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ON THE REPRESENTATIONS OF THE LOCAL CURRENT ALGEBRA AND THE GROUP OF DIFFEOMORPHISMS (II)

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Dedicated to Professor Su Bu-chin on the Occasion of his 80th Birthday and his 50th Year of Educational Work

1. Let X be a k-dimensional connected C^∞-manifold. Following [5, 6], we consider the group $\mathrm{Diff}(X)$ of all C^∞-bijections φ which are identical mappings outside some compact sets K_φ. The group $\mathrm{Diff}(X)$ is a topological group if it is endowed by the usual Schwartz's topology. The unitary representations of this group is closely connected with the theory of quasi-invariant measures, statistical mechanics, the representation of the local current algebra in the quantum theory of fields, etc. (cf. [1—5]). In [6] there is some series of elementary unitary representations of the group $\mathrm{Diff}(X)$. The aim of the present paper is to find another series of elementary unitary representations by means of the tangent bundle.

In § 1, the preliminary of the tangent bundle connected with the representations is discussed. In § 2, the representations connected with finite configuations are given. In § 3, the representations connected with the infinite configuations are given. In § 4, the representations connected with the Poisson measures are discussed.

Let $C^\infty(p)$ be the family of all C^∞-functions defined on a neighbourhood of the point $p\in X$, where the different functions in $C^\infty(p)$ may have different domains of definition, and s be a fixed natural number. Now we consider the following linear functional $t(\cdot)$ on $C^\infty(p)$, which depends on the partial derivatives of $f\in C^\infty(p)$ up to the order s only, namely, if $x=(x_1, x_2, \cdots, x_k)$ is the local coordinate in the neighbourhood of the point p, with coordinate x^0 at p, then there are real numbers $t^{i_1 i_2\cdots i_k}$ such that

$$t(f)=\sum_{1\leq i_1+\cdots+i_k\leq s} t^{i_1\cdots i_k}\frac{\partial^{i_1+\cdots+i_k}}{\partial x_1^{i_1}\cdots\partial x_k^{i_k}} f(p(x))\Big|_{x=x^0} \tag{1}$$

for $f\in C^\infty(p)$. The vecter spaces of all linear functionals of the type (1) is denoted by $\mathscr{T}_s(p)$. The dimension of $\mathscr{T}_s(p)$ is

$$N_s=\sum_{1\leq k\leq s} n(n+1)\cdots(n+k)/k!.$$

Let $\mathscr{T}(p)$ be the tangent space of X at the point p, $\otimes^k\mathscr{T}(p)$ be the space of all

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contravariant vectors of order k. Obviously,

$$\mathscr{T}_s(p)=\bigoplus_{k=1}^{s}\otimes^k\mathscr{T}(p).$$

Let $B^{(s)}$ be N_s-dimensional Euclidean space, $X^{(s)}=\{(p,t)\,|\,p\in X,\ t\in\mathscr{T}_s(p)\}$ be projection

$$\pi:\ (p,t)\mapsto p,\quad (p,t)\in X^{(s)},$$

$G^{(s)}$ be the group of all non-singular linear transforms in $B^{(s)}$, and $(X^{(s)}, B^{(s)}, \pi, G)$ be the fiber bundle like the tangent bundle. We choose a basis t_m, $m=1, 2, \cdots, N$ of $\mathscr{T}_s(p)$ arbitrarily, then we choose the cooresponding dual basis f_m, $m=1, 2, \cdots, N$, such that

$$t_l(f_m)=\delta_{lm}.$$

We can choose the neighbourhood $O(p)$ of the point p sufficiently small such that $O(p)$ is diffeomorphic to a sphere in the k-dimensional Euclidean space with the local coordinate $x=(x_1, \cdots, x_k)$ and there exists the dual basis $t_l^{(q)}$, $l=1, 2, \cdots, N_s$ is $\mathscr{T}_s(q)$ for any $q\in O(p)$ such that

$$t_l^{(q)}(f_m)=\delta_{lm}.$$

If $t\in\mathscr{T}_s(q)$, there is a set of numbers η_m such that $t=\sum\eta_m t_m^{(q)}$, hence we have an open set $O_p^{(s)}=\{(q,t)\,|\,q\in O(p),\ t\in\mathscr{T}_s(q)\}$ of X, in which the local coordinate is

$$(x_1, \cdots, x_k, \eta_1, \cdots, \eta_{N_s}\},$$

where $(x_1, \cdots, x_k)$ is the local coordinate of q. Of course, $X^{(s)}$ is also a manifold.

If $\psi\in\mathrm{Diff}(X)$, $p\in X$, $t\in\mathscr{T}_s(p)$, then $(d\psi)t$ is an element in $\mathscr{T}_s(\psi(p))$ satisfying

$$((d\psi)t)(f)=t(f\circ\psi),\quad f\in C^\infty(\psi(p)),$$

and $d\psi$: $t\mapsto(d\psi)t$ is a linear transformation on $\mathscr{T}_s(p)$. Hence we can define a C^∞-diffeomorphism $\tilde{\psi}$ as follows

$$\tilde{\psi}(p,t)=(\psi(p),\ (d\psi)t),\quad (p,t)\in X^{(s)}.$$

Let $(\mathrm{Diff}(X))^\sim=\{\tilde{\psi}\,|\,\psi\in\mathrm{Diff}(X)\}$.

If $p\in X$, $D(p)=\{\psi\,|\,\psi(p)=p,\ \psi\in\mathrm{Diff}(X)\}$ and $dD(p)=\{d\psi\,|\,\psi\in D(p)\}$, then $dD(p)$ is obviously a group of linear transformations in $\mathscr{T}_s(p)$. The space $\mathscr{T}_s(p)$ can be decomposed into multually disjoint, invariant and transitive with respect to the group $dD(p)$ sets $W_j(p)$, $j=0, 1, \cdots$ with $W_0(p)=\{0\}$. For any $q\in X$, if $\psi\in\mathrm{Diff}(X)$ and $\psi(p)=q$, then $(d\psi)W_j(p)$ is invariant and transtive with respect to $dD(q)$. The set $(d\psi)W_j(p)$ is denote by $W_j(q)$. Hence the submanifold

$$X_j^{(s)}=\{(p,t)\,|\,p\in X,\ t\in W_j(p)\},\quad j=1, 2, \cdots$$

of $X^{(s)}$ is invariant and transitive with respective to $(\mathrm{Diff}(X))^\sim$ (actually the restriction of $(\mathrm{Diff}(X))^\sim$ in $X_j^{(s)}$). The trivial case is $W_0(p)=\{0\}$ and $X_0^{(s)}=X$.

For any $t_0\in W_j(p)$, the isotropic group at t_0 is $dD(p, t_0)=\{d\psi\,|\,\psi\in D(p),\ (d\psi)t_0=t_0\}$ The submanifold is diffeomorphic to the manifold $dD(p)/dD(p, t_0)$, which is a left coset. Thus every $W_j(p)$ is a C^∞-manifold, for $j\neq 0$.

Suppose that m is a smooth measure in X, Diff (X, m) is the subgroup of all diffeomorphisms ψ in Diff(X) satisfying

$$\psi m = m.$$

According to the measure m, we can choose the neighbourhood of p and the local coordinate there such that

$$dm(p(x)) = \prod_{j=1}^{k} dx_j.$$

Let $D(p, m)$ be the group of all diffeomorphisms ψ in a certain Diff$(O(p), m)$ and $dD(p, m) = \{d\psi \mid \psi \in D(p, m)\}$. In the following we only consider the case that $W_j(p)$ is also transitive with respect to $dD(p, m)$ and there is a smooth measure in it, which is aslo invariant with respective to $dD(p, m)$. In this non-trival case, the manifold is called suitable. We shall then fixed a suitable $W_j(p)$ and denote it by $T(p)$. This does exist, for example, if $T(p) = \mathscr{T}(p) - \{0\}$ and the smooth measure ν in it is

$$d\nu(t) = \prod_{j=1}^{k} dt^j,$$

where $t = \sum_{j=1}^{k} t^j \frac{\partial}{\partial x_j}$. If $T(p)$ is suitable, and $\psi \in$ Diff(X), then $T(\psi(p)) = (d\psi) T(p)$ is also suitable. The manifold

$$(X, T) = \{(p, t) \mid p \in X,\ t \in T(p)\}$$

is also denoted by $\tilde{X}$. If $\tilde{\nu}$ is any smooth measure in T, then there is a measure ξ in X which is equivalent to the product measure $m \times \tilde{\nu}$. The measure ξ is denote by $\tilde{m}$ when $\tilde{\nu}$ is ν. In the following sections we shall construct the unitary representations by means of $\tilde{X}$ and $(\mathrm{Diff}(X))^{\sim}$.

2. By the method similar to that in § 1 in [6], we construct the unitary representations of Diff$\tilde{X}$. For the convienence of the reader we shall give the details.

Let $\tilde{X}^n$ be the topological product of n-copies of $\tilde{X}$, $\xi_n = \xi \times \cdots \times \xi$, where ξ is a smooth measure in $\tilde{X}$. Let $L^2_{\xi_n}(\tilde{X}^n, W)$ be the Hilbert space of all W-valued, measurable and square integrable functions F on $\tilde{X}^n$ with norm

$$\|F\|^2 = \int \|F(q_1, q_2, \cdots, q_n)\|^2_W \, d\xi(q_1) \cdots d\xi(q_n) < +\infty,$$

where W is a Hilbert space.

Now we construct a unitary represtation U of Diff(X) in $L^2_{\xi_n}(\tilde{X}^n, W)$ as follows

$$(U(\psi)F)(q_1, \cdots, q_n) = \prod_{j=1}^{n} J_\psi^{1/2}(q_j) F(\tilde{\psi}^{-1} q_1, \cdots, \tilde{\psi}^{-1} q_n),$$

where $J_\psi(q) = d\xi(\tilde{\psi}^{-1} q)/d\xi(q)$. In particular, when $\xi = m \times \tilde{\nu}$, we have

$$J_\psi(p, t) = \frac{dm(\psi^{-1} p)}{dm(p)} \frac{d\tilde{\nu}((d\psi^{-1})t)}{d\tilde{\nu}(t)}, \quad t \in T(p).$$

If $\xi = m \times \nu$ and $\psi \in$ Diff(X, m), then

$$(U(\psi)F)(q_1, \cdots, q_n) = F(\tilde{\psi}^{-1} q_1, \cdots, \tilde{\psi}^{-1} q_n).$$

Let ρ be an irreducible unitary representation of the symmetric group of order n

in W. Let $H_{n,\rho}$ be the subspace of all functions F in $L^2_{\xi_n}(\widetilde{X}^n, W)$ satisfying

$$F(q_{\sigma(1)}, \cdots, q_{\sigma(n)}) = \rho(\sigma)^{-1}F(q_1, \cdots, q_n), \quad \sigma \in S_n.$$

The restriction of U in $H_{n,\rho}$ is denoted by $V^{\rho,T,\xi}$ or simplely by V^ρ. This is also a unitatry representation of Diff(X). In particoular, V^ρ coincides with that in [6], when $T=\{0\}$. But in general these two representations are different.

Theorem 1. *If* $\dim X>1$, *then the restriction of* $V^{\rho,T,\widetilde{m}}$ *in* Diff (X) *is irreducible.*

Proof By the similar method in the proof of the Theorem 2 of § 1 in [6], we replace the Lemma 1 there by the following lemma

Lemma 1 *If* $p_1, \cdots, p_n$ *are* n *different points,* $t_j \in T(p_j)$, $j=1, 2, \cdots, m$ *then there exist the neighbourhoods* O_j *of the points* (p_j, t_j) *in* $\widetilde{X}$, $j=1, 2, \cdots, n$, *such that*

(1) *the closure* $\bar{O}_j$ of O_j *is* C^∞*-diffeomorphic to a closed sphere,* $\bar{O}_i \cap \bar{O}_j = \phi$ *for* $i \neq j$, *and* $\widetilde{m}(O_1) = \cdots = \widetilde{m}(O_n)$.

(2) *for any permutation* $k_1, k_2, \cdots, k_n$ *of* $1, 2, \cdots, n$, *there exists a* $\psi \in$ Diff(X, m) *such that* $\widetilde{\psi}(\bar{O}_i) = \bar{O}_{k_i}$, $i=1, 2, \cdots, n$.

Proof Without lose of generality, we may suppose that X is an open sphere in the Euclidean space, and m is the Lebesgue measure. From [6], there exists $\psi_{ij} \in$ Diff (X, m) for any two points x_i, $x_j (x_i \neq x_j)$, such that (1) $\psi_{ij} D^\varepsilon_{x_i} = D^\varepsilon_{x_j}$, $\psi_{ij} D^\varepsilon_{x_j} = D^\varepsilon_{x_i}$ for a certain sufficient small positive number ε, where D^ε_x is an open sphere with center x and radius ε, (2) ψ_{ij} is an identical mapping in a certain neighbourhood of every x_k for $k \neq i$, $k \neq j$.

Now we construct a mapping $\varphi_{ij} \in$ Diff(X, m) satisfying the following condition: there exists a small positive number ε such that $\varphi_{ij} D^\varepsilon_{x_i} = D^\varepsilon_{x_i}$, $\varphi_{ij} D^\varepsilon_{x_j} = D^\varepsilon_{x_j}$ and

$$(d\varphi_{ij})(d\psi_{ij}) t_i = t_j, \quad (d\varphi_{ij})(d\psi_{ij}) t_j = t_i.$$

In fact, by the transitivity of T, there is a mapping $\varphi \in D(x_i, m)$ such that $(d\varphi) \cdot (d\psi_{ij}) t_j = t_i$. Let

$$\hat{\varphi} = \psi_{ij}^{-1} \circ \varphi^{-1} \circ \psi_{ij}^{-1}.$$

Obviously, $\hat{\varphi} \in D(x_j, m)$ and $(d\hat{\varphi})(d\psi_{ij}) t_i = t_j$. We can modify the mappings φ and $\hat{\varphi}$ in the neighbourhood of x_j and extend them suitablely so that φ_{ij} satisties the above conditions. From § 1, we know that the mapping $\varphi_{ij} \circ \psi_{ij}$ preserves the measure $\widetilde{m}$.

We take a suitable neighbourhood O_i of (x_i, t_i) such that $\pi O_i = D^\varepsilon_{x_i}$. Let $\widetilde{\varphi}_{ij} \circ \widetilde{\psi}_{ij} O_i = O_j$. Hence

$$\widetilde{\varphi}_{ij} \circ \widetilde{\psi}_{ij} O_j = \widetilde{\hat{\varphi}} \circ \widetilde{\psi}_{ij} O_j = (\widetilde{\varphi} \circ \widetilde{\psi}_{ij})^{-1} O_j = O_i.$$

Theorem 2. *If* $\dim X>1$, *and* ξ *is an arbitrary smooth measure in* $\widetilde{X}$, *then* $V^{\rho,T,\xi}$ *is an irreducible unitary represention of* Diff(X).

Proof We construct a unitary operator Ξ from $L^2_{\xi_n}(\widetilde{X}^n, W)$ onto $L^2_{\widetilde{m}_n}(\widetilde{X}^n, W)$ as follows

$$\Xi: F(q_1, \cdots, q_n) \mapsto F(q_1, \cdots, q_n) \prod_{j=1}^{n} \left(\frac{d\xi(q_j)}{d\widetilde{m}(q_j)} \right)^{1/2}.$$

Then $V^{\rho,T,\xi}=\Xi^{-1}V^{\rho,T,\tilde{m}}\Xi$. However, $V^{\rho,T,\tilde{m}}$ is an irreducible unitary representation of Diff(X) by Theorem 1. Thus $V^{\rho,T,\xi}$ is irreducible unitary representation also.

3. Let $B_{\tilde{X}}$ be the set of all finite configurations in X, $\Gamma_{\tilde{X}}$ be the set of all infinite and locally finite configurations in $\tilde{X}$, μ be a measure in $\Gamma_{\tilde{X}}$ which is quasi-invariant with respect to (diff(X))$^{\sim}$, and $L^2_\mu(\Gamma_{\tilde{X}})$ be the Hilbert space of all measurable and μ-square integrable functions on $\Gamma_{\tilde{X}}$. We construct the unitary representation U_μ defined by

$$(U_\mu(\psi)F)(\gamma)=J_\psi^{1/2}(\gamma)F(\gamma),\quad \gamma\in\Gamma_{\tilde{X}},\quad F\in L^2_\mu(\Gamma_{\tilde{X}}),$$

where $J_\psi(\gamma)=d\mu(\tilde{\psi}^{-1}\gamma)/d\mu(\gamma)$.

Theorem 3. *If μ is a quasi-invariant and ergodic measure in $\Gamma_{\tilde{X}}$ with respect to* (Diff(X))$^{\sim}$, *ρ is an irreducible unitary representation of the symmectric group S_n, then $U_\mu\otimes V^{\rho,T,\xi}$ is also an irreducible representation of* Diff(X).

Let ρ be an irreducible representation of S_n in W, $n=0, 1, 2, \cdots$. We use the similar notation $\Gamma_{\tilde{X},n}$ as in [6]. Suppose that $\tilde{\mu}$ is the Campbell measure of μ in $\Gamma_{\tilde{X},n}$. Let $L^2_{\tilde{\mu}}(\Gamma_{\tilde{X},n}, W)$ be the Hilbert space of all the W-valued measurable and square integrable functions F on $\Gamma_{\tilde{X},n}$, with

$$\|F\|^2=\int_{\Gamma_{\tilde{X}n}}\|F(c)\|^2_W\, d\tilde{\mu}(c)<+\infty.$$

We construct the unitary representation

$$(U(\psi)F)(\gamma, q_1, \cdots, q_n)=J_\psi^{1/2}(\gamma)F(\tilde{\psi}^{-1}\gamma, \tilde{\psi}^{-1}q_1, \cdots, \tilde{\psi}^{-1}q_n).$$

Let $H_{\mu,n,\rho}$ be the subspace of all functions F in $L^2_{\tilde{\mu}}(\Gamma_{\tilde{X},n}, W)$ which satisfies the condition

$$F(\gamma;\ q_{\sigma(1)}, \cdots, q_{\sigma(n)})=\rho(\sigma)^{-1}F(\gamma; q_1, \cdots, q_n),\quad \sigma\in S_n,$$

and $\tilde{U}^\rho_\mu$ be the restriction of the representation of $U(\psi)$ in the space $H_{\mu,n,\rho}$.

Theorem 4. *If μ is an ergodic and quasi-invariant measure in $\Gamma_{\tilde{X}}$ with respact to* (Diff(X))$^{\sim}$ *and ρ is an irreducible unitary representibtion of S_n, then the unitatry representation of $\tilde{U}^\rho_\mu$ is irreducible.*

The proof of these two theorems is similar to that in § 3 of [6].

4. In this section we consider the unitatry represntation constructed by the Poisson measure.

Let $\Delta_{\tilde{X}}=B_{\tilde{X}}\cup\Gamma_{\tilde{X}}$. We add the point O to the set $T(p)$ and define $\nu(\{0\})=0$. Let $\tilde{\Delta}_{\tilde{X}}$ be the set of all those configurations in $\Delta_{\tilde{X}}$ of which there are only finit points (p, t) with have non-vanishing t. Let ξ be a smooth measure on X,, $\lambda>0$ be a parameter, and μ^ξ_λ be the Poisson measure with the parameter λ cooresponding to the smooth measure ξ.

Theorem 5. *If $\xi=m\times\tilde{\nu}$ is equivalentk to a smooth measure and $\tilde{\nu}\sim\nu, \tilde{\nu}(T)<+\infty$, then the Poisson measure μ^ξ_λ is concentrated in $\tilde{\Delta}_{\tilde{X}}$ and quasi-invariant with respect to* (Diff(X))$^{\sim}$ *with the Radon-Nikodym's derivative*

$$\frac{d\mu_\lambda^\xi(\tilde{\psi}^{-1}\nu)}{d\mu_\lambda^\xi(\nu)} = \prod_{(p,t)\in\nu} \frac{dm(\tilde{\psi}^{-1}p)}{dm(p)} \frac{d\tilde{\nu}((d\psi^{-1})t)}{d\tilde{\nu}(t)}, \quad \nu\in\tilde{\Delta}_{\tilde{X}} \tag{2}$$

for $\psi\in \mathrm{Diff}(X)$.

Proof We have to consider the case of $m(X)=+\infty$ only, since in the opposite case the measure μ_λ^ξ is concentrated in the set $B_{\tilde{X}}$. We constuct a sub-manifold $Y\subset X$ with $m(Y)<+\infty$. Let $\tilde{Y}=\{(p,t)\,|\,p\in X,\ t\in T(p),\ t\neq 0\}$. By the definition of the Poisson measure, we have

$$\mu_\lambda^\xi\{\nu\in\Delta_{\tilde{X}}\,|\,|\nu\cap\tilde{Y}|=n\}=\frac{(\lambda m(Y)\tilde{\nu}(T))^n}{n!}e^{-\lambda m(Y)\tilde{\nu}(T)}.$$

Hence

$$\mu_\lambda^\xi(\tilde{\Delta}_{\tilde{X}})\geqslant\sum_{n=1}^{\infty}\mu_\lambda^\xi(\{\nu\in\Delta_{\tilde{X}}\,|\,|\gamma\cap\tilde{Y}|=n\})=(e^{\lambda m(Y)\tilde{\nu}(T)}-1)e^{-\lambda m(Y)\tilde{\nu}(T)}.$$

When the submanifold Y varies and $m(Y)\to\infty$, we have $\mu_\lambda^\xi(\tilde{\Delta}_{\tilde{X}})=1$. The other part of Theorem 5 can be proved by this equality.

Theorem 6. *If* $m(X)=+\infty$, $\tilde{\nu}(T)<+\infty$ *and* $\xi=m\times\tilde{\nu}$, *then the cooresponding Poisson measure* μ_λ^ξ *is concentrated in* $\tilde{\Gamma}_{\tilde{X}}=\tilde{\Delta}_{\tilde{X}}\cap\Gamma_{\tilde{X}}$, *quasi-invariant and ergodic with respect to the group* $(\mathrm{Diff}(X))^\sim$.

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关于局部流代数与可微分变换群的表示(II)

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摘 要

设 X 是一个 k 维的连通的 C^∞ 流形, $\mathrm{Diff}(X)$ 是 $X\to X$ 的可微分变换且在无限远附近不动的一一映照全体所成的群. 本文继[5]以后, 利用 X 上的张量丛给出一类新的既约酉表示, 这种酉表示密切地联系于拟不变测度, 特别是 Poisson 测度。