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QUALITATIVE THEORY OF THE QUADRATIC SYSTEMS IN THE COMPLEX SPACE

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Dedicated to Professor Su Bu-chin on the Occasion of his 80th Birthday and his 50th Year of Educational Work

In the past 25 years many works have been done by Chinese and Soviet mathematicians on the qualitative theory of the real quadratic differential equation

$$\frac{dy}{dx}=\frac{Q_2(x, y)}{P_2(x, y)}, \tag{1}$$

where x and y are real variables, P_2 and Q_2 are polynomials of degree $\leqslant 2$ with real coefficients. (See [1, 2]). Meanwhile, in the Soviet school, researches about global and local aspects of the general polynomial system

$$\dot{Z}=P(Z) \tag{2}$$

in complex variables have also been carried out gradually (see [3]), where $P(Z)$ is polynomial in $Z=(Z_1, \cdots, Z_n)$ of degree $\leqslant N$, and $Z\in C^n$. Up to now, one can see very little affinity between these two areas, although they have obviously a common objective: to solve the second part of the Hilbert's 16th problem.

As an intermediate work, the purpose of this paper is to carry out a rudimentary investigation of the qualitative property of solutions or integral surfaces of equation (1), where, instead, we will assume throughout this paper that x, y are complex variables, while P_2 and Q_2 still remain to be polynomials of degree $\leqslant 2$ with real coefficients.

§ 1. Some properties of solutions of a complex differential equation with real coefficients

Consider the equation

$$\frac{dy}{dx}=\frac{Q(x, y)}{P(x, y)}, \tag{3}$$

where P and Q are power series of the complex variables x and y with real coefficients. Let $x=x_1+ix_2$, $y=y_1+iy_2$, then on separating real and imaginary parts, any integral $f(x, y)=0$ of (3) can be thought as a 2-dimensional surface

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$$S: \ f_1(x_1, x_2, y_1, y_2)=0, \quad f_2(x_1, x_2, y_1, y_2)=0 \tag{4}$$

in the 4-dimensional (x_1, x_2, y_1, y_2) real space.

Let

$$F(x, y, C)=0 \tag{5}$$

be the general integral of (3), then, generally speaking, (4) can be obtained from (5) by giving C a definite value, real or complex. Furthermore, a solution of (3) may be a multiple-valued function, which can have even infinitely many branches. Hence in the sequel we must fix a single-valued branch of this solution, otherwise, even the uniqueness of solutions of a boundary value problem will not be ensured.

Example 1. Consider a very simple special case of (3)

$$\frac{dy}{dx}=-\mu\frac{y}{x}, \tag{6}$$

where $\mu>0$ is real and irrational. The general solution of (6) is

$$x^\mu y=C. \tag{7}$$

Let $x=re^{i\theta}=x_1+ix_2$, $y=y_1+iy_2$, $C=C_1+iC_2$, then one can get from (7) a family of integral surfaces

$$\begin{cases} r^\mu(y_1\cos\mu(\theta+2k\pi)-y_2\sin\mu(\theta+2k\pi))=C_1, \\ r^\mu(y_1\sin\mu(\theta+2k\pi)+y_2\cos\mu(\theta+2k\pi))=C_2 \end{cases} \tag{8}$$

$$(k=0, \ \pm1, \ \pm2, \ \cdots),$$

for any fixed value of C.

In order to find the intersection of (8) with the real plane $x_2=y_2=0$, one needs only to put in (8) $x_2=y_2=\theta=0$, $r=x_1$. Now, there are two different cases.

Case A. $k=0$. Then we get from (8)

$$x_1^\mu y_1=C_1, \quad 0=C_2,$$

Which means that the intersection curve of (8) for $C=C_1$ and $k=0$ with the real plane is just the integral curve of (6) which corresponds to the same constant C_1, if (6) is considered to have real variables. Moreover, we see that if $C_2\neq0$, then (8) for $k=0$ does not intersect the real plane.

Case B. $k\neq0$. Here the intersection curve must satisfy both the two equations

$$x_1^\mu y_1\cos 2k\pi\mu=C_1, \quad x_1^\mu y_1\sin 2k\pi\mu=C_2. \tag{9}$$

For $C_2\neq0$, (9) yields

$$C_1=C_2\cot 2k\pi\mu, \tag{10}$$

from which C_1 is determined by k, μ and C_2. The intersection curve (9) is then

$$x_1^\mu y_1=\frac{C_1}{\cos 2k\pi\mu}=C_1'=|C|. \tag{11}$$

That is to say, for $C_2\neq0$, $k\neq0$, only when C_1 satisfies (10), the integral surface (8) will intersect the real plane at a curve (11), but here $C_1'\neq C_1$, different from that of Case A.

It is easily seen that for $C_2=0$ and $k\neq 0$, (8) will intersect the real plane only when $C_1=0$, and thus the intersection will always be $x_1=0$ and $y_1=0$ for all $k\neq 0$.

To sum up, we see that for a fixed real number $C_1'\neq 0$, one can determine infinitely many pairs of C_1 and C_2, say, (C_1^k, C_2^k) by (11) and (10), corresponding to $k=\pm 1, \pm 2, \cdots$, such that all the surfaces (8) with $C=C_1^k+iC_2^k$ will intersect the real plane at the same curve (11). In other words, if we take

$$x_1^\mu y_1=C_1' \quad \text{when} \quad x_2=y_2=0$$

as a boundary condition for the equation (6), then the solution is not unique. Since in this paper we are only interested in the intersection curve of integral surfaces of a complex equation (3) with the real plane, we will agree here after to take only the fundamental single-valued branch of any multi-valued solution.

It is interesting to note if we put $\mu=\sigma+i\tau$ in (6), such that σ, τ and σ/τ are all positive irrational, then parallel to (8) and (9) we will have now

$$\begin{cases} C_1=r^\sigma e^{-\tau(\theta+2k\pi)}[y_1\cos(\tau\ln r+\sigma(\theta+2k\pi))-y_2\sin(\tau\ln r+\sigma(\theta+2k\pi))], \\ C_2=r^\sigma e^{-\tau(\theta+2k\pi)}[y_1\sin(\tau\ln r+\sigma(\theta+2k\pi))+y_2\cos(\tau\ln r+\sigma(\theta+2k\pi))] \end{cases} \tag{$\overline{8}$}$$

and (for $x_2=y_2=0$, $\theta=0$, $\gamma=x_1$)

$$\begin{cases} C_1=x_1^\sigma y_1 e^{-2k\pi\sigma}\cos(\tau\ln x_1+2k\pi\sigma), \\ C_2=x_1^\sigma y_1 e^{-2k\pi\sigma}\sin(\tau\ln x_1+2k\pi\sigma). \end{cases} \tag{$\overline{9}$}$$

Thus for any fixed $C=C_1+iC_2$, ($\overline{9}$) implies

$$x_1^\sigma y_1=|C|e^{2k\pi\tau}, \quad |C|=\sqrt{C_1^2+C_2^2}. \tag{$\overline{11}$}$$

For $k=0, \pm 1, \pm 2, \cdots$ the family of intersection curves ($\overline{11}$) will be dense in the real plane, which reveals the density property of equation (6).

It is easy to prove the following fundamental.

Theorem 1. *If an integral surface S of* (3) *intersects the real plane at a curve l (it may consist of more than one connected components), then l is an integral curve of the corresponding real equation*

$$\frac{dy_1}{dx_1}=\frac{Q(x_1, y_1)}{P(x_1, y_1)}. \tag{12}$$

Proof Rewrite (3) in the form of a dynamic system (t real)

$$\frac{dx}{dt}=P(x, y), \quad \frac{dy}{dt}=Q(x, y). \tag{13}$$

Separating real and imaginary parts gives

$$\begin{aligned} &\frac{dx_1}{dt}=P_r(x_1, x_2, y_1, y_2), \quad \frac{dx_2}{dt}=P_j(x_1, x_2, y_1, y_2), \\ &\frac{dy_1}{dt}=Q_r(x_1, x_2, y_1, y_2), \quad \frac{dy_2}{dt}=Q_j(x_1, x_2, y_1, y_2). \end{aligned} \tag{14}$$

Notice that for any positive integer n, we have

$$(x_1+ix_2)^n=[x_1^n+x_2(\cdots)]+ix_2[\cdots], \quad \text{etc.},$$

(14) gives when $x_2=y_2=0$

$$\frac{dx_1}{dt}=P_r(x_1,\ 0,\ y_1,\ 0)=P(x_1,\ y_1),\qquad \frac{dx_2}{dt}=P_j(x_1,\ 0,\ y_1,\ 0)\equiv 0,$$

$$\frac{dy_1}{dt}=Q_r(x_1,\ 0,\ y_1,\ 0)=Q(x_1,\ y_1),\qquad \frac{dy_2}{dt}=Q_j(x_1,\ 0,\ y_1,\ 0)\equiv 0,$$

which means that any trajectory of (14), starting from a point of l will always remain in the real plane, travelling the whole or a connected component of l, and hence is also a trajectory of the dynamical system obtained from (12).

Remark. If we put in (14) $x_1=y_1=0$ (or $x_1=y_2=0$, or $x_2=y_1=0$), in general we will not get $\frac{dx_1}{dt}=\frac{dy_1}{dt}\equiv 0$ $\left(\text{or } \frac{dx_1}{dt}=\frac{dy_2}{dt}\equiv 0,\ \text{or } \frac{dx_2}{dt}=\frac{dy_1}{dt}\equiv 0\right)$. So, even if the plane $x_1=y_1=0$, $x_1=y_2=0$ or $x_2=y_1=0$ intersects S at a curve l', it need not be a trajectory of (14).

Theorem 1 implies that, in order to investigate the property of intersection curves of the general solution surfaces $F(x,\ y,\ C)=0$ of (3) with the real plane, it is sufficient to consider C to be a real constant only. Because at this time $F(x_1,\ y_1,\ C)=0$ will represent a general solution of (12), and the locus of this family of curves will already fill up the whole real plane.

§ 2. Properties of Solutions of a Complex Quadratic System with Real Coefficients

For the complex quadratic differential system with real coefficients corresponding to equation (1), i. e. (with t real)

$$\frac{dx}{dt}=P_2(x,\ y),\quad \frac{dy}{dt}=Q_2(x,\ y),\tag{1*}$$

the definition of a critical point is just the same as that for a real quadratic system. But now a critical point may have complex coordinates. Hereafter we will call it a real (complex) critical point, if it lies (does not lie) on the real plane $x_2=y_2=0$.

We can find in [4] many interesting properties of the trajectories of a real quadratic system, and now we are going to do the same investigation with regard to (1*).

The general equations of a 2-dimensional plane in the real 4-dimensional space are

$$\alpha_1x_1+\alpha_2x_2+\alpha_3y_1+\alpha_4y_2+\alpha_5=0,\quad \beta_1x_1+\beta_2x_2+\beta_3y_1+\beta_4y_2+\beta_5=0,\tag{15}$$

Of course, (15) can not always be written as an equation of a complex line

$$ax+by+c=0,\tag{16}$$

since (16) is equivalent to

$$a_1x_1-a_2x_2+b_1y_1-b_2y_2+c_1=0,\quad a_1x_2+a_2x_1+b_1y_2+b_2y_1+c_2=0,\tag{17}$$

where $a=a_1+ia_2$, $b=b_1+ib_2$, $c=c_1+ic_2$. Hence we call (15) a general (2-dim.) plane, and (16) a special (2-dim.) plane.

The proof of the following theorem is just the same as that of the corresponding theorem for a real quadratic system.

Theorem 2. *The special plane* (16) *has at most 2 tangent points*[1] *with integral surfaces of* (1*), *otherwise,* (16) *is itself a solution of* (1).

Theorem 3. *If* (1) *has no special plane solution, then any three critical points of* (1*) *can not be collinear.*

Proof Suppose on the contrary, there are 3 critical points P_1, P_2 and P_3 lying on a straight line L. Obriously, we can find a special plane (16), passing through L. Since any P_i can be taken as a tangent point of (16) with some integral surface of (1), Theorem 2 implies that (16) is a solution of (1), contrary to the hypothesis.

Remark. If (1) has special plane solution, then (1*) may even have 4 collinear critical points. For example, if $P_2(x, y)=x^2-y^2$, $Q(x, y)=xy$, then (1) has a solution $y=0$, while the origin is a critical point of multiplicity 4 of (1*).

The following theorem is a generalization of the well-known property; "Any closed orbit of a real quadratic differential system is a convex oval" in [4]:

Theorem 4. *If a plane Π_1 intersects an integral surface S of* (1) *at a closed orbit l of* (1*), *then l must be convex.*

Proof Let (15) be the equations of Π_1. Introduce the following coordinate transformation with real coefficients

$$\begin{aligned} x_1' &= \alpha_2 x_1 - \alpha_1 x_2 + \alpha_4 y_1 - \alpha_3 y_2, \\ x_2' &= \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 y_1 + \alpha_4 y_2 + \alpha_5, \\ y_1' &= \beta_2 x_1 - \beta_1 x_2 + \beta_4 y_1 - \beta_3 y_2, \\ y_2' &= \beta_1 x_1 + \beta_2 x_2 + \beta_3 y_1 + \beta_4 y_2 + \beta_5. \end{aligned} \tag{18}$$

(18) can also be written as

$$x' = ax + by + c, \quad y' = \lambda x + \mu y + \nu, \tag{19}$$

where $x'=x_1'+ix_2'$, $y'=y_1'+iy_2'$, $x=x_1+ix_2$, $y=y_1+iy_2$, $a=\alpha_2+i\alpha_1$, $b=\alpha_4+i\alpha_3$, $c=i\alpha_5$, $\lambda=\beta_2+i\beta_1$, $\mu=\beta_4+i\beta_3$, $\nu=i\beta_5$. Suppose in the new coordinates (1*) becomes

$$\frac{dx_1'}{dt}=R_1(x_1', y_1', x_2', y_2'), \quad \frac{dy_1'}{dt}=R_2(x_1', y_1', x_2', y_2'),$$
$$\frac{dx_2'}{dt}=R_3(x_1', y_1', x_2', y_2'), \quad \frac{dy_2'}{dt}=R_4(x_1', y_1', x_2', y_2').$$

Now Π_1 has equation $x_2'=y_2'=0$, i. e., it is the new real plane, and l is a closed orbit of the new real quadratic system

$$\frac{dx_1'}{dt}=R_1(x_1', y_1', 0, 0), \quad \frac{dy_1'}{dt}=R_2(x_1', y_1', 0, 0).$$

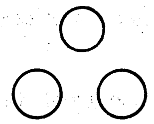
Hence from [4], l must be convex.

1) i. e., a point at which $\dfrac{Q_2(x, y)}{P_2(x, y)}=-\dfrac{a}{b}$, and hence any critical point on (16) can be considered as a tangent point of this plane.

Remark. If the closed curve l in Theorem 4 is not an orbit of (1*), then we can not prove its convexity.

Similar to [4], we have also the following two theorems, the proof is omitted.

Theorem 5. *No (2-dim.) plane can intersect any integral surface of* (1) *at three disjoint closed orbits of* (1*) *with relative position*



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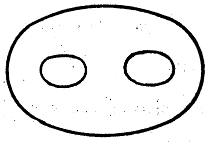

Theorem 6. *If an integral surface S of* (1) *intersects the real plane at a closed curve (it must be an orbit of* (1*)), *then S necessarily has common points with the plane*

$$D: \quad \frac{\partial}{\partial x} P_2 + \frac{\partial}{\partial y} Q_2 = 0. \tag{20}$$

Remark 1. The common part of S and D need not be a 1-dimensional curve.

Example 2. The equation (a, b, c real)

$$\frac{dy}{dx} = \frac{x(ax+by+c)}{-y(ax+by+c)+1-x^2-y^2}$$

has $x^2+y^2=1$ as one of its integral surface, which intersects the real plane $x_2=y_2=0$ at a circle $x_1^2+y_1^2=1$. It can easily be proved that now

$$\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} = (b-2)x - ay = 0$$

has only two points in common with $x^2+y^2=1$.

Remark 2. Suppose the condition of Theorem 4 is satisfied, since (20) is invariant under the coordinate transformation (19), D will still have common points with S. However, if the (closed) intersection curve of S and Π_1 is not an orbit of (1*), then S may have no point in common with the plane D.

Example 3. The equation

$$\frac{dy}{dx} = \frac{x+(x^2-y^2-1)}{y+(x^2-y^2-1)} \tag{21}$$

has an integral surface

$$S: \quad x^2-y^2=1 \quad \text{or} \quad x_1^2-y_1^2-x_2^2+y_2^2=1, \quad x_1x_2-y_1y_2=0,$$

it intersects the plane $x_2=y_1=0$ at the circle $x_1^2+y_2^2=1$, but the plane

$$\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} = 2(x-y) = 0$$

has no common point with S. It is easily seen that the circle $x_2=y_1=0$, $x_1^2+y_2^2=1$ is not an orbit of the corresponding dynamical system.

§ 3. Variation of periodic orbits with respect to the parameter.

In this section we will give three examples in order to show that the investigation of a polynomial system with real coefficients and complex variables x and y will make clearer the property of periodic orbits of the corresponding real system.

Example 4. The system

$$\frac{dx}{dt}=-y+x(x^2+y^2-\lambda),\quad \frac{dy}{dt}=x+y(x^2+y^2-\lambda) \tag{22}$$

has an integral surface

$$x^2+y^2=\lambda, \tag{23}$$

which in the (x_1, x_2, y_1, y_2) space can be written as

$$x_1^2+y_1^2-x_2^2-y_2^2=\lambda,\quad x_1x_2+y_1y_2=0. \tag{24}$$

When $\lambda>0$ (23) intersects the real plane at a circle $x_1^2+y_1^2=\lambda$, which is the unique (unstable) limit cycle of the corresponding real system. This circle tends to the origin as $\lambda\to 0$. If $\lambda<0$, then (23) has no common point with the real plane. The origin in the real plane is a stable focus when $\lambda>0$, an unstable focus when $\lambda\leqslant 0$.

Rewrite (22) as

$$\begin{cases}\dfrac{dx_1}{dt}=-y_1+x_1(x_1^2+y_1^2-x_2^2-y_2^2-\lambda)-2x_2(x_1x_2+y_1y_2),\\ \dfrac{dy_1}{dt}=x_1+y_1(x_1^2+y_1^2-x_2^2-y_2^2-\lambda)-2y_2(x_1x_2+y_1y_2),\\ \dfrac{dx_2}{dt}=-y_2+x_2(x_1^2+y_1^2-x_2^2-y_2^2-\lambda)+2x_1(x_1x_2+y_1y_2),\\ \dfrac{dy_2}{dt}=x_2+y_2(x_1^2+y_1^2-x_2^2-y_2^2-\lambda)+2y_1(x_1x_2+y_1y_2).\end{cases} \tag{25}$$

Putting in (25) $x_1=y_1=0$ gives

$$\frac{dx_1}{dt}=\frac{dy_1}{dt}\equiv 0,\quad \frac{dx_2}{dt}=-y_2+x_2(-x_2^2-y_2^2-\lambda),\quad \frac{dy_2}{dt}=x_2+y_2(-x_2^2-y_2^2-\lambda). \tag{26}$$

From this we see that all intersection curves of integral surfaces of (22) with the imaginary plane $x_1=y_1=0$ are trajectories of (25), and it is easily seen that when $\lambda>0$ the imaginary plane has no common point with (24), when $\lambda=0$ they have a unique common point (0, 0, 0, 0), when $\lambda<0$, they intersect at the circle $x_2^2+y_2^2=-\lambda$, which is the unique (stable) limit cycle of (26) in the imaginary plane.

It is well known that $\lambda=0$ is a bifurcation value of the so-called Hopf bifurcation of (22), when (22) is considered to be a real system. But for a complex system (22), $\lambda=0$ is only a bifurcation value of the integral surface (24). When λ varies from positive to negative, the shape of (24) undergoes a sudden change, so that the intersection circle of (24) with $x_2=y_2=0$ shrinks to the origin, and then another intersection circle of (24) with $x_1=y_1=0$ appears from the origin and grows gradually. Moreover, all these intersection circles are trajectories of the corresponding 4-dimensional systems (25).

A somewhat different example is the following:

Example 5. Consider the system

$$\frac{dx}{dt}=-y(ax+by+c)+\lambda-x^2-y^2,\quad \frac{dy}{dt}=x(ax+by+c). \tag{27}$$

Now, (23) or (24) is again an integral surface. It has been proved in [5] that if $\lambda>0$ and

$$c^2>\lambda(a^2+b^2),$$

then (27) has a unique limit cycle $x_1^2+y_1^2=\lambda$ in the real plane, which will shrink to the origin when $\lambda\to0$. Let us consider the case when $\lambda<0$. Put $\lambda=-\mu^2$ and rewrite (27) as

$$\begin{cases}\dfrac{dx_1}{dt}=-a(x_1y_1-x_2y_2)-b(y_1^2-y_2^2)-cy_1+x_2^2+y_2^2-x_1^2-y_1^2-\mu^2,\\ \dfrac{dx_2}{dt}=-a(x_1y_2+x_2y_1)-2by_1y_2-cy_2-2(x_1x_2+y_1y_2),\\ \dfrac{dy_1}{dt}=a(x_1^2-x_2^2)+b(x_1y_1-x_2y_2)+cx_1,\\ \dfrac{dy_2}{dt}=2ax_1x_2+b(x_1y_2+x_2y_1)+cx_2.\end{cases} \tag{28}$$

Let $x_1=y_1=0$ in (28), we get

$$\begin{gathered}\frac{dx_1}{dt}=y_2(ax_2+by_2)+x_2^2+y_2^2-\mu^2,\quad \frac{dy_1}{dt}=-x_2(ax_2+by_2),\\ \frac{dx_2}{dt}=-cy_2,\quad \frac{dy_2}{dt}=cx_2,\end{gathered} \tag{29}$$

so that we can not get the desired closed orbit in $x_1=y_1=0$, contrary to that happened in Example 4. Now let us examine the equation satisfied by the projection on (x_2, y_2) plane of the family of trajectories on (23). We solve from (24)

$$y_1=\pm\frac{x_2\sqrt{x_2^2+y_2^2-\mu^2}}{\sqrt{x_2^2+y_2^2}},\quad x_1=\frac{\mp y_2\sqrt{x_2^2+y_2^2-\mu^2}}{\sqrt{x_2^2+y_2^2}}, \tag{30}$$

substituting in the second and fourth equations of (28) gives

$$\begin{cases}\dfrac{dx_2}{dt}=\pm\dfrac{[a(y_2^2-x_2^2)\sqrt{x_2^2+y_2^2-\mu^2}-2bx_2y_2\sqrt{x_2^2+y_2^2-\mu^2}]}{\sqrt{x_2^2+y_2^2}}-cy_2,\\ \dfrac{dy_2}{dt}=\mp\dfrac{\sqrt{x_2^2+y_2^2-\mu^2}}{\sqrt{x_2^2+y_2^2}}[2ax_2y_2+b(y_2^2-x_2^2)]+cx_2.\end{cases} \tag{31}$$

From (31) we see that the intersection circle $x_2^2+y_2^2=\mu^2$ of (24) and the (x_2, y_2) plane is a closed orbit. Introduce in (31) the curvilinear coordinates

$$x_2=\mu(1-n)\cos\theta,\quad y_2=\mu(1-n)\sin\theta, \tag{32}$$

under which $n=0$ corresponds to $x_2^2+y_2^2=\mu^2$. In order that (31) can be a real system, assume $n\leqslant0$. Noticing that under (32) we have

$$x_2^2+y_2^2-\mu^2=\mu^2(n^2-2n),\quad y_2^2-x_2^2=-\mu^2(1-n)^2\cos2\theta,$$
$$2x_2y_2=\mu^2(1-n)^2\sin2\theta$$

and

$$\frac{dx_2}{dt}=-\mu(1-n)\sin\theta\frac{d\theta}{dt}-\mu\cos\theta\frac{dn}{dt},\quad \frac{dy_2}{dt}=\mu(1-n)\cos\theta\frac{d\theta}{dt}-\mu\sin\theta\frac{dn}{dt} \tag{33}$$

and putting $\tau=\mu(1-n)t$, we can solve from (31) and (33)

$$\begin{cases}\dfrac{d\theta}{d\tau}=c\pm\mu\sqrt{n(n-2)}\,(-a\sin\theta+b\cos\theta),\\[2mm] \dfrac{dn}{d\tau}=\pm(1-n)\mu\sqrt{n(n-2)}\,(a\cos\theta+b\sin\theta).\end{cases} \tag{34}$$

Since $\frac{d\theta}{d\tau}$ has the same sign as C when $|n|$ is sufficiently small, we see from (34) that $x_2^2+y_2^2=\mu^2$ is a limit cycle of (31) from outside. For $\lambda=-\mu^2$ there is no point on (24) satisfying $x_2^2+y_2^2<\mu^2$, the projecting system (31) is thus only defined for $x_2^2+y_2^2\geqslant\mu^2$. However, we have in (30), (31) and (34) both plus and minus signs, so $x_2^2+y_2^2=\mu^2$ is still a twosided limit cycle on (24).

Remark. For $\lambda=\mu^2>0$, if we solve x_2, y_2 from (24), substitute in the first and third equations in (28) and introduce in (x_1, y_1) plane the same curvilinear coordinates as (32), and also the change of time scale: $\tau=\mu(1-n)t$, we will finally get

$$\begin{cases}\dfrac{d\theta}{d\tau}=\dfrac{c}{\mu(1-n)}+a\cos\theta+b\sin\theta,\\[2mm] \dfrac{dn}{d\tau}=\dfrac{n(2-n)}{1-n}(b\cos\theta-a\sin\theta).\end{cases}$$

From the condition $c^2>\mu^2(a^2+b^2)$ we see that $x_1^2+y_1^2=\mu^2$ is also a limit cycle on (24). But different from (31), the projection of trajectories of this integral surface on the (x_1, y_1) plane satisfies the equations

$$\begin{cases}\dfrac{dx_1}{dt}=-y_1(ax_1+by_1+c)-\dfrac{x_1(x_1^2+y_1^2-\mu^2)(ay_1-bx_1)}{x_1^2+y_1^2},\\[2mm] \dfrac{dy_1}{dt}=x_1(ax_1+by_1+c)-\dfrac{y_1(x_1^2+y_1^2-\mu^2)(ay_1-bx_1)}{x_1^2+y_1^2},\end{cases}$$

which are defined for all (x_1, y_1).

It is also easily seen that in Example 4, the integral surface (24) is filled with closed orbits, and there is no limit cycle for the dynamical sub-system on (24).

In the above two examples we see that with the variation of the parameter limit cycle in the real plane shrinks to the origin and then reappears in the imaginary plane or in a certain integral surface. In the following we will give a third example, which shows when two limit cycles in the real plane approach to each other, coincide and then disappear, how we can find them out again.

Example 6. Consider the system

$$\begin{cases}\dfrac{dx}{dt}=[-y+x(x^2+y^2-1)^2]\cos\lambda-[x+y(x^2+y^2-1)^2]\sin\lambda,\\[2mm] \dfrac{dy}{dt}=[x+y(x^2+y^2-1)^2]\cos\lambda+[-y+x(x^2+y^2-1)^2]\sin\lambda.\end{cases} \tag{35}$$

As a real system, (35) has been discussed somehow in [5]. One finds that (35) can be obtained from

$$\frac{dx}{dt}=-y+x(x^2+y^2-1)^2,\quad \frac{dy}{dt}=x+y(x^2+y^2-1)^2 \tag{36}$$

by rotating its vector field through an angle λ. For $0<\lambda<\frac{\pi}{4}$, (35) has two limit cycles

$$x^2+y^2=1\pm\sqrt{\tan\lambda}. \tag{37}$$

They coincide and become a semi-stable cycle $x^2+y^2=1$ as $\lambda\to 0$, and then disappear as λ becomes negative. Let us examine (35) as a complex system for any real λ, by using the method of the former two examples. The integral surface of (35)

$$(x^2+y^2-1)^2=\tan\lambda \tag{38}$$

can be written as

$$\begin{cases}(x_1^2-x_2^2+y_1^2-y_2^2-1)^2-4(x_1x_2+y_1y_2)^2=\tan\lambda,\\ (x_1^2-x_2^2+y_1^2-y_2^2-1)(x_1x_2+y_1y_2)=0.\end{cases} \tag{39}$$

If $\tan\lambda\geqslant 0$, then (39) is equivalent to

$$x_1^2-x_2^2+y_1^2-y_2^2-1=\pm\sqrt{\tan\lambda},\quad x_1x_2+y_1y_2=0, \tag{40}$$

if $\tan\lambda\leqslant 0$, (39) is equivalent to

$$x_1^2-x_2^2+y_1^2-y_2^2-1=0,\quad x_1x_2+y_1y_2=\pm\frac{1}{2}\sqrt{-\tan\lambda}. \tag{41}$$

In the following we discuss only the latter case. First, write out the equivalent 4-dimensional system of (35)

$$\begin{cases}\frac{dx_1}{dt}=\{-y_1+x_1(x_1^2-x_2^2+y_1^2-y_2^2-1)^2-4x_1(x_1x_2+y_1y_2)^2\\ \qquad -4x_2(x_1^2-x_2^2+y_1^2-y_2^2-1)(x_1x_2+y_1y_2)\}\cos\lambda\\ \qquad -\{x_1+y_1(x_1^2-x_2^2+y_1^2-y_2^2-1)^2-4y_1(x_1x_2+y_1y_2)^2\\ \qquad -4y_2(x_1^2-x_2^2+y_1^2-y_2^2-1)(x_1x_2+y_1y_2)\}\sin\lambda\\ \qquad =R_1(x,y)\cos\lambda-R_2(x,y)\sin\lambda,\\ \frac{dy_1}{dt}=R_2(x,y)\cos\lambda+R_1(x,y)\sin\lambda,\\ \frac{dx_2}{dt}=\{-y_2+x_2(x_1^2-x_2^2+y_1^2-y_2^2-1)^2-4x_2(x_1x_2+y_1y_2)^2\\ \qquad +4x_1(x_1^2-x_2^2+y_1^2-y_2^2-1)(x_1x_2+y_1y_2)\}\cos\lambda\\ \qquad -\{x_2+y_2(x_1^2-x_2^2+y_1^2-y_2^2-1)^2-4y_2(x_1x_2+y_1y_2)^2\\ \qquad +4y_1(x_1^2-x_2^2+y_1^2-y_2^2-1)(x_1x_2+y_1y_2)\}\sin\lambda\\ \qquad =R_3(x,y)\cos\lambda-R_4(x,y)\sin\lambda,\\ \frac{dy_2}{dt}=R_4(x,y)\cos\lambda+R_3(x,y)\sin\lambda.\end{cases} \tag{42}$$

Solve x_1, y_1 from (41), we get

$$x_1=\frac{1}{2(x_2^2+y_2^2)}\left[\sqrt{-\tan\lambda}\,x_2-y_2\sqrt{4y_2^2(x_2^2+y_2^2+1)+\tan\lambda}\,\right],$$

$$y_1=\frac{1}{2(x_2^2+y_2^2)}\left[\sqrt{-\tan\lambda}\,y_2+x_2\sqrt{4y_2^2(x_2^2+y_2^2+1)+\tan\lambda}\,\right].$$

Substituting in (42) gives

$$\begin{cases} \frac{dx_2}{dt} = [-y_2+x_2\tan\lambda]\cos\lambda - [x_2+y_2\tan\lambda]\sin\lambda = \frac{-y_2}{\cos\lambda}, \\ \frac{dy_2}{dt} = [x_2+y_2\tan\lambda]\cos\lambda + [-y_2+x_2\tan\lambda]\sin\lambda = \frac{x_2}{\cos\lambda}. \end{cases} \tag{43}$$

Its phase-portrait, after projecting on the (x_2, y_2) plane, is a family of circles. Similarly, if we replace (41) by (40), solve from it x_2, y_2, or x_1, y_1, and then substitute in (42), the result will be the same as (43). So the situation is just the same as in Example 4, i. e., we can not find in the integral surface (38) limit cycles which disappear in the real plane as λ varies from positive to negative.

It is worth noting that if we put in (42) $x_1=y_1=0$, then we get

$$\begin{cases} \frac{dx_2}{dt} = [-y_2+x_2(x_2^2+y_2^2+1)^2]\cos\lambda - [x_2+y_2(x_2^2+y_2^2+1)^2]\sin\lambda \\ \frac{dy_2}{dt} = [x_2+y_2(x_2^2+y_2^2+1)^2]\cos\lambda + [-y_2+x_2(x_2^2+y_2^2+1)^2]\sin\lambda, \\ \frac{dx_1}{dt} \equiv \frac{dy_1}{dt} \equiv 0. \end{cases} \tag{44}$$

Transform the first two in polar coordinates

$$\frac{dr}{dt} = r[(r^2+1)^2\cos\lambda - \sin\lambda], \quad \frac{d\theta}{dt} = \cos\lambda + (r^2+1)^2\sin\lambda. \tag{45}$$

From (45) we see that when λ increases from $\frac{\pi}{4}$, a limit cycle

$$r = \sqrt{\sqrt{\tan\lambda} - 1}$$

bifurcates from the origin in the (x_2, y_2) plane, which is just come out of the one shrinking to the origin in the (x_1, y_1) plane when $\lambda = \frac{\pi}{4}$. But the other limit cycle still remains in (x_1, y_1) plane.

In order to find the missing semi-stable cycle of the real plane when λ varies from positive to negative, let us look for a plane

$$x_1 = kx_2, \quad y_1 = ky_2, \tag{46}$$

where k is real, such that it will intersect (41) at a circle. For this purpose, we substitute (46) in (41), and demand that

$$(k^2-1)(x_2^2+y_2^2) = 1 \quad \text{and} \quad k(x_2^2+y_2^2) = \pm\frac{1}{2}\sqrt{-\tan\lambda} \tag{47}$$

will be the same equation. This gives

$$\sqrt{-\tan\lambda}\, k^2 \pm 2k - \sqrt{-\tan\lambda} = 0. \tag{48}$$

Obviously, we must take $|k|>1$, and then (48) will give only two values of k

$$k = \pm\frac{1+\sqrt{1-\tan\lambda}}{\sqrt{-\tan\lambda}} \quad (\lambda<0). \tag{49}$$

Substituting (49) in (46), and (46) in the latter two equations of (42) gives

$$\begin{cases}\dfrac{dx_2}{dt}=\{-y_2+x_2[(5k^4-10k^2+1)(x_2^2+y_2^2)^2-(6k^2-2)(x_2^2+y_2^2)+1]\}\cos\lambda \\ \qquad -\{x_2+y_2[(5k^4-10k^2+1)(x_2^2+y_2^2)^2-(6k^2-2)(x_2^2+y_2^2)+1]\}\sin\lambda \\ \qquad =M_1(x_2,\ y_2)\cos\lambda-M_2(x_2,\ y_2)\sin\lambda, \\ \dfrac{dy_2}{dt}=M_2(x_2,\ y_2)\cos\lambda+M_1(x_2,\ y_2)\sin\lambda.\end{cases} \tag{50}$$

Now, let us solve the equation

$$(5k^4-10k^2+1)(x_2^2+y_2^2)^2-(6k^2-2)(x_2^2+y_2^2)+1-\tan\lambda=0$$

in $x_2^2+y_2^2$, this gives two solutions

$$x_2^2+y_2^2=\frac{1}{k^2-1}\quad\text{and}\quad x_2^2+y_2^2=\frac{(k^2+1)^2}{(5k^4-10k^2+1)(k^2-1)}. \tag{51}$$

The first represents a circle in $(x_2,\ y_2)$ plane when $|k|>1$, while the second represents a circle when $|k|>\frac{1}{5}\sqrt{5+2\sqrt{5}}\sim 1.37$. Substitute (51) in (50), notice that the value in the [ ] equals $\tan\lambda$, so that (50) becomes (43). The circle[1)]

$$x_1=kx_2,\quad y_1=ky_2,\quad x_2^2+y_2^2=\frac{1}{k^2-1} \tag{52}$$

lies in (41), along which also stands

$$\frac{d}{dt}(x_1-kx_2)\equiv\frac{d}{dt}(y_1-ky_2)\equiv 0. \tag{53}$$

This shows that (52) is actually a closed orbit of (35). But (53) does not hold along the circle

$$x_1=kx_2,\quad y_1=ky_2,\quad x_2^2+y_2^2=\frac{(k^2+1)^2}{(5k^4-10k^2+1)(k^2-1)} \tag{54}$$

(except when $|k|=\sqrt{3}$, for this value (54) coincides with (52)), since $x_1^2-x_2^2+y_1^2-y_2^2-1\neq 0$ on (54). Therefore, (54) is not a closed orbit of (35).

In polar coordinates, (50) has the form

$$\begin{cases}\dfrac{dr}{dt}=r(5k^4-10k^2+1)\left(r^2-\dfrac{1}{k^2-1}\right)\left(r^2-\dfrac{(k^2+1)^2}{(5k^4-10k^2+1)(k^2-1)}\right)\cos\lambda, \\ \dfrac{d\theta}{dt}=1+(5k^4-10k^2+1)\left(r^2-\dfrac{1}{k^2-1}\right)\left(r^2-\dfrac{(k^2+1)^2}{(5k^4-10k^2+1)(k^2-1)}\right),\end{cases}$$

hence (52) is really a limit cycle in the plane (46). Since k has two values in (49), (52) actually represents two limit cycles, each in a plane (46). They are just the limit cycles which disappear in the real plane when λ changes from positive to negative. This is because as $\lambda\to 0$, we have $|k|\to\infty$, and in (52) $x_2^2+y_2^2\to 0$, but

$$x_1^2+y_1^2=\frac{k^2}{k^2-1}\to 1.$$

Remark. The last equation in (52) can be written as

$$x_2^2+y_2^2=\frac{-1+\sqrt{1-\tan\lambda}}{2}$$

1) For the sake of convenience, we call it circle, too. Actually, only its projection on $(x_2,\ y_2)$ plane is a circle.

and from the first two we get

$$x_1^2+y_1^2=\frac{k^2}{k^2-1}=\frac{1+\sqrt{1-\tan\lambda}}{2}.$$

Then by means of (41), we see that the two circles in (52) satisfy the equations

$$x^2+y^2=x_1^2+y_1^2-x_2^2-y_2^2+2i(x_1x_2+y_1y_2)=1\pm i\sqrt{-\tan\lambda}=1\pm\sqrt{\tan\lambda} \qquad (55)$$

respectively, and this is just the same as (37), although λ is negative now. Notice that the plus and minus signs in the right hand side of (55) only explain that these two circles are situated in different planes of (46), but not signify that they have different radii.

The reason we can find the disappeared limit cycles of the real plane in any other place in the above three examples is that, in every example the equation of an integral surface $f(x, y)=0$ is known. And if this integral surface intersects the real plane, the intersection curve is always a limit cycle, i. e., $f(x_1, y_1)=0$ is the equation of this cycle. On the other hand, if only the existence but not the equation of the limit cycle is known, it will still be a difficult problem to find out the disappeared limit cycle of the real plane as the parameter varies. Thus, for example the real quadratic differential system of type (I) (see [6])

$$\frac{dx}{dt}=-y+\delta x+lx^2+xy, \quad \frac{dy}{dt}=x$$

when $\delta l>0$, and

$$\frac{dx}{dt}=-y+\delta x+xy+ny^2, \quad \frac{dy}{dt}=x$$

when $\delta n>0$ all belong to this case. We have confidence in this problem that the same situation would appear for these systems.

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二次微分系统复域定性理论

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摘　要

本文研究具实的系数与自变量，以及复的因变量的二次微分系统

$$\frac{dx}{dt}=P_2(x, y), \quad \frac{dy}{dt}=Q_2(x, y)$$

的定性理论(假设其中 $x=x_1+ix_2$, $y=y_1+iy_2$)，把它的解理解为四维 (x_1, y_1, x_2, y_2) 相空间中的二维曲面，推广了作者在1957年所得到的二次微分系统实域定性理论中的一些基本结果．同时我们又研究了三个含实参数，且在实平面 $x_2=y_2=0$ 中有极限环的方程组，看看当参数经过某些分歧值，极限环从实平面中消失后，它们又在复空间中的那些地方出现了．本文所研究的问题与常微分方程的分枝理论有密切关系，其中未解决的问题尚待继续探讨。