v_i^{(\alpha)}\sigma^*$$

for all $i\in I'$, $\alpha\in I'$. Hence $\sigma^* = \bar{\sigma}\in M(\{S_j\}_{j\in I'}, \mathscr{L}(P', \mathfrak{M});\ I')$. This proves that

$$\mathscr{L}(P, \mathfrak{M}) \subseteq M(\{S_j\}_{j\in I'}, \mathscr{L}(P', \mathfrak{M});\ I').$$

On the other hand, let $p\in P$, $v\in \mathfrak{M}$, $[\{S_j\sigma_j'\}_{j\in J}]\in M(\{S_j\}_{j\in I'}, \mathscr{L}(P', \mathfrak{M});\ I')$, then $(pv)[\{S_j\sigma_j'\}_{j\in J}] = p(v[\{S_j\sigma_j'\}_{j\in J}])$, it follows that $[\{S_j\sigma_j'\}_J]\in \mathscr{L}(P, \mathfrak{M})$. This proves $M(\{S_j\}_{j\in I'}, \mathscr{L}(P', \mathfrak{M});\ I') \subseteq \mathscr{L}(P, \mathfrak{M})$.

Similarly to Theorem 1.1 we can prove the remainder of this theorem.

Theorem 1.4. *Under the same assumptions as in Theorem 1.3, $\mathscr{L}(P, \mathfrak{M})$ has a minimal expression about Θ and $\mathscr{L}(P', \mathfrak{M})$ and the cardinal number of this expression is $[P':P]_L$.*

Proof It follows from Theorem 1.3.

Now let $F = \sum_{\alpha\in I_1} P'f^{(\alpha)}$, $P' = \sum_{\beta\in I_2} Pg'^{(\beta)}$, then $\mathfrak{M} = \sum_{I} Fu_i = \sum_{\alpha\in I_1, \beta\in I_2} P(g'^{(\beta)}f^{(\alpha)}u_i)$. Let $v_i^{(\alpha,\beta)} = g'^{(\beta)}f^{(\alpha)}u_i$, $\alpha\in I_1$, $\beta\in I_2$, then $\{v_i^{(\alpha,\beta)}\}$ is a P-basis of $\mathfrak{M}$. Write

$$M(\Theta E_L', \mathscr{L}(F, \mathfrak{M});\ I_1\times I_2) = \{[\{S_k r_{jL} w_{kj}\}_{\substack{k\in I_1'\\ j\in I_2'}}]\},$$

where $I_1'\subseteq I_1$, $I_2'\subseteq I_2$, $S_k\in\Theta$, $r_{jL}\in E_L'$, $\omega_{jk}\in \mathscr{L}(F, \mathfrak{M})$.

Definition 1.7. *Let $\mathfrak{M} = \sum_{I} Fu_i$, E' be the algebra of G, and let Θ be the same as above. Then $\mathfrak{M}$ is called a $(\aleph_\nu, 3)$-type space over P if $\mathfrak{M}$ is a soluable space and there exists a set $M(\Theta E_L', \mathscr{L}(F, \mathfrak{M});\ I)$ which is $\aleph_\nu$-transitive over P.*

Theorem 1.5. *Let $\mathfrak{M} = \sum_{i\in I} Fu_i$, G be a group of automorphims of F. Let $P = I(G)$ as above, E' be the algebra of G, $P' = C_F(E')$, $\Theta = \{S_j = (S_j, \psi_j) \mid \psi_j\in G\}$, and $[F:P]_L = \aleph_\nu$. Suppose that $\mathfrak{M}$ is a $(\aleph_\nu, 3)$-type space over P, then there exist subsets $\{S_j\}_{j\in I_1}$ of Θ and $\{r_{jL}\}_{j\in I_2}$ of E_L' such that $\mathscr{L}(P, \mathfrak{M}) = \oplus M(\{S_k r_{jL}\}_{\substack{k\in I_1\\ j\in I_2}}, \mathscr{L}(F, \mathfrak{M});\ I_1\times I_2)$ holds, where Card. $I_1\times I_2 = \aleph_\nu$.*

Conversely, if $\mathscr{L}(P, \mathfrak{M}) = \oplus M(\{S_k r_{jL}\}_{\substack{k\in I_1'\\ j\in I_2'}}, \mathscr{L}(F, \mathfrak{M});\ I_1'\times I_2')$, where $S_k\in\Theta$, $r_j\in E'$, then there exist subsets $I_1\subset I_1'$ and $I_2\subset I_2'$ such that $\mathscr{L}(P, \mathfrak{M}) = \oplus M(\{S_k r_{jL}\}_{\substack{k\in I_1\\ j\in I_2}}, \mathscr{L}(F, \mathfrak{M});\ I_1\times I_2)$ and Card. $I_1\times I_2 = \aleph_\nu$.

Proof Let $\{y_{\beta,\alpha}\}_{\beta\in I_1, \alpha\in I_2}$ be a set of F-linearly independent elements of $\mathfrak{M}$, then by the assumptions, there exists an element $\sigma \in M(\Theta E'_L, \mathscr{L}(F, \mathfrak{M}); I_1 \times I_2)$ such that

$$y_{\beta,\alpha} = v_i^{(\alpha,\beta)}\sigma, \quad \beta \in I_1, \quad \alpha \in I_2.$$

Since $\sigma = [\{S_k r_{jL} w_{kj}\}_{\substack{k\in I_1\\ j\in I_2}}]$, where $S_k \in \Theta$, $r_{jL} \in E'_L$, $w_{kj} \in \mathscr{L}(F, \mathfrak{M})$, we have

$$y_{\beta,\alpha} = [\{g'^{(\beta)\psi_k I_{r_j^{-1}}} f^{(\alpha)\psi_k I_{r_j^{-1}}}(u_i S_k r_{jL} w_{kj})\}_{\substack{k\in I_1\\ j\in I_2}}] = [\{a_{k,j}^{(\beta,\alpha)} x_{kj}(i)\}_{\substack{k\in I_1\\ j\in I_2}}],$$

where $a_{k,j}^{(\beta,\alpha)} = g'^{(\beta)\psi_k I_{r_j^{-1}}} f^{(\alpha)\psi_k I_{r_j^{-1}}}$, $x_{kj}(i) = u_i S_k r_{jL} w_{kj}$, $I_{r_j^{-1}}$ is an inner isomorphism. Analogous to the proofs of Theorem 1.1 and 1.3 we can prove all assertions of our theorem.

Theorem 1. 6. *Let the assumptions be as in Theorem* 1.5, *then* $\mathscr{L}(P, \mathfrak{M})$ *has a minimal expression about* $\Theta E'_L$ *and* $\mathscr{L}(F, \mathfrak{M})$. *And the cadinal number of the minimal expression is equal to* $[F: P]_L$.

Remark. Let the assumptions and symbols be as in Theorem 1.5. Then from Theorem 1.5 it follows that

$$\mathscr{L}(P, \mathfrak{M}) = \oplus M(\{S_k r_{jL}\}_{\substack{k\in I_1\\ j\in I_2}}, \mathscr{L}(F, \mathfrak{M}); I_1 \times I_2). \qquad (1.11)$$

Putting $S^*_{kj} = S_k r_{jL}$, it is easy to see that S^*_{kj} is a semilinear transformation of $\mathfrak{M} = \sum_{i\in\Gamma} F u_i$, and $\psi^*_{kj} = \psi_k I_{r_j^{-1}}$ is the associated isomorphism, where $S_k = (S_k, \psi_k)$, $I_{r_j^{-1}}$ is an inner isomorphism, $r_j \in E'$.

§ 2.

We take an example to explain the preceding theory.

We assume that $\mathfrak{M} = \sum F u_i$, G is a group of automorphisms of F, $P = I(G)$, E' is the algebra of G, $P' = C_F(E')$ and $[F: P]_L = n < \infty$. Denote the class of all finite subsets $\{x_i\}_{i\in I}$ of $\mathfrak{M}$ by D, where I always denotes a finite set. Then we can define a function $[\ \ ]$ of D into $\mathfrak{M}$, i. e., $[\{x_i\}_{i\in I}] = \sum_{i\in I} x_i$. Clearly $[\ \ ]$ satisfies all conditions of Definition 1.1. Now we check it. We have $(f \sum_{i\in I} x_i)\sigma = (\sum_{i\in I} f x_i)\sigma$ for $f \in F$, $\sigma \in \mathbf{E}$, hence condition (ii) is satisfied. It is easy to see from [1] and [2] that the condition (iii) of Definition 1.1 is satisfied. Therefore the above defined function is a soluable one with finite order.

On the other hand, we can introduce a similar function to $\mathbf{E}$, the complete ring of endomorphisms of $(\mathfrak{M}, +)$. Let D^* be the class of all finite subsets of $\mathbf{E}$. Then we define a function $[\ \ ]$ of D^* into $\mathbf{E}$, i. e. $[\{\varepsilon_i\}_{i\in I}] = \sum_{i\in I} \varepsilon_i$. Clearly we have $x \sum_{i\in I} \varepsilon_i = \sum_{i\in I} x\varepsilon_i$ for any $x \in \mathfrak{M}$.

Now let E' be the algebra of G, $P' = C_F(E')$, $E'_L \mathscr{L}(F, \mathfrak{M}) = \{\sum_{j<\infty} r_{jL}\omega_j \mid r_j \in E', \omega_j \in \mathscr{L}(F, \mathfrak{M})\}$. Then $E'_L \mathscr{L}(F, \mathfrak{M})$ is finite transitive in $\mathfrak{M}$ over P', i. e., let $\{x_i\}_{i\in I}$ be a finite set of P'-linearly independent elements, $\{y_i\}_{i\in I}$ be a subset of $\mathfrak{M}$, then there exists an element $\sigma \in E'_L \mathscr{L}(F, \mathfrak{M})$ such that $x_i\sigma = y_i$ for $i \in I$. Hence from the proof of Theorem 1.1 it follows that $\mathfrak{M}$ is a $([F:P']_L, 1)$-type space with the above finite soluable function[ ]. Without going into the matter in detail it is clear that Theorem 1.1 coincides with the Theorem 1 in [1]. In this case we have

$$\oplus M(\{r_{jL}\}_{j\in I}, \mathscr{L}(F, \mathfrak{M}); I) = \sum_{j\in I} \oplus r_{jL}\mathscr{L}(F, \mathfrak{M}),$$

Card. $I = [F:P']_L$.

In the same way we write $\Theta\mathscr{L}(P', \mathfrak{M}) = \sum_{S_j\in\Theta} S_j\mathscr{L}(P', \mathfrak{M})$. It is clear that $\Theta\mathscr{L}(P', \mathfrak{M})$ is finite transitive in $\mathfrak{M}$ over P. Owing to the soluable function [ ] with finite order mentioned above, it is easy to see that $\mathfrak{M}$ is a $([P': P]_L, 2)$-type space over P. In this case Theorem 1.1 coincides again with the Theorem 1 in [1].

Let $\Theta E'_L\mathscr{L}(F, \mathfrak{M}) = \sum S_k r_{jL}\mathscr{L}(F, \mathfrak{M})$, then it is clear that $\Theta E'_L\mathscr{L}(F, \mathfrak{M})$ is finite transiive in $\mathfrak{M}$ over P. Similarly we know that $\mathfrak{M}$ is a $([F: P]_L, 3)$-type space over P. Theorem 1.1 coincides with the Theorem 1 in [1] once more.

§ 3.

In this section, we are going to investigate infinite Galois theory of division ring F.

Let $\tilde{\mathbf{E}}$ denote the set of all endomorphisms of the additive group $(F, +)$. Let D_ν be the class of all subsets $\{f_i\}_I$ of F whose cardinal number $\leqslant \aleph_\nu$ and $\tilde{D}_\nu$ be the class of all subsets $\{\tilde{s}_i\}_I$ of $\mathbf{E}$ whose cardinal number $\leqslant \aleph_\nu$. As above, let [ ] denote a function from D_ν to F and at the same time let [ ] also denote a function from $\tilde{D}_\nu$ to $\mathbf{E}$ such that $a[\{\tilde{s}_i\}_I] = [\{a\tilde{s}_i\}_I]$ holds for all $a\in F$. Write $F_R = \{f_R \mid f\in F,\ af_R = af,\ a\in F\}$. It is clear that $F_R \subseteq \tilde{\mathbf{E}}$ and F is a ring isomorphic to F_R. If $\{f_{iR}\}_I$ is any subset of F_R and Card. $I \leqslant \aleph_\nu$, then $\{f_{iR}\}\in\tilde{D}_\nu$ and $[\{f_{iR}\}_I]\in\mathbf{E}$. For the convenience of future, we demand that $[\{f_{iR}\}_I]\in F_R$. And we can easily prove that $[\{f_{iR}\}_I] = [\{g_{iR}\}_I]$ if and only if $[\{f_i\}_I] = [\{g_i\}_I]$.

Let $[\{a_jv_j\}_I, [\{b_jv_j\}_I]$ be two elements of $\mathfrak{M}$ and Card. $I \leqslant \aleph_\nu$, where $a_j, b_j\in F$, $v_j\in V$. If $\{v_j\}_{I_1}$ is any maximal set of F-linearly independent elements of $\{v_j\}_I$ and we write $\{v_j)_{I_2} = \{v_j\}_I - \{v_j\}_{I_1}$, then for any element $v_j\in\{v_j\}_{I_2}$ we have $v_j = \sum_{i\in I_1} g_i^{(j)} v_i$, where all but a finite number of $g_i^{(j)}$ are zero. Hence we have

$$\begin{aligned} [\{a_jv_j\}_I] &= [\{a_iv_i\}_{I_1} \cup \{a_j \sum_{i\in I_1} g_i^{(j)} v_i\}_{j\in I_2}], \\ [\{b_jv_j\}_I] &= [\{b_iv_i\}_{I_1} \cup \{b_j \sum_{i\in I_1} g_i^{(j)} v_i\}_{j\in I_2}], \end{aligned} \tag{3.1}$$

where $\cup$ represents the union of sets.

Definition 3.1. *Let $\mathfrak{M} = \sum_{i\in I} Fu_i$ be a ν-soluable space (see §1), $\tilde{\mathbf{E}}$ be the complete ring of endomorphisms of $(F, +)$, and let $\tilde{D}_\nu$ be the class of all subsets $\{\tilde{s}_i\}_{i\in I}$ with Card. $I \leqslant \aleph_\nu$. The function[ ]: $\tilde{D}_\nu \to \tilde{\mathbf{E}}$ will be called a ν-function about F if it satisfies the following conditions:*

(i) *If* $\{f_{iR}\}_{i\in I}\in \tilde{D}_v$, $f_{iR}\in F_R$, *then* $[\{f_{iR}\}_{i\in I}]\in F_R$.

(ii) $[\{a_j v_j\}_{j\in I}]=[\{b_j v_j\}_{j\in I}]$ *if and only if*

$$[a_{iR}\cup\{a_{jR}g_{iR}^{(j)}\}_{j\in I_2}]=[b_{iR}\cup\{b_{jR}g_{iR}^{(j)}\}_{j\in I_2}]$$

for all $i\in I_1$ (*see* (3.1)).

Now we consider the element $[\{\tilde{\varepsilon}_i\}_{i\in I}]$ of $\tilde{\mathbf{E}}$. It is easy to see that for any element $\tilde{\sigma}\in\tilde{\mathbf{E}}$ we have $[\{\tilde{\varepsilon}_i\}_{i\in I}]\tilde{\sigma}=[\{\tilde{\varepsilon}_i\tilde{\sigma}\}_{i\in I}]$. Let $[\{\tilde{\varepsilon}_i\}_{i\in I}]$ and $[\{\tilde{\delta}_i\}_{i\in I'}]$ be elements of $\tilde{\mathbf{E}}$, and K is a subring of F, then we say that $[\{\tilde{\varepsilon}_i\}_{i\in I}]$ is equal to $[\{\tilde{\delta}_i\}_{i\in I'}]$ over K denoted by $[\{\tilde{\varepsilon}_i\}_{i\in I}]\underset{K}{=\!=}[\{\tilde{\delta}_i\}_{i\in I'}]$ if and only if $k[\{\tilde{\varepsilon}_i\}_{i\in I}]=k[\{\tilde{\delta}_i\}_{i\in I'}]$ for all $k\in K$.

Let K be a subring of F, $\tilde{\varepsilon}$ be an element of $\tilde{\mathbf{E}}$. We say that $\tilde{\varepsilon}F_R$ is (right) F_R-cyclic module relative to K_R if there exists an automorphism σ of F_R such that $k_R\tilde{\varepsilon}=\tilde{\varepsilon}k_R^{\sigma}$ for any $k_R\in K_R$. When $K_R=F_R$, then $\tilde{\varepsilon}F_R$ is shortly said to be F_R-cyclic module.

For the sake of simplicity we use the element 1_K of $\tilde{\mathbf{E}}$ to denote the identity of K, i. e. $k1_K=k$ for all $k\in K$. It is easy to see that $1_K F_R$ is an F_R-cyclic module relative K_R.

Definition 3. 2. *Let* λ, $\lambda'\in\tilde{\mathbf{E}}$. *Then two* F_R-*cyclic modules* λF_R *and* $\lambda' F_R$ *are said to be* (K_R, F_R)-*bimodule* (*i. e. left* K_R-*module and also right* F_R-*module*) *isomorphic relative to* K *denoted by* $\lambda F_R\underset{K}{\cong}\lambda' F_R$ *if there exists an element* $\delta_R\in F_R$ *for the element* λ *such that the relation* $\sum_i k_{iR}\lambda f_{iR}\underset{K}{=\!=}\sum_i k_{iR}^*\lambda f_{iR}^*$ *holds if and only if the relation* $\sum_i k_{iR}\lambda'\delta_R f_{iR}\underset{K}{=\!=}\sum_i k_{iR}^*\lambda'\delta_R f_{iR}^*$ *holds. Conversely, if there exists an element* $\delta'_R\in F_R$ *for the element* λ' *such that the relation* $\sum_j \bar{k}_{jR}\lambda'\bar{f}_{jR}\underset{K}{=\!=}\sum_j \bar{k}_{jR}^*\lambda'\bar{f}_{jR}^*$ if and only if

$$\sum_j \bar{k}_{jR}\lambda\delta'_R\bar{f}_{jR}\underset{K}{=\!=}\sum_j \bar{k}_{jR}^*\lambda\delta'_R\bar{f}_{jR}^*.$$

Clearly we have (i) $\lambda F_R\underset{K}{\cong}\lambda F_R$ (ii) if $\lambda F_R\underset{K}{\cong}\lambda' F_R$ then $\lambda' F_R\underset{K}{\cong}\lambda F_R$, and (iii) if $\lambda F_R\underset{K}{\cong}\lambda' F_R$, $\lambda' F_R\underset{K}{\cong}\lambda'' E_R$, then $\lambda F_R\underset{K}{\cong}\lambda'' F_R$.

Now we let $\{\tilde{\varepsilon}_i\}_{i\in I}$ be a subset of $\tilde{\mathbf{E}}$, $F_R\subset\tilde{\mathbf{E}}$, and

$$M(\{\tilde{\varepsilon}_i\}_{i\in I},\ F_R;\ I)=\{[\{\tilde{\varepsilon}_i f_{iR}\}_{i\in I}]\,|\,f_{iR}\in F_R\}.$$

Then we can introduce the following definition.

Definition 3. 3. *Let* $\{\tilde{\varepsilon}_i\}_{i\in I}\in\tilde{D}_v$, $H=M(\{\tilde{\varepsilon}_i\}_{i\in I}, F_R; I)$. H *is said to be a homogeneous Galois* (K_R, F_R)-*bimodule if and only if there exists a subset* $\{\lambda_j\}_{j\in J}$ *satisfying the following conditions*:

(i) $\lambda_j F_R$ *is* F_R-*cyclic module and* $H=\sum_{j\in J}\lambda_j F_R$[3],

(ii) *For any* F_R-*module* $\lambda_j F_R$ *we have* $\lambda_j F_R\underset{K}{\cong}1_K F_R$.

Lemma 3. 1. *Let* H *be a homogeneous Galois* (K_R, F_R)-*bimodule*,

$$A=\{\varphi\in H\,|\,\varphi k_R\underset{K}{=\!=}k_R\varphi \text{ for all } k_R\in K_R\},$$

3) $\sum_{j\in J}\lambda_j F_R=\{\sum_j\lambda_j f_{jR}\,|\,f_{jR}\in F_R\}$ and $\sum_j\lambda_j f_{jR}$ express a sum of elements λ_j and f_{jR} in ring $\tilde{\mathbf{E}}$.

then $AF_R = H$.

Proof Since $H = \sum_{j\in J} \lambda_j F_R$, it is enough to show that $\lambda_j \in AF_R$. In fact, $\lambda_j F_R \underset{K}{\cong} 1_K F_R$ by the assumption, hence from the relation $k_R 1_K \underset{K}{\cong} 1_K k_R$ for any $k_R \in K_R$ it follows that $k_R \lambda_j \delta_R \underset{K}{=\!=} \lambda_j \delta_R k_R$, where δ_R is an element of F_R. Hence $\lambda_j \delta_R \in A$. Therefore $\lambda_j \in AF_R$.

Lemma 3.2. *Let H be a homogeneous Galois (K_R, F_R)-bimodule, $M_R = C_{F_R}(K_R)$. Let $\varphi_1, \cdots, \varphi_n$ be elements of A. Suppose we have a non-trivial relation $\sum_{i=1}^{n} \varphi_i f_{iR} = 0$, $f_{iR} \in F_R$, then there exist $m_{1R}, \cdots, m_{nR} \in M_R$ such that $\sum_{i=1}^{n} \varphi_i m_{iR} = 0$ is a non-trivial relation too.*

Proof Without loss of generality we may assume that $\sum_{i=1}^{n} \varphi_i f_{iR} = 0$ and the relation is a shortest one in the sense that the number of non-zero elements of $\{f_{iR}\}$ is least. If $\varphi_1 = 0$, then our lemma is clear. Hence we assume $\varphi_1 \neq 0$, $\varphi_1 + \varphi_2 f_{2R} + \cdots + \varphi_n f_{nR} = 0$ and $f_{iR} \neq 0$ for $i = 2, \cdots, n$. Then we have

$$k_R(\varphi_1 + \varphi_2 f_{2R} + \cdots + \varphi_n f_{nR}) - (\varphi_1 + \varphi_2 f_{2R} + \cdots + \varphi_n f_{nR}) k_R = 0$$

for all $k_R \in K_R$. Hence $\sum_{i=2}^{n} \varphi_i (k_R f_{iR} - f_{iR} k_R,) = 0$. Therefore $k_R f_{iR} - f_{iR} k_R = 0$ for $i = 2, \cdots, n$. This proves that $f_{iR} \in M_R$ for $i = 2, \cdots, n$.

Lemma 3.3. *Let $H = M(\{\varepsilon_i\}_{i\in I}, F_R; I)$ is a homogeneous Galois (K_R, F_R)-bimodule, $\widetilde{K} = I(A(K))$*[4]*, then H is a homogeneous Galois $(\widetilde{K}_R, F_R)$-bimodule.*

Proof By Definition 3.3 it is enough to show that $\lambda_j F_R \underset{\widetilde{K}}{\cong} 1_{\widetilde{K}} F_R$ for any $\lambda_j F_R$. We attend first to the following. Let σ be an automorphism of F. Owing to the isomorphism of F onto F_R, it is clear that σ can be regarded as an automophism of F_R because we can define f_R^σ: $f^* f_R^\sigma = f^* f^\sigma$ for $f^* \in F$. Clearly σ is an automorphism of F_R and we have $(f_R)^\sigma = (f^\sigma)_R$. This proves that $A(K) = A(K_R)$.

Since $k_R 1_K = 1_K k_R$ for $k_R \in K_R$, by assumptions it follows that $k_R \lambda \delta_R \underset{K}{=\!=} \lambda \delta_R k_R$ for $\delta \in F$, where $\lambda = \lambda_j$. Since by assumption λF_R is an F_R-cyclic module, we have $k_R \lambda \delta_R = \lambda \delta_R k_R^{\sigma I_{\delta_R^{-1}}} \underset{K}{=\!=} \lambda \delta_R k_R$. By the definition of F_R-cyclic module we know $k_R \lambda = \lambda k_R^\sigma$ for any automorphism σ of F_R. Operating the identity $1 \in K$ on two sides of above equation we get $(1\lambda)\delta k_R = (1\lambda)\delta k_R^{\sigma I_{\delta^{-1}}}$. Hence $f k_R^{\sigma I_{\delta_R^{-1}}} = f k_R$ for any $f \in F$. From this follows $k_R^{\sigma I_{\delta_R^{-1}}} = k_R$ for $k_R \in K_R$. Therefore $\sigma I_{\delta^{-1}} \in A(K)$. This proves $\tilde{k}_R^{\sigma I_{\delta_R^{-1}}} = \tilde{k}_R$, $\tilde{k}_R \in \widetilde{K}_R$, Since λF_R is F_R-cyclic module, it follows that

$$\sum_l \tilde{k}_{lR} 1_{\widetilde{K}} f_{lR} \underset{\widetilde{K}}{=\!=} \sum_l \tilde{k}_{lR} f_{lR} \underset{\widetilde{K}}{=\!=} \sum_l \tilde{k}_{lR}^{\sigma I_{\delta_R^{-1}}} f_{lR} = \sum_l \delta_R^{-1} \tilde{k}_{lR}^\sigma \delta_R f_{lR}. \tag{3.2}$$

Now we make a correspondence σ': $\sum_l \tilde{k}_{lR} 1_{\widetilde{K}} f_{lR} \to \sum_l \tilde{k}_{lR} \lambda \delta_R f_{lR}$. We want to show

4) $A(K) = \{\sigma \in G \mid k^\sigma = k$ for all $k \in K\}$, $I(A(K)) = \{f \in F \mid f^\sigma = f$ for all $\sigma \in A(K)\}$.

that if $0 \underset{\tilde{K}}{=\!=} \sum_l \tilde{k}_{lR} 1_{\tilde{K}} f_{lR}$, then $0 \underset{\tilde{K}}{=\!=} \sum_l \tilde{k}_{lR}\lambda\delta_R f_{lR}$. If $\tilde{k}^* \sum_l \tilde{k}_{lR} 1_{\tilde{K}} f_{lR} = 0$ for any element $\tilde{k}^* \in \tilde{K}$, then by (3.2) we have $0 = \sum_l \tilde{k}^* \delta^{-1} \tilde{k}^{\sigma}_{lR} \delta_R f_{lR} = \tilde{k}^* \delta^{-1} \sum_l \tilde{k}^{\sigma}_{lR} \delta_R f_{lR}$. Hence

$$\sum_l \tilde{k}^{\sigma}_{lR} \delta_R f_{lR} = 0.$$

On the other hand we have

$$\sum_l \tilde{k}^* \tilde{k}_{lR} \lambda \delta_R f_{lR} = \sum_l \tilde{k}^* \lambda \tilde{k}^{\sigma}_{lR} \delta_R f_{lR} = (\tilde{k}^* \lambda) \sum_l \tilde{k}^{\sigma}_{lR} \delta_R f_{lR} = 0.$$

It follows that $\sum_l \tilde{k}_{lR} \lambda \delta_R f_{lR} \underset{\tilde{K}}{=\!=} 0$. Conversely, if $0 \underset{\tilde{K}}{=\!=} \sum_l \tilde{k}_{lR} \lambda \delta_R f_{lR}$, then according to the inverse course of above proof we can obtain $0 \underset{\tilde{K}}{\cong} \sum_l \tilde{k}_{lR} 1_{\tilde{K}} f_{lR}$. Therefore under the above correspondence σ': $\sum_l \tilde{k}_{lR} 1_{\tilde{K}} f_{lR} \to \sum_l \tilde{k}_{lR} \lambda \delta_R f_{lR}$ the $(\tilde{K}, F_R)$-bimodules λF_R and $1_K F_R$ are $(\tilde{K}_R, F_R)$-bimodule isomorphic relative to $\tilde{K}$. Therefore we complete our proof.

Having above preparations we can turn back to the Galois theory of division ring F.

Let K be a division subring of F, G a group of automorphisms of F and $K \supset P = I(G)$. From Theorem 1.5 it follows that if $\mathfrak{M}$ is a $(\aleph_\nu, 3)$-type space over P, then $\mathscr{L}(P, \mathfrak{M}) = \oplus M(\{S_j\}_{j\in I}, \mathscr{L}(F, \mathfrak{M}); I)$, where S_j are F-semi-linear isomorphisms for $j \in I$ and Card. $I = \aleph_\nu$. Hence $\sigma = [\{S_j\omega_j\}_{j\in \bar{I}}]$ for any element $\sigma \in \mathscr{L}(P, \mathfrak{M})$, where $\omega_j \in \mathscr{L}(F, \mathfrak{M})$. In order to indicate that σ is relative to the set $\bar{I}$, we write specifically $\sigma = \sigma_{\bar{I}}$. Let I' be a subset of I, $\sigma_{I'} = [\{S_j\omega_j\}_{j\in I'}]$, we call $\sigma_{I'}$ an associative element of σ. If $\sigma_{I'} = [\{S_j\omega_j\}_{j\in I'}]$ is an arbitrary associative element of σ, then we can form a set $H(\sigma_{I'}) = M(\{\psi_j\}_{j\in I'}, F_R; I') = \{[\{\psi_j f_{jR}\}_{j\in I'}] \mid f_{jR} \in F_R\}$, where $S_j = (S_j, \psi_j)$, $j \in I'$.

We are now ready to give the following definition.

Definition 3. 4. *Let K be a division subring of F and $K \supset P$, $\mathscr{L}(K, \mathfrak{M})$ be the complete ring of K-linear transformations of $\mathfrak{M}$, $\mathscr{L}(P, \mathfrak{M}) = \oplus M(\{S_j\}_{j\in I}, \mathscr{L}(F, \mathfrak{M}); I)$ by §1. Let $\sigma_{I'}$ be an associative element of $\sigma \in \mathscr{L}(P, \mathfrak{M})$, and let $H(\sigma_{I'}) = M(\{\psi_j\}_{j\in I'}, F_R; I')$, where ψ_j is associative isomorphism of S_j, i. e., $S_j = (S_j, \psi_j)$. We say that K is a homogenize division subring if and only if $H(\sigma_{I'})$ is a homogeneous Calois(K_R, F_R)-bimodule, where $H(\sigma_{I'})$ contains 1_K.*

Lemma 3. 4. *Let $\mathfrak{M} \underset{i\in\Gamma}{=\!=} Fu_i$ be a $(\aleph_\nu, 3)$-type space over P with ν-function about F (see Definition 3.1) and let K be a homogenize division subring of F and $K \supset P$. Write $\tilde{K} = I(A(K))$. Then $\mathscr{L}(K, \mathfrak{M}) = \mathscr{L}(\tilde{K}, \mathfrak{M})$, $K = \tilde{K}$.*

Proof Let H be a homogeneous Galois (K_R, F_R)-bimodule. As Lemma 3.1. $A = \{\varphi \in H \mid \varphi k_R \underset{K}{=\!=} k_R \varphi$ for all $k_R \in K_R\}$. Let $\tilde{A} = \{\tilde{\varphi} \in H \mid \tilde{\varphi}\tilde{k}_R \underset{\tilde{K}}{=\!=} \tilde{k}_R \tilde{\varphi}$ for all $\tilde{k}_R \in \tilde{K}_R\}$. It is clear that $\tilde{A} \subseteq A$. By Lemmas 3.1 and 3.3 $H = \tilde{A}F_R = AF_R$. Write $\tilde{M}_R = C_{F_R}(\tilde{K}_R)$. Clearly $\tilde{M}_R = M_R = C_{F_R}(K_R)$. Hence A and $\tilde{A}$ are both right vector spaces over M_R and $\widetilde{A}$ is a subspace of A. If $\varphi\in A$, then from $H=\widetilde{A}F_R\supset A$ it follows that there exist $\tilde{\varphi}_1, \cdots, \tilde{\varphi}_n\in\widetilde{A}$ such that $\varphi=\sum_{i=1}^{n}\tilde{\varphi}_i f_{iR}$, $f_{iR}\in F_R$. By Lemma 3.2 there exist $m_{1R}, \cdots, m_{nR}\in M_R$ such that $\varphi=\sum_{i=1}^{n}\tilde{\varphi}_i m_{iR}\in\widetilde{A}$. Hence $\widetilde{A}=A$.

We want to show $\mathscr{L}(K, \mathfrak{M})\subseteq\mathscr{L}(\widetilde{K}, \mathfrak{M})$. Since $\mathscr{L}(P, \mathfrak{M})=\oplus M(\{S_j\}_{j\in I}, \mathscr{L}(F, \mathfrak{M}); I)$, Card. $I=\aleph_\nu=[F: P]_L$, for any element $\sigma\in\mathscr{L}(K, \mathfrak{M})$ it has the form $\sigma=[\{S_j\omega_j\}_{j\in\bar{I}}]$, where $S_j=(S_j, \psi_j)$, $\omega_j\in\mathscr{L}(F, \mathfrak{M})$. Take an element $u\in\mathfrak{M}$ such that $u\sigma\neq 0$, then for any $k\in K$ we have

$$(ku)\sigma=[\{(ku)S_j\omega_j\}_{j\in\bar{I}}]=[\{k^{\psi_i}v_i\}_{i\in I_1}\cup\{k^{\psi_j}\sum_{i\in I_1}g_i^{(j)}v_i\}_{j\in I_2}],$$

$$k(u\sigma)=[\{k(uS_j\omega_j)\}_{j\in\bar{I}}]=[\{kv_i\}_{i\in I_1}\cup\{k\sum_{i\in I_1}g_i^{(j)}v_i\}_{j\in I_2}],$$

where $v_j=uS_j\omega_j$, $\{v_i\}_{i\in I_1}$ is the set of maximal F-linearly independent elements of $\{v_j\}_{j\in\bar{I}}$ and $\{v_j\}_{j\in I_2}=\{v_j\}_{j\in\bar{I}_2}-\{v_i\}_{i\in I_1}$. Since $\mathfrak{M}$ has ν-function about F, we obtain the following equation

$$[k^{\psi_i}\cup\{k^{\psi_j}g_i^{(j)}\}_{j\in I_2}]=[k\cup\{kg_i^{(j)}\}_{j\in I_2}],\ i\in I. \tag{3.3}$$

That is

$$[\psi_i\cup\{\psi_j g_{iR}^{(j)}\}_{j\in I_2}]\underset{K}{=\!=}[1_R\cup\{g_{iR}^{(j)}\}_{j\in I_2}]\in F_R \tag{3.4}$$

for any $i\in I_1$. If $[k\cup\{kg_i^{(j)}\}_{j\in I_2}]=0$ for all $i\in I_1$, then $u\sigma=0$, this is impossible. Hence we may assume $[1_R\cup\{g_{1R}^{(j)}\}_{j\in I_2}]=b_R^{-1}\neq 0$. Now we let $H=\{[\psi_1 f_R\cup\{\psi_j f'_R\}_{j\in\bar{I}_2}]\,|f_R, f'_R\in F_R;\ \bar{I}_2\subseteq I_2\}$. It is easy to see that H contains 1_R. It is clear that $k1_R=k=k1_K$ for any $k\in K$. Therefore H is a homogeneous Galois (K_R, F_R)-bimodule by the assumptions. From the property of homogenize subring K and Definition 3.3 it follows that ψ_1F_R and ψ_jF_R are all (K_R, F_R)-bimodule isomorphis with 1_KF_R relative to K, $j\in I_2$.

By Lemma 3.3 H is a homogeneous Galois $(\widetilde{K}_R, F_R)$-bimodule. From (3.4) it follows that

$$k[\psi_i b_R\cup\{\psi_j g_{iR}^{(j)}b_R\}_{j\in I_2}]=k\in K, \tag{3.5}$$

where $b_R^{-1}=[\{1_R\cup\{g_{1R}^{(j)}\}_{j\in I_2}]$. Since $[\psi_1 b_R\cup\{\psi_j g_{1R}^{(j)}b_R\}_{j\in I_2}]\underset{K}{=\!=}1_R$, we have

$$k_R[\psi_1 b_R\cup\{\psi_j g_{1R}^{(j)}b_R\}_{j\in I_2}]\underset{K}{=\!=}[\psi_1 b_R\cup\{\psi_j g_{1R}^{(j)}b_R\}_{j\in I_2}]k_R.$$

Therefore $[\psi_1 b_R\cup\{\psi_j g_{1R}^{(j)}b_R\}_{j\in I_2}]\in A$. By Lemma 3.2, $A=\widetilde{A}$. And we have

$$\tilde{k}_R[\psi_1 b_R\cup\{\psi_j g_{1R}^{(j)}b_R\}_{j\in I_2}]\underset{\widetilde{K}}{=\!=}[\psi_1 b_R\cup\{\psi_j g_{1R}^{(j)}b_R\}_{j\in I_2}]\tilde{k}_R,$$

for any $\tilde{k}_R\in\widetilde{K}_R$. Hence

$$\tilde{k}[\psi_1 b_R\cup\{\psi_j g_{1R}^{(j)}b_R\}_{j\in I_2}]=\tilde{k},$$

$$\tilde{k}[\psi_1\cup\{\psi_j g_{1R}^{(j)}\}_{j\in I_2}]=\tilde{k}b_R^{-1}=\tilde{k}[1_R\cup\{g_{1R}^{(j)}\}_{j\in I_2}], \tag{3.6}$$

for all $\tilde{k}\in\widetilde{K}$.

If $[1_R\cup\{g_{iR}^{(j)}\}_{j\in I_2}]=0$, by (3.4) it follows that

$$k_R[\psi_i\cup\{\psi_j g_{iR}^{(j)}\}_{j\in I_2}]\underset{K}{=\!=}0\underset{K}{=\!=}[\psi_i\cup\{\psi_j g_{iR}^{(j)}\}_{j\in I_2}]k_R.$$

Since $A=\widetilde{A}$, it follows that

$$\tilde{k}[\psi_i \cup \{\psi_j g_{iR}^{(j)}\}_{j\in I_2}] = 0 = \tilde{k}[1_R \cup \{g_{iR}^{(j)}\}_{j\in I_2}] \tag{3.7}$$

for all $\tilde{k}\in\tilde{K}$. In view of the property of ν-function [ ] of $\mathfrak{M}$ about F and (3.6), (3.7) we have

$$(\tilde{k}u)\sigma = [\{\tilde{k}^{\psi_j}v_j\}_{j\in I}] = [\{\tilde{k}^{\psi_i}v_i\}_{i\in I_1} \cup \{\tilde{k}^{\psi_j}v_j\}_{j\in I_2}] = [\{\tilde{k}v_i\}_{i\in I_1} \cup \{\tilde{k}v_j\}_{j\in I_2}] = \tilde{k}(u\sigma).$$

If $u\sigma=0$, then $v\sigma\neq 0$ for $\sigma\neq 0$. Hence $(u+v)\sigma\neq 0$. From the preceding results it follows that $(\tilde{k}(u+v))\sigma=\tilde{k}((u+v)\sigma)$, therefore $(\tilde{k}u)\sigma+(\tilde{k}v)\sigma=\tilde{k}(u\sigma)+\tilde{k}(v\sigma)$. But $v\sigma\neq 0$, it follows that $(\tilde{k}v)\sigma=\tilde{k}(v\sigma)$, and $(\tilde{k}u)\sigma=\tilde{k}(u\sigma)$. This proves $\sigma\in\mathscr{L}(\tilde{K}, \mathfrak{M})$, hence $\mathscr{L}(K, \mathfrak{M})\subseteq\mathscr{L}(\tilde{K}, \mathfrak{M})\subseteq\mathscr{L}(K, \mathfrak{M})$. This completes the proof.

Definition 3.5. *Let $\tilde{G}$ be a subgroup of G, $\tilde{K}=I(\tilde{G})$. Let E' be the algebra of G. Then $\tilde{G}$ is called a homogenize group if and only if for any element $\sigma=[\{S_j\omega_j\}_{j\in I}]$ of $\mathscr{L}(\tilde{K}, \mathfrak{M})$, where $S_j=(S_j, \psi_j)$, we have*

(i) *there exists δ_j in E' such that $\psi_j I_{\delta_j}\in\tilde{G}$,*

(ii) *if $1_{\tilde{K}}\in H(\sigma_{I'})=M(\{\psi_j\}_{I'}, F_R;\ I')$, then $H(\sigma_{I'})$ is a homogenize Galois (K_R, F_R)-bimodule, where $I'\subset I$.*

(iii) *if $S=(S, \psi)\in\mathscr{L}(\tilde{K}, \mathfrak{M})$, $S=[\{S_j\omega_j\}_{j\in I}]$, $S_j=(S_j, \psi_j)$, $\omega_j\in\mathscr{L}(F, \mathfrak{M})$ and $\psi_j F_R \underset{\tilde{K}}{\cong} 1_K F_R$ for $j\in I$, then $\psi\in\tilde{G}$.*

Lemma 3.5. *Let $\mathfrak{M}$ be a $(\aleph_\nu, 3)$-type space over P (see § 1) with ν-function about F. Let K be a homogenize subring of F, $\tilde{G}=A(K)$, then $\tilde{G}$ is a homogenize subgroup.*

Proof By Lemma 3.4 $K=\tilde{K}=I(\tilde{G})$. Let $\sigma\in\mathscr{L}(K, \mathfrak{M})$, then $\sigma=[\{S_j\omega_j\}_{j\in I}]$, where $S_j=(S_j, \psi_j)$, $\omega_j\in\mathscr{L}(F, \mathfrak{M})$. We first check (i) of Definition 3.5. In fact, we choose an arbitrary element $S_j=(S_j, \psi_j)$ and an element u such that $v_j=uS_j\omega_j\neq 0$. Let $v_i=uS_i\omega_i$ and $\{v_i\}_{i\in I_1}$ be a set of maximal F-linearly independent elements and $v_j\notin\{v_i\}_{i\in I_1}$. Write $\{v_i\}_{i\in I_2}=\{v_i\}_{i\in I}-\{v_i\}_{i\in I_1}$. Then by (3.4) there exists an element $1\in I_1$ such that

$$[\psi_1\cup\{\psi_j g_{1R}^{(j)}\}_{j\in I_2}] \underset{K}{=\!=} [1_R\cup\{g_{1R}^{(j)}\}_{j\in I_2}]\neq 0.$$

Set $H=\{[\psi_1 f_R\cup\{\psi_j f'_R\}_{j\in \bar{I}_2}]\mid f_R, f'_R\in F_R;\ \bar{I}_2\subseteq I_2\}$, then $1_K\in H$. From the property of homogenize subring of K it follows that $\psi_1 F_R$ and $\psi_j F_R$ are all (K_R, F_R)-bimodule isomorphic with $1_K F_R$ relative to K for $j\in I_2$. As the proof of Lemma 3.3 we have $\psi_j I_{\delta_{jR}} \underset{K}{=\!=} 1_K$, hence $\psi_j I_{\delta_j}\in\tilde{G}$. Since $P^{\psi_j}=p$ for any element p of P, we have $p^{I_{\delta_j}}=p$, and $\delta_j\in E'$. This proves that (i) of Definition 3.5 is satisfied. From the property of homogenize subring of K it follows that (ii) of Definition 3.5 is also satisfied. Now we check (iii) of Definition 3.5. If $S=(S, \psi)\in\mathscr{L}(K, \mathfrak{M})$, then $k^\psi=k$ for any element $k\in K$. Hence $\psi\in A(K)=\tilde{G}$. This proves $\tilde{G}$ is a homogenize subgroup.

Lemma 3.6. *Let $\tilde{G}$ be a homogenize subgroup of G, then $\tilde{K}=I(\tilde{G})$ is a homogenize subring and $A(\tilde{K})=\tilde{G}$.*

Proof From the property of homogenize subgroup and Definition 3.5. it follows that $\widetilde{K}$ is a homogenize subring. Now we need to prove the last assertion. Let $\widetilde{\widetilde{G}}=A(\widetilde{K})$; then $\widetilde{\widetilde{G}}$ is a homogenize subgroup by Lemma 3.5. Let $\widetilde{\widetilde{K}}=I(\widetilde{\widetilde{G}})$, then $\widetilde{\widetilde{K}}$ is a homogenize subring. Hence $\mathscr{L}(\widetilde{K}, \mathfrak{M})=\mathscr{L}(\widetilde{\widetilde{K}}, \mathfrak{M})$ by Lemma 3.4. Let $\psi\in\widetilde{\widetilde{G}}$, then $\psi\in\widetilde{G}$ by the property of homogenize subgroup $\widetilde{G}$ (see Definition 3.5). Therefore $\widetilde{\widetilde{G}}=\widetilde{G}$.

We are now ready to establish the following fundamental theorem of infinite Galois theory of division ring F.

Theorem 3.1. *(Fundamental theorem). Let F be a division ring, G be a group of automorphisms of F, $P=I(G)$ and $[F: P]_L=\aleph_\nu$. Let G' be an arbitrary homogenize subgroup of G, K be an arbitrary homogenize subring of F and $K\supset P$. Then the correspondences $G'\rightarrow I(G')$ and $K\rightarrow A(K)$ are inverses of each other.*

§ 4.

We will show that the theory established in § 3 implies the usual finite Galois theory of division ring F. Let $\mathbf{E}$ be the complete set of endomorphisms of $(F, +)$, $\{f_i\}_{i\in I}$ be a finite subset of F, then the function [ ] is defined as follows: $[\{f_i\}_{i\in I}]=\sum_{i\in I} f_i$, where $\sum$ expresses the finite sum of F. Clearly $[\{f_i\}_{i\in I}]\in F$. Let $\mathfrak{M}=\sum Fu_i$ and $\{v_i\}_{i\in I}$ be a finite subset of $\mathfrak{M}$. Let $\{v_i\}_{i\in I_1}$ be a set of maximal F-linearly independent elements of $\{v_i\}_{i\in I}$, $\{v_i\}_{i\in I_2}=\{v_i\}_I-\{v_i\}_{I_1}$, then any element v_j of $\{v_j\}_{I_2}$ can be expressed as $v_j=\sum_{i\in I_1} g_i^{(j)}v_i$. It is clear that $\sum_{j\in I} a_jv_j=\sum_{j\in I} b_jv_j$ holds if and only if $b_i+\sum_{j\in I_2} b_jg_i^{(j)}=a_i+\sum_{j\in I_2} a_jg_i^{(j)}$. This proves that the above defined function [ ] satisfies all conditions of Definition 3.1. Therefore $\mathfrak{M}$ has a 0-function (i. e. $\nu=0$) about F.

Theorem 4.1. *Let $\mathfrak{M}=\sum Fu_i$, G be a group of automorphisms of F, $P=I(G)$. Let $[F: P]_L<\infty$, then*

(i) *if K is a division subring of F and $K\supset P$, then K is a homogenize subring (see § 3),*

(ii) *$\widetilde{G}$ is a homogenize subgroup of G if and only if $\widetilde{G}$ is an N-subgroup*[5].

Proof From [1] it follows that $\mathscr{L}(P, \mathfrak{M})=\sum_{j\in I}\oplus S_j\mathscr{L}(F, \mathfrak{M})$, where $I=\{1, \cdots, n\}$ and $\mathscr{L}(K, \mathfrak{M})=\sum_{j\in I_1}\oplus S_j\mathscr{L}(F, \mathfrak{M})$, $I_1\subset I$. Moreover, $A(K)=\widetilde{G}=\{\psi_j | S_j=(S_j, \psi_j)$ for $j\in I_1\}$. Now we want to prove that K is a homogenize subring in the meaning of definition 3.4. In fact, if $\sigma\in\mathscr{L}(K, \mathfrak{M})$, then $\sigma=\sum_{j\in I_1} S_j\omega_j$. We still denote the associative element of σ by σ_P and write $H(\sigma_{\overline{R}})=\sum_{j\in I_1'}\psi_jF_R$. It is enough to show

1) Let R be a ring with an identity, Φ its center and let G be a group of automorphisms in R. Then we call the subalgebra E' over Φ generated by the (regular) elements c such that $I_c\in G$ the algebra of G. We say that G is an N-subgroup if and only if for every regular $c\in E'$, $I_{c'}\in G$.

that $H(\sigma_{I'_1})$ is a homogeneous Galois (K_R, F_R)-bimodule. Since ψ_j is an automorphism of F, it is easy to see that $\psi_j F_R$ is an F_R-cyclic module. Hence the condition (i) of Definition 3.3 is satisfied. It remainds to show that $\psi_j F_R \underset{K}{\cong} 1_K F_R$. But it is clear because $k^{\psi_i}=k$ holds for every ψ_j, $j\in I'_1$. Hence $\psi_j \underset{K}{=\!=} 1_K$ for $j\in I'_1$. This proves $I(\sigma_{I'_1})$ is a homogeneous Galois (K_R, F_R)-bimodule. Therefore K is a homogenize division subring.

Now we want to show the assertion (i) of Theorem 4.1. Let $\widetilde{G}$ be a homogenize subgroup and E' be the algebra of $\widetilde{G}$, $\delta'\in E'$, then $\delta'_L\in \mathscr{L}(\widetilde{K}, \mathfrak{M})$, where $\widetilde{K}=I(\widetilde{G})$. By (iii) of Definition 3.5, the associative isomorphism $I_{\delta'}$ of δ'_L belongs to $\widetilde{G}$, hence $\widetilde{G}$ is an N-subgroup. Conversely, let $\widetilde{G}$ be an N-subgroup, $\widetilde{K}=I(\widetilde{G})$, then from [1] it follows that every $S_j=(S_j, \psi_j)$ in the form of the element $\sigma=\sum_{j\in I} S_j\omega_j$ of $\mathscr{L}(\widetilde{K}, \mathfrak{M})$ mplies $\psi_j\in\widetilde{G}$. Let $\widetilde{G}_0$ be a subgroup of inner automorphisms of $\widetilde{G}$, then $\psi_j I_\delta \in\widetilde{G}$ for any $I_\delta\in\widetilde{G}_0$. Hence the condition (i) of Definition 3.5 is satisfied. Let $I'\subset I$, $\sigma_{I'}=\sum_{j\in I'} S_j\omega_j$, $H(\sigma_{I'})=\sum_{j\in I'}\psi_j F_R$, then clearly we have $\psi_j F_R \underset{\widetilde{K}}{\cong} 1_{\widetilde{K}} F_R$, where $\widetilde{K}=I(\widetilde{G})$ since $\tilde{k}^{\psi_j}=\tilde{k}$ for $\psi_j\in\widetilde{G}$. This implies that the condition (ii) of Definition 3.5 is satisfied. On the other hand, if $S=(S,\psi)\in\mathscr{L}(\widetilde{K}, \mathfrak{M})$, then from the well known proprety that every N-subgroup is a Galois subgroup, it follows that $\psi\in\widetilde{G}$. This proves $\widetilde{G}$ is a homogenize subgroup. This completes the proof.

From Theorems 4.1 and 3.1 we can now obtain again the finite Galois theory for division rings.

Theorem 4. 2. *Let F be a division ring, G be a group of automorphisms of F, $P=I(G)$ and $[F: P]_L<\infty$. Let G' be any N-subgroup of G, and K be any division subring of F containing P. Then the correspondences $G'\to I(G')$ and $K\to A(K)$ are inverses of each other.*

References

[1] Xu Yonghua, A finite structure theorem between primitive rings and its application to Galois theory, *Chin. Ann. of Math.* **1**: 2 (1980), 183—197.

[2] Xu Yonghua, A theory of ring that are isomorphic to the complete rings of linear transformations (VI), *Acta Math. Sinica*, **23** (1980), 646—657.

关于除环的无限 Galois 理论

许 永 华

(复旦大学数学研究所)

摘 要

本文分二部分，第一部分把作者所建立的线性变换完全环之间的有限结构定理扩展到无限的情形. 第二部分应用此扩展了的结构定理研究除环上的无限 Galois 理论. 我们的理论包含通常除环上的有限 Galois 理论.