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A NOTE ON THE APPROXIMATE SOLUTION OF THE CAUCHY PROBLEM BY NUMBER-THEORETIC NETS

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Dedicated to Professor Su Bu-chin On the Occaion of his 80th Birthday and his 50th Year of Educational Work

§ 1. Introduction

We use $\boldsymbol{x}=(x_1, \cdots, x_s)$ to denote a vector with real coeffients and $\boldsymbol{m}=(m_1, \cdots, m_s)$, $\boldsymbol{l}=(l_1, \cdots, l_s)$ and $\boldsymbol{a}=(a_1, \cdots, a_s)$ the vectors with integral components. We use the notations $\bar{x}=\max(1, |x|)$, $\|\boldsymbol{m}\|=\bar{m}_1\cdots\bar{m}_s$, $(\boldsymbol{m}, \boldsymbol{x})=\sum_{i=1}^{s} m_i x_i$ the scalar product of $\boldsymbol{m}$ and $\boldsymbol{x}$ and $Q(\boldsymbol{x})$ a polynomial of $\boldsymbol{x}$. We also use $C(\xi, \cdots, \eta)$ to denote a positive constant depending on $\xi, \cdots, \eta$ only, but not always with the same value.

Consider the problem of approximate solution of the equation

$$\frac{\partial u}{\partial t}=Q\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_s}\right)u, \quad 0\leqslant t\leqslant T, \quad -\infty<x_\nu<\infty \ (1\leqslant \nu\leqslant s) \tag{1}$$

with the initial condition

$$u(0, \boldsymbol{x})=\varphi(\boldsymbol{x})=\sum C(\boldsymbol{m})e^{2\pi i(\boldsymbol{m},\boldsymbol{x})},$$

where the Fourier coeffients $C(\boldsymbol{m})$ satisfy

$$|C(\boldsymbol{m})|\leqslant C/\|\boldsymbol{m}\|^{\alpha}$$

in which $C(>0)$ and $\alpha(>1)$ are two constants.

We use p to denote prime number and $N=[p^{\frac{2\alpha}{4\alpha-1}}(\ln p)^{\frac{-(2\alpha-1)(s-1)}{4\alpha-1}}]$, where $[x]$ denotes the integral part of x. We also use the following notations:

1° $f(t, \boldsymbol{x})^T$ denotes the set of numbers $f\left(t, \frac{\boldsymbol{a}k}{p}\right)$, $1\leqslant k\leqslant p$,

2° $\Gamma f^T=\sum_{\|\boldsymbol{m}\|\leqslant N}\tilde{C}(t, \boldsymbol{m})e^{2\pi i(\boldsymbol{m},\boldsymbol{x})}$,

where

$$\tilde{C}(t, \boldsymbol{m})=\frac{1}{p}\sum_{k=1}^{p} f\left(t, \frac{\boldsymbol{a}k}{p}\right)e^{-2\pi i(\boldsymbol{a},\boldsymbol{m})k/p},$$

3° $D^T_{r_1,\cdots,r_s}f^T=\left(\frac{\partial^r}{\partial x_1^{r_1}\cdots\partial x_s^{r_s}}\Gamma f^T\right)^T$,

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where $r_1+\cdots+r_s=r$, $r_i\geqslant 0(1\leqslant i\leqslant s)$,

4° $\|f\|^2=\int_{G_s}|f|^2 d\boldsymbol{x}$,

where G_s denotes the s-dimensional unit cube $0\leqslant x_i\leqslant 1$ $(1\leqslant i\leqslant s)$.

Theorem 1. *Suppose that $Q(\boldsymbol{x})$ is a polynomial such that the solution of (1) satisfies $\|u(t, \boldsymbol{x})\|\leqslant c(s)\|\varphi(\boldsymbol{x})\|$.*) Then for any given p, there exists an $\boldsymbol{a}=\boldsymbol{a}(p)$ such that*

$$R=\|u(t, \boldsymbol{x})-\Gamma v(t, \boldsymbol{x})^T\|\leqslant Cc(\alpha, s)p^{\frac{-\alpha(2\alpha-1)}{4\alpha-1}}(\ln p)^{\frac{2\alpha^2(s-1)}{4\alpha-1}}, \tag{2}$$

Where $v(t, \boldsymbol{x})^T$ denotes the solution of the system of the ordinary differential equation

$$\frac{dv(t, \boldsymbol{x})^T}{dt}=Q(D^T_{1,0,\cdots,0}, \cdots, D^T_{0,\cdots,0,1})v(t, \boldsymbol{x})^T \tag{3}$$

with initial condition

$$v(0, \boldsymbol{x})=D^T_{0,\cdots,0}\varphi(\boldsymbol{x})^T.$$

This gives a modification of a result due to Рабенький В. С.[1] which will be obtained, if the right hand side of (2) is replaced by $Cc(\alpha, s)p^{\frac{1-\alpha}{2}}(\ln p)^{\frac{(\alpha+1)(s-1)}{2}}$.

If p and $\boldsymbol{a}(p)$ in Theorem 1 are changed by F_{n+1} and $(1, F_n)$ respectively for the case $s=2$, where $F_n=\frac{1}{\sqrt{s}}\left(\left(\frac{1+\sqrt{s}}{2}\right)^n-\left(\frac{1-\sqrt{s}}{2}\right)^n\right)$, $(n=1, 2, \cdots)$ denote the Fibonacci sequence, then the right hand side of (2) may be improved slightly by $Cc(\alpha)F_n^{\frac{-\alpha(2\alpha-1)}{4\alpha-1}}(\ln 3F_n)^{\frac{3\alpha-1}{4\alpha-1}}$.

The vectoro $\boldsymbol{a}$ is called a good lattice point modulo p by Hlawka, E. or an optimal coefficient modulo p by Коробов, Н. М. and a table of good lattice points is contained in many books for the purpose of practical use, for example the book of Hua Loo Keng and Wang Yuan[2]

§ 2. Several lemmas.

Lemma 1. *For any given p, there exists $\boldsymbol{a}$ such that any non-zero solution $\boldsymbol{l}$ of the congruence*

$$(\boldsymbol{a}, \boldsymbol{l})\equiv 0 \pmod p$$

satisfies

$$\|\boldsymbol{l}\|>c(s)p/(\ln p)^{s-1}, \tag{4}$$

and

$$\sum_{(\boldsymbol{a},\boldsymbol{l})\equiv 0(\mathrm{mod}\ p)}' \frac{1}{\|\boldsymbol{l}\|^\alpha}\leqslant c(\alpha, s)p^{-\alpha}(\ln p)^{\alpha(s-1)}, \tag{5}$$

where $\sum'$ denotes a sum with an exception $\boldsymbol{l}=\boldsymbol{0}$, (Cf, Бахвалов, Н. С. [3]).

Lemma 2. *Suppose that $\|\boldsymbol{l}\|\geqslant 3^s$ and that $1\leqslant M\leqslant\|\boldsymbol{l}\|/3^s$. Then*

$$\sum_{\|\boldsymbol{m}\|\leqslant M}\frac{1}{\|\boldsymbol{l}+\boldsymbol{m}\|^\alpha}<c(\alpha, s)M^\alpha\|\boldsymbol{l}\|^{-\alpha}$$

(Cf. Wang Yuan [4]).

*) For example, $Q(\boldsymbol{x})$ is a positive definite quadratic form.

In the following, the vector $\boldsymbol{a}$ is taken such that (4) and (5) are satisfied.

Lemma 3. *We have*

$$\Gamma(e^{2\pi i(\boldsymbol{l},\boldsymbol{x})})^T=\sum_{\substack{\|\boldsymbol{m}\|<N\\(\boldsymbol{a},\boldsymbol{l}-\boldsymbol{m})\equiv0(\mathrm{mod}\ p)}}e^{2\pi i(\boldsymbol{m},\boldsymbol{x})}.$$

In particular,

$$\Gamma(e^{2\pi i(\boldsymbol{l},\boldsymbol{x})})^T=e^{2\pi i(\boldsymbol{l},\boldsymbol{x})}$$

for $\|\boldsymbol{l}\|\leqslant N$ *and* $p>c(s)$.

Proof

$$\Gamma(e^{2\pi i(\boldsymbol{l},\boldsymbol{x})})^T=\sum_{\|\boldsymbol{m}\|\leqslant N}\frac{1}{p}\sum_{k=1}^{p}e^{2\pi i(\boldsymbol{a},\boldsymbol{l})k/p}e^{-2\pi i(\boldsymbol{a},\boldsymbol{m})k/p}e^{2\pi i(\boldsymbol{m},\boldsymbol{x})}=\sum_{\substack{\|\boldsymbol{m}\|\leqslant N\\(\boldsymbol{a},\boldsymbol{l}-\boldsymbol{m})\equiv0(\mathrm{mod}\ p)}}e^{2\pi i(\boldsymbol{m},\boldsymbol{x})}$$

The Lemma is proved.

Lemma 4. *Suppose that* $\varphi(\boldsymbol{x})=e^{2\pi i(\boldsymbol{m},\boldsymbol{x})}$, *where* $\|\boldsymbol{m}\|\leqslant N$. *Then* $R=0$.

Proof Suppose that

$$u(t,\ \boldsymbol{x})=u(t)e^{2\pi i(\boldsymbol{m},\boldsymbol{x})}$$

and

$$v(t,\ \boldsymbol{x})^T=v(t)\,(e^{2\pi i(\boldsymbol{m},\boldsymbol{x})})^T,$$

where $u(0)=v(0)=1$. Substituting into (1) and (3), we have

$$\frac{\partial u}{\partial t}=u'(t)e^{2\pi i(\boldsymbol{m},\boldsymbol{x})}=Qu=u(t)Q(2\pi i\boldsymbol{m})e^{2\pi i(\boldsymbol{m},\boldsymbol{x})}$$

and

$$v'(t)\,(e^{2\pi i(\boldsymbol{m},\boldsymbol{x})})^T=v(t)Q(2\pi i\boldsymbol{m})\,(e^{2\pi i(\boldsymbol{m},\boldsymbol{x})})^T$$

by Lemma 3. Hence

$$u'(t)=u(t)Q(2\pi i\boldsymbol{m})$$

and

$$v'(t)=v(t)Q(2\pi i\boldsymbol{m}).$$

Since $u(t)$ and $v(t)$ satisfy the same ordinary differential equation with the same initial value, therefore $u(t)\equiv v(t)$ and the Lemma follows.

Lemma 5. *Let*

$$\varphi_2(\boldsymbol{x})=\sum_{\|\boldsymbol{m}\|>N}C(\boldsymbol{m})e^{2\pi i(\boldsymbol{m},\boldsymbol{x})}.$$

Then

$$\|\Gamma\varphi_2(\boldsymbol{x})^T\|\leqslant Cc(\alpha,\ s)p^{\frac{-\alpha(2\alpha-1)}{4\alpha-1}}(\ln p)^{\frac{4\alpha^2(s-1)}{4\alpha-1}}$$

Proof It follws from Lemma 3 that

$$\|\Gamma\varphi_2(\boldsymbol{x})^T\|^2=\int_{G_s}\Big|\sum_{\|\boldsymbol{l}\|>N}C(\boldsymbol{l})\Gamma(e^{2\pi i(\boldsymbol{l},\boldsymbol{x})})^T\Big|^2d\boldsymbol{x}$$

$$=\int_{G_s}\Big|\sum_{\|\boldsymbol{m}\|\leqslant N}\sum_{\substack{\|\boldsymbol{l}\|>N\\(\boldsymbol{a},\boldsymbol{l}-\boldsymbol{m})\equiv0(\mathrm{mod}\ p)}}C(\boldsymbol{l})e^{2\pi i(\boldsymbol{m},\boldsymbol{x})}\Big|^2d\boldsymbol{x}=\sum_{\|\boldsymbol{m}\|\leqslant N}\Big(\sum_{\substack{\|\boldsymbol{l}\|>N\\(\boldsymbol{a},\boldsymbol{l}-\boldsymbol{m})\equiv0(\mathrm{mod}\ p)}}C(\boldsymbol{l})\Big)^2.$$

Let $\boldsymbol{l}-\boldsymbol{m}=\boldsymbol{n}$. Then

$$\|\Gamma\varphi_2(\boldsymbol{x})^T\|^2\leqslant C^2\sum_{\|\boldsymbol{m}\|\leqslant N}\Big(\sum_{(\boldsymbol{a},\boldsymbol{n})\equiv0(\mathrm{mod}\ p)}{}'\frac{1}{\|\boldsymbol{n}+\boldsymbol{m}\|^\alpha}\Big)^2$$

$$=C^2\sum_{\|\boldsymbol{m}\|\leqslant N}\sum_{(\boldsymbol{a},\boldsymbol{n})\equiv0(\mathrm{mod}\ p)}{}'\Big(\frac{\|\boldsymbol{n}\|}{\|\boldsymbol{m}\|\,\|\boldsymbol{n}+\boldsymbol{m}\|}\Big)^\alpha\frac{\|\boldsymbol{m}\|^\alpha}{\|\boldsymbol{n}\|^\alpha}\sum_{(\boldsymbol{a},\boldsymbol{l})\equiv0(\mathrm{mod}\ p)}{}'\frac{1}{\|\boldsymbol{l}+\boldsymbol{m}\|^\alpha}.$$

Since

$$\frac{\|\boldsymbol{n}\|}{\|\boldsymbol{m}\|\,\|\boldsymbol{n}+\boldsymbol{m}\|}\leqslant 2^s$$

and we may suppose that $p>c(s)$, by Lemmas 1 and 2 we have

$$\|\Gamma\varphi_2(\boldsymbol{x})^T\|^2\leqslant C^2c(\alpha,\ s)N^{2\alpha}p^{-2\alpha}(\ln p)^{2\alpha(s-1)}$$

$$\leqslant C^2c(\alpha,\ s)p^{\frac{-2\alpha(2\alpha-1)}{4\alpha-1}}(\ln p)^{\frac{4\alpha(s-1)}{4\alpha-1}}.$$

The Lemma is proved.

Lemma 6. *If* $v(0,\ \boldsymbol{x})^T=(\Gamma\varphi_2(\boldsymbol{x})^T)^T$, *then*

$$\|\Gamma v(t,\ \boldsymbol{x})^T\|\leqslant Cc(\alpha,\ s)p^{\frac{-\alpha(2\alpha-1)}{4\alpha-1}}(\ln p)^{\frac{2\alpha^2(s-1)}{4\alpha-1}}.$$

Proof For any given $\boldsymbol{l}$, we shall prove that the congruence

$$(\boldsymbol{a},\ \boldsymbol{l}-\boldsymbol{m})\equiv 0\ (\mathrm{mod}\ p) \tag{6}$$

has at most 1 solution $\boldsymbol{m}$ satisfying

$$\|\boldsymbol{m}\|\leqslant N.$$

In fact, if there are two different vectors $\boldsymbol{m}$ and $\boldsymbol{m}'$ satisfying (6), then $\boldsymbol{m}-\boldsymbol{m}'\neq\mathbf{0}$,

$$(\boldsymbol{a},\ \boldsymbol{m}-\boldsymbol{m}')\equiv 0(\mathrm{mod}\ p)$$

and

$$\|\boldsymbol{m}-\boldsymbol{m}'\|\leqslant 2^sN$$

which leads to a contradiction with Lemma 1. Hence by Lemma 3, we have

$$\Gamma(e^{2\pi i(\boldsymbol{l},\boldsymbol{x})})^T=0 \text{ or } e^{2\pi i(\boldsymbol{m},\boldsymbol{x})},$$

where $\|\boldsymbol{m}\|\leqslant N$. Consequently, it follows from Lemma 4 that the solution $u(t,\ \boldsymbol{x})$ of the partial differential equation

$$\begin{cases}\dfrac{\partial u}{\partial t}=Q\left(\dfrac{\partial}{\partial x_1},\ \cdots,\ \dfrac{\partial}{\partial x^s}\right)u,\\ u(0,\ \boldsymbol{x})=\Gamma\varphi_2(\boldsymbol{x})^T\end{cases}$$

and $\Gamma v(t,\ \boldsymbol{x})^T$ are identical, where $v(t,\ \boldsymbol{x})^T$ is the solution of the ordinary differential equation

$$\begin{cases}\dfrac{dv(t,\ \boldsymbol{x})^T}{dt}=Q(D^T_{1,0,\cdots,0},\ \cdots,\ D^T_{0,\cdots,0,1})v(t,\ \boldsymbol{x})^T,\\ v(0,\ \boldsymbol{x})^T=(\Gamma\varphi_2(\boldsymbol{x})^T)^T.\end{cases}$$

Hence by Lemma 5, we have

$$\|\Gamma v(t,\ \boldsymbol{x})^T\|=\|v(t,\ \boldsymbol{x})\|\leqslant c(s)\|u(0,\ \boldsymbol{x})\|\leqslant c(s)\|\Gamma\varphi_2(\boldsymbol{x})^T\|$$

$$\leqslant Cc(\alpha,\ s)p^{\frac{-\alpha(2\alpha-1)}{4\alpha-1}}(\ln p)^{\frac{2\alpha^2(s-1)}{4\alpha-1}}.$$

The Lemma is proved.

§ 3. The proof of Theorem 1.

Let

$$\varphi(\boldsymbol{x})=\varphi_1(\boldsymbol{x})+\varphi_2(\boldsymbol{x}),$$

where

$$\varphi_1(\boldsymbol{x})=\sum_{\|\boldsymbol{m}\|\leqslant N}C(\boldsymbol{m})e^{2\pi i(\boldsymbol{m},\boldsymbol{x})}$$

and

$$\varphi_2(\boldsymbol{x}) = \sum_{\|\boldsymbol{m}\|>N} C(\boldsymbol{m}) e^{2\pi i(\boldsymbol{m},\boldsymbol{x})}.$$

Let $u_1(t, \boldsymbol{x})$ and $u_2(t, \boldsymbol{x})$ denote the solutions of the equation (1) with the initial conditions $u_1(0, \boldsymbol{x}) = \varphi_1(\boldsymbol{x})$ and $u_2(0, \boldsymbol{x}) = \varphi_2(\boldsymbol{x})$ respectively. Further let $v_1(t, \boldsymbol{x})^T$ and $v_2(t, \boldsymbol{x})^T$ be the solutions of the equation (3) with the initial conditions $v_1(0, \boldsymbol{x})^T = D^T_{0,\cdots,0}\varphi_1(\boldsymbol{x})^T$ and $v_2(0, \boldsymbol{x})^T = D^T_{0,\cdots,0}\varphi_2(\boldsymbol{x})^T$ respectively. Then

$$u(t, \boldsymbol{x}) = u_1(t, \boldsymbol{x}) + u_2(t, \boldsymbol{x})$$

and

$$v_1(t, \boldsymbol{x})^T = v_1(t, \boldsymbol{x})^T + v_2(t, \boldsymbol{x})^T.$$

It follows that

$$\|u_1(t, \boldsymbol{x}) - \Gamma v_1(t, \boldsymbol{x})^T\| = 0$$

by Lemma 4 and that

$$\|\Gamma v_2(t, \boldsymbol{x})^T\| \leqslant Cc(\alpha, s) p^{\frac{-\alpha(2\alpha-1)}{4\alpha-1}} (\ln p)^{\frac{2\alpha^2(s-1)}{4\alpha-1}}$$

by Lemma 5. Since

$$\|u_2(t, \boldsymbol{x})\| \leqslant \|u_2(0, \boldsymbol{x})\| = \left(\int_{G_s} |\varphi_2(\boldsymbol{x})|^2 d\boldsymbol{x}\right)^{1/2} \leqslant C\left(\sum_{\|\boldsymbol{m}\|>N} \frac{1}{\|\boldsymbol{m}\|^{2\alpha}}\right)^{1/2}$$
$$\leqslant Cc(\alpha, s) N^{-\frac{2\alpha-1}{2}} (\ln p)^{\frac{s-1}{2}} \leqslant Cc(\alpha, s) p^{-\frac{\alpha(2\alpha-1)}{4\alpha-1}} (\ln p)^{\frac{2\alpha^2(s-1)}{4\alpha-1}},$$

we have

$$\|u(t, \boldsymbol{x}) - \Gamma v(t, \boldsymbol{x}\|^T\|^2 = \|u_1(t, \boldsymbol{x}) + u_2(t, \boldsymbol{x}) - \Gamma v_1(t, \boldsymbol{x})^T - \Gamma v_2(t, \boldsymbol{x})^T\|^2$$
$$\leqslant 3(\|u_1(t, \boldsymbol{x}) - \Gamma v_1(t, \boldsymbol{x})^T\| + \|u_2(t, \boldsymbol{x})\|^2 + \|\Gamma v_2(t, \boldsymbol{x})^T\|^2)$$
$$\leqslant C^2 c(\alpha, s) p^{\frac{-2\alpha(2\alpha-1)}{4\alpha-1}} (\ln p)^{\frac{4\alpha^2(s-1)}{4\alpha-1}}.$$

The Theorem Ts proved.

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关于用数论网格求柯西问题渐近解的一个注记

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摘　要

Рябенький, В. С. 曾提出用数论网格构造的常微分方程组的解来构造偏微分方程

$$\begin{cases} \dfrac{\partial u}{\partial t}=Q\left(\dfrac{\partial}{\partial x_1}, \cdots, \dfrac{\partial}{\partial x_s}\right)u, \ 0\leqslant t\leqslant T, \ -\infty<x_\nu<\infty\,(1\leqslant \nu\leqslant s), \\ u(0, \boldsymbol{x})=\varphi(\boldsymbol{x}) \end{cases}$$

的近似解 $u^*(t, \boldsymbol{x})$ 的方法，此处 $Q(\boldsymbol{x})$ 为多项式. 当 $Q(\boldsymbol{x})$ 与 $\varphi(\boldsymbol{x})$ 适合某些条件时，他并给出了用 $u^*(t, \boldsymbol{x})$ 逼近 $u(t, \boldsymbol{x})$ 的误差估计.

本文改进 Рябенький 的结果，即将误差估计中的误差主阶 $p^{-\frac{\alpha-1}{2}}$ 改进为 $p^{-\frac{\alpha(2\alpha-1)}{4\alpha-1}}$，此处 p 为所用常微分方程的个数，而 α 为刻划 $\varphi(\boldsymbol{x})$ 光滑度的大于 1 的常数.