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LOCAL ISOMETRIC IMBEDDING OF RIEMANNIAN MANIFOLDS M^n INTO A SPACE OF CONSTANT CURVATURE S^{n+1}

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Dedicated to Professor Su Bu--chin on the Occasion of his 80th Birthday and his 50th Year of Educational Work

1. The problem of isometrically imbedding an n-dimensional Riemannian manifold M^n into Euclidean space E^{n+p} has received considerable attention. This problem, on which a series of results have been obtained both locally and globally, is still being studied now, thus as different sets of the necessary and sufficient conditions have been studied. These conditions are intrinsic and expressible by the Riemannian tensor of M^n. For example, Allendoerfer, C. B., obtained the conditions for an Einstein space of class one,[1] Jaak Vilms obtained the conditions for a Riemannian space of class one,[2] and so on. The enverloping space E^{n+1} they discussed is Euclidean and the conditions obtained take different forms as the different mothods they used. These conditions are too complicated for applications. The purpose of this paper is to find a set of necessary and sufficient intrinsic conditions for the Riemannian manifold M^n being a hypersurface of a space of constant curvature,and give some applications of these conditions. Since it is a problem of local imbedding, the following discussions are given on a coordinate neighborhood at a given point. Throughout this paper, the term Riemannian manifold or Riemannian space is understood to be a Riemannian manifold with a real fundamental quadratic form

$$\varphi=g_{\alpha\beta}dy^\alpha dy^\beta, \quad g=\det|g_{ij}|\neq 0,$$

as the basis of the metric, either positive definite or not, that is, the element of length ds of the space is defined by

$$ds^2=eg_{\alpha\beta}dy^\alpha dy^\beta \quad (e=\pm 1).$$

Besides a set of necessary and suffiicient conditions of this manifold, we describe roughly the main results of this paper as follows: If any Riemannian manifold M^n can be isometrically imbedded into two spaces S^{n+1} of constant curvature, it can be isometrically imbedded into space S^{n+1} of any constant curvature, and M^n must be a conformally flat space of class one. Conversely, if conformally flat M^n can be

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isometrically imbedded in one S^{n+1} of constant curvature, it can be isometrically imbedded into an S^{n+1} of any constant curvature. This class of spaces involves the spaces of constant curvature as its special class. In this sense, the former is an extension of the latter. we give in the end of this paper the characteristic Riemannian curvature form of this class of spaces.

2. The Gauss equations for a hypersurface M^n of a space S^{n+1} 1) of constant curvature K_0 is given by[3]

$$R_{ijkl}=e(b_{ik}b_{jl}-b_{il}b_{jk})+K_0(g_{ik}g_{jl}-g_{il}g_{jk}), \tag{1}$$

where $e=\pm 1$, g_{ij} the Riemannian metric of M^n, R_{ijkl} the Riemannian curvature tensor. If we put

$$T_{ijkl}=R_{ijkl}-K_0(g_{ik}g_{jl}-g_{il}g_{jk}), \tag{2}$$

the equation (1) becomes

$$T_{ijkl}=e(b_{ik}b_{jl}-b_{il}b_{jk}) \quad (e=\pm 1). \tag{3}$$

After a complicate calculation, we find from (3) the following identity

$$2eb_{il}b_j({}_qT_{mp})_{kh}=T_{hil}({}_qT_{mp})_{kj}+T_{ikl}({}_qT_{mp})_{hj}+T_{hkl}({}_qT_{mp})_{ij}, \tag{4}$$

where $b_j({}_qT_{mp})_{kh}$ indicates the sum of three terms obtained one by one by a cyclically permutation of $\begin{pmatrix} q & m & p \\ m & p & q \end{pmatrix}$ and so on. If the identity (4) is multiplied by $g^{hq}g^{pk}$ and summed for h, q, p, k, and put

$$T=R+n(n-1)K_0, \tag{5}$$

$$T_i^j=R_i^j+(n-1)K_0\delta_i^j, \quad T_{ij}=R_{ij}+(n-1)K_0g_{ij}, \tag{6}$$

$$T^p_{jkl}=R^p_{jkl}-K_0(\delta^p_kg_{jl}-\delta^p_lg_{jk}), \tag{7}$$

we have

$$2b_{jl}b_{ip}\left(T^p_m-\frac{1}{2}\delta^p_mT\right)=e\left(T_{jl}T_{im}-T_{ip}T^p_{jlm}+T_{hl}T^h_{mij}+T^p_{jlq}T^q_{ipm}-\frac{1}{2}T^q_{klm}T^k_{qij}\right). \tag{8}$$

If we put

$$P_{jlim}\equiv T_{jl}T_{im}-T_{ip}T^p_{jlm}+T_{pl}T^p_{mij}+T^p_{jlq}T^q_{ipm}-\frac{1}{2}T^q_{klm}T^k_{qij}, \tag{9}$$

the equation (8) becomes

$$2b_{jl}b_{ip}\left(T^p_m-\frac{1}{2}\delta^p_mT\right)=eP_{jlim} \tag{10}$$

and consequently we have

$$P_{jlim}P_{rshk}=b_{jl}b_{rs}(2b_{ip}T^p_m-b_{im}T)(2b_{hp}T^p_k-b_{hk}T).$$

By means of (10), this equation is reducible to

$$(2P_{iphk}T^p_m-P_{imhk}T)b_{jl}b_{rs}=eP_{jlim}P_{rshk}. \tag{11}$$

Lemma 1. *If*

$$H_{hijk}=a_{hj}a_{ik}-a_{hk}a_{ij} \quad (h, i, j, k=1, \cdots, n) \tag{12}$$

1) Both S^{n+1} and M^n are Riemannian spaces in the sense as stated in § 1, that is, their fundamental quadratic forms may be indefinite.

and

$$H_{hijk}+H_{hjki}+H_{hkij}=0 \tag{13}$$

and if the rank r of the matrix (H_{hijk}) *is* $\geqslant 3$, *we have* $a_{ji}=a_{ij}$.

Proof From (12) and (13) it follows that

$$a_{hj}(a_{ik}-a_{ki})+a_{hk}(a_{ji}-a_{ij})+a_{hi}(a_{kj}-a_{jk})=0. \tag{14}$$

Since the rank of the matrix (a_{ij}) is equal to that of (H_{hijk}), that is, equal to r, we suppose, without loss of generality, that $\det|a_{ij}|\neq 0$ $(i,\ j=1,\ \cdots,\ r)$, such that a set of a^{hj} can be defined by the following equations

$$a^{hj}a_{hi}=\delta_i^j \quad (h,\ i,\ j=1,\ \cdots,\ r).$$

Firstly, if we take $h,\ i,\ j,\ k=1,\ \cdots,\ r$, in the equation (14) and contracting it by a^{hj}, we get

$$(r-2)(a_{ik}-a_{ki})=0.$$

Since $r\geqslant 3$, it follows that $a_{ik}=a_{ki}$ $(i,\ k=1,\ \cdots,\ r)$.

Secondly, take in (14) $h,\ i,\ k=1,\ \cdots,\ r,\ j=r+1,\ \cdots,\ n$; we have

$$a_{hk}(a_{ji}-a_{ij})+a_{hi}(a_{kj}-a_{jk})=0.$$

Contracting it by a^{hi}

$$(r-1)(a_{kj}-a_{jk})=0,$$

and consequently

$$a_{kj}=a_{jk} \quad (k=1,\ \cdots,\ r,\ j=r+1,\ \cdots,\ n),$$

Finally, take in (14) $k,\ j=r+1,\ \cdots,\ n$; $h,\ i=1,\ \cdots,\ r$, we find

$$a_{kj}=a_{jk} \quad (k,\ j=r+1,\ \cdots,\ n).$$

Hence

$$a_{ij}=a_{ji} \quad (i,\ j=1,\ \cdots,\ n)$$ q. e. d.

The following lemma is due to Prof. Hu Hesheng: [5]

Lemma 2. *If a Riemannian manifold* M^n *admits isometric imbedding into a space* S^{n+1} *of constant curvature* K_0 *and if the rank of the matrix* (T_{ijkl}) *is* $\geqslant 4$, *the Codazzi equation is the algebraic consequence of the Gauss equation.*

Now we state the following

Theorem 1. *Let the rank of the matrix* (T_{ijkl}) *of a Riemannian manifold* M^n *is* $\geqslant 4$, $T\neq 0$, M^n *admits local isometric immedding into* S^{n+1} *of constant curvature* K_0 *if and only if the following equations hold and are non trivial*

$$(2P_{hphk}T_k^p-P_{hkhk}T)T_{abcd}=P_{achk}P_{bdhk}-P_{adhk}P_{bchk}, \tag{15}$$

for a, b, c, $d=1,\ \cdots,\ n$, *and for a certain set of h, k.*

In order to prove this theorem, we state and prove the following

Lemma 3. *If a Riemannian manifold* M^n *admits isometric imbedding into a space* S^{n+1} *of constant curvature, and if the rank of the matrix* (T_{ijkl}) *is* $\geqslant 3$, $T\neq 0$, *there exists at least a set of h, k such that*

$$2P_{hphk}T_k^p-P_{hkhk}T\neq 0. \tag{16}$$

If for any h, k

$$2P_{hphk}T^p_k - P_{hkhk}T = 0,$$

we have from (11)

$$P_{jlim} = 0. \tag{17}$$

From (10) it follows that

$$b_{ip}\Big(T^p_m - \frac{1}{2}\delta^p_m T\Big) = 0 \tag{18}$$

or

$$b_{jl}b_{ip}T^p_m - \frac{1}{2}b_{jl}b_{im}T = 0.$$

By means of this equation and the Gauss equation (3) we obtain

$$T_{ijpl}T^p_m - \frac{1}{2}TT_{ijml} = 0$$

or

$$T_{km}T^k_{lij} - \frac{1}{2}TT_{ijml} = 0.$$

Contracting it by g^{li},

$$\Big(T^k_j - \frac{1}{2}T\delta^k_j\Big)T_{km} = 0. \tag{19}$$

Let the rank of the matrix (T_{km}) be r. By the theory of linear algebra there exists at a point P of M^n a coordinate system in which the matrix (T_{km}) takes the following form

$$\left(\begin{array}{ccc|c} T_{11} & \cdots & T_{1r} & \\ \cdots & \cdots & \cdots & 0 \\ T_{r1} & \cdots & T_{rr} & \\ \hline & 0 & & 0 \end{array}\right)$$

In this coordinate system (19) may be written as

$$g^{kl}\Big(T_{lj} - \frac{1}{2}Tg_{lj}\Big)T_{km} = 0 \quad (j,\ k,\ m = 1,\ \cdots,\ r;\ \ l = 1,\ \cdots,\ n). \tag{20}$$

Since $\det|T_{ij}| \neq 0$ $(i,\ j = 1,\ \cdots,\ r)$, a set of T^{mp} can be defined by

$$T_{km}T^{mp} = \delta^p_k. \tag{21}$$

Contracting (20) by T^{mp}

$$g^{pl}\Big(T_{lj} - \frac{1}{2}Tg_{lj}\Big) = 0,$$

and contracting again by T^{jq},

$$g^{pq} = \frac{1}{2}TT^{pq} \quad (p,\ q = 1,\ \cdots,\ r).$$

Therefore

$$T = g^{ls}T_{ls} = g^{pq}T_{pq} = \frac{1}{2}TT^{pq}T_{pq} = \frac{r}{2}T.$$

Since $T\neq 0$, we have $r=2$. Then

$$T^p_m=g^{pl}T_{lm}=0 \quad (m\geqslant 3).$$

From (18) we find

$$b_{im}=0 \quad (m=3, \cdots, n;\ i=1, \cdots, n).$$

It means that the rank of the matrix (b_{ij}) is $\leqslant 2$. But it is known that the rank of the matrix (T_{hijk}) is equal to that of (b_{ij}),[4] and consequently the rank of (T_{hijk}) is $\leqslant 2$. This is a contradiction. Therefore there is at least a set of h, k satisfying (16).

q. e. d.

We proceed to prove the Theorem 1.

Under the conditions stated in Lemma 3, it is seen that each equality of (15) is non trival. But the equalities (15) are the consequence of the conditions (11) and (3) which are the necessary conditions for M^n to admit local isometric imbedding into S^{n+1}, and therefore (15) are also the necessary conditions.

We now prove that (15) are the sufficient conditions. According to (16), we can take $e=1$ or -1 such that

$$e(2P_{hphk}T^p_k-P_{hkhk}T)>0.$$

Define

$$b_{jl}=\frac{P_{jlhk}}{\sqrt{e(2P_{hphk}T^p_k-P_{hkhk}T)}}. \tag{22}$$

From (22) and (15) we have

$$T_{hijk}=e(b_{hj}b_{ik}-b_{hk}b_{ij}) \quad (e=\pm 1). \tag{23}$$

Since T_{hijk} satisfies the conditions (13), and the rank of the matrix (T_{hijk}) is $\geqslant 4$, by Lemma 1 we have $b_{jl}=b_{lj}$, and cons equently the equations (23) can be taken as the Gauss equations of a hypersurface of S^{n+1} of constant curvature K_0.

By Lemma 2, the Codazzi equations of this hypersurface are algebraic consequence of the Gauss equations. That is, M^n admits local isometric imbedding into a space S^{n+1} of constant curvature K_0. q. e. d.

When $K_0=0$, we denote the corresponding P_{hijk} by Q_{hijk}. As a consequence of Theorem 1 we have

Corollary 1. *Let the metric rank of a Riemannian manifold V^n, i. e., the rank of the matrix (R_{hijk}) is $\geqslant 4$, $R\neq 0$, a necessary and sufficient condition for V^n to be of class one is as follows*

$$(2Q_{hphk}R^p_k-Q_{hkhk}\,R)R_{abcd}=Q_{achk}Q_{bdhk}-Q_{adhk}Q_{bchk}, \tag{24}$$

where a, b, c, $d=1, \cdots, n$ and h, k is a set of indices such that the equality is non trivial.

Corollary 2. *If a Riemannian manifold M^n admits isometric imbedding into a space S^{n+1} of constant curvature K_0, then*

$$P_{jlhk}=P_{ljhk}, \tag{25}$$

$$P_{pqcd}P_{rsab}=P_{pqab}P_{rscd}, \tag{26}$$

$$2T_{jilp}\left(T^p_m - \frac{1}{2}T\delta^p_m\right) = P_{jilm} - P_{iljm}, \tag{27}$$

$$2P_{ahcd}T^h_b - P_{abcd}T = 2P_{chab}T^h_d - P_{cdab}T. \tag{28}$$

Proof From (11) we have (22), and since $b_{jl} = b_{lj}$ we obtain (25). Interchanging the indices j and r, l and s in (11), we obtain (26). Substracting the equations (10) from the equation obtained by interchanging in (10) the indices i and j, making use of (3), we get the equation (27). From (11) it follows that

$$2P_{ahcd}\left(T^h_b - \frac{1}{2}T\delta^h_b\right)T_{rqsp} = P_{rsab}P_{qpcd} - P_{rpab}P_{qscd}. \tag{29}$$

From (26) it is seen that the right-hand member of (29) is invariant by interchanging the indices a and c, b and d, and consequently the equation (28) is easily obtained.

3. We give some applications of Theorem 1. After a direct computation we find

$$P_{jilm} = Q_{jilm} + (n-1)(n-2)K_0^2 g_{jl}g_{im} + (n-1)K_0 g_{im}R_{jl} + (n-3)K_0 g_{jl}g_{im}, \tag{30}$$

$$\begin{aligned}2T_{jilp}\left(T^p_m - \frac{1}{2}T\delta^l_m\right) = {} & 2R_{jilp}R^p_m - RR_{jilm} + (n-1)(n-2)K_0^2(g_{jl}g_{im} - g_{il}g_{jm}) \\ & + [R(g_{jl}g_{im} - g_{jm}g_{il}) - 2(g_{jl}R_{im} - g_{il}R_{jm}) \\ & - (n-1)(n-2)R_{jilm}]K_0.\end{aligned} \tag{31}$$

Substituting (30) and (31), into (27) we have

$$2R_{jilp}R^p_m - RR_{jilm} = Q_{jilm} - Q_{iljm} + (n-1)(n-2)K_0C_{ijml}, \tag{32}$$

where

$$\begin{aligned}C_{ijml} \equiv R_{ijml} & - \frac{1}{n-2}(g_{il}R_{jm} + g_{jm}R_{il} - g_{jl}R_{im} - g_{im}R_{jl}) \\ & - \frac{R}{(n-1)(n-2)}(g_{im}g_{jl} - g_{il}g_{jm})\end{aligned} \tag{33}$$

is the conformal curvature tensor of M^n.

Theorem 2. *If a Riemannian manifold M^n $(n \geqslant 4)$ admits isometric imbedding into two $(n+1)$-dimensional spaces of constant curvatures K_0 and K_1 respectively, and $K_0 \neq K_1$, M^n is conformally flat.*

Since (32) is satisfied by K_0 and K_1, $K_0 \neq K_1$, and $n \geqslant 4$, we have consequently $C_{ijml} = 0$, and M^n is conformally flat.

4. When a Riemannian manifold M^n is of constant curvature a, we have

$$R_{hijk} = a(g_{hj}g_{ik} - g_{hk}g_{ij}). \tag{34}$$

In this case (30) becomes

$$P_{jilm} = (n-1)(n-2)(a-K_0)^2 g_{jl}g_{im}, \tag{35}$$

and from (2), we have

$$T_{ijkl} = (a-K_0)(g_{ik}g_{jl} - g_{il}g_{jk}).$$

Substituting these equations into (15), we find that either the left-hand member or the right-hand member of (15) is equal to

$$(n-1)^2(n-2)^2(a-K_0)^4 g^2_{hk}(g_{ac}g_{bd} - g_{ad}g_{bc})$$

which can not vanish identically for $n \geqslant 4$, and $a - K_0 \neq 0$.

When $a-K_0\neq 0$, it is seen that the rank of the matrix (T_{ijkl}) is equal to that of the matrix (g_{ij}) and consequently the rank is $\geqq 4$. Since in this case

$$T=-n(n-1)(a-K_0)\neq 0,$$

we obtain by Theorem 1 the following

Theorem 3. *Any Riemannian manifold M^n $(n\geqslant 4)$ of constant curvature a admits isometric imbedding into a space S^{n+1} of any constant curvature $K_0(\neq a)$.*

From (11), we have (22) and consequently

$$e(2P_{hphk}T^p_k-P_{hkhk}T)>0 \quad (e=\pm 1)$$

is a necessary condition that M^n admits isometric imbedding into S^{n+1}. When M^n is of constant curvature a and S^{n+1} of constant curvature K_0, this condition becomes

$$e(2P_{hphk}T^p_k-P_{hkhk}T)=e(n-1)^2(n-2)^2g^2_{hk}(a-K_0)^3>0.$$

Moreover, if the fundamental quadratic form $g_{\alpha\beta}dy^\alpha dy^\beta$ of S^{n+1} is positive definite, that is, $e=1$, from this condition it follows that $a-K_0>0$, and we have the following known result:

Any Riemannian manifold M^n $(n\geqslant 4)$ of constant curvature a does not admit isometric imbedding into a space S^{n+1} of constant curvature K_0, if $K_0>a$ and the fundamental quadratic form of S^{n+1} is positive definite.

5. Substituting (2) and (30) into (15) we find that (15) is an algebraic equation in K_0 of the following form

$$(n-1)^2(n-2)^2g^2_{hk}C_{abcd}K^3_0+\cdots=0. \tag{36}$$

From Theorem 2, if M^n $(n\geqslant 4)$ admits isometric imbedding into two $(n+1)$-dimensional spaces of constant curvature, M^n is conformally flat, i. e., $C_{abcd}=0$, and (36) is a quadratic equation in K_0, we have the following result:

If a Riemannian manifold M^n admits isometric imbedding into three $(n+1)$-dimensional spaces of different constant curvatures, and if the rank of the matrix

$$(R_{ijkl}-a(g_{ik}g_{jl}-g_{il}g_{jk}))$$

is $\geqslant 4$, $R+n(n-1)a\neq 0$, where a is a constant, then M^n admits isometric imbedding into a space of constant curvature a, and M^n is of class one and conformally flat.

6. From the foregoing results, a Riemannian manifold which is not conformally flat does not admit isometric imbedding into two spaces S^{n+1} of different constant curvatures, and any M^n of constant curvature admits isometric imbedding into a space S^{n+1} of any constant curvature. It arises a question: does there exist a conformally flat M^n which is not of constant curvature but admits isometric imbedding into two spaces of different constant curvature? The answer is affirmative. In fact, we can prove that if a conformally flat M^n admits isometric imbedding into a space S^{n+1} of constant curvature, it admits also isometric imbedding into a space S^{n+1} of any constant curvature. In other words, this M^n is of class one. Conversely, any conformally flat M^n of class one admits isometric imbedding into a space S^{n+1} of any constant curvature. According to Theorem 3, any Riemannian manifold M^n $(n\geqslant 4)$ of constant curvature admits isometric imbedding into a space S^{n+1} of any constant curvature. And any M^n of constant curvature is conformally flat and of class one. So that the conformally flat M^n of class one may be seen as an extension of the spaces of constant curvature.

We proceed to establish this conclusion.

It is well known that the manifold M^n $(n\geqslant 4)$ is conformally flat if and only if $C_{ijkl}=0$ or by (33)

$$R_{ijkl}=g_{il}d_{jk}+g_{jk}d_{il}-g_{ik}d_{jl}-g_{jl}d_{ik}, \tag{37}$$

where

$$d_{jk}=\frac{1}{n-2}R_{jk}-\frac{R}{2(n-1)(n-2)}g_{jk}. \tag{38}$$

It is easy to show that (38) is algebraically consequence of (37).

If we put

$$\varDelta\equiv g^{ij}d_{ij},\quad d^i_l=g^{ih}d_{hl},\quad D_{ij}\equiv g^{pk}d_{pi}d_{kj}=d^k_i d_{kj},\quad D\equiv g^{ij}D_{ij} \tag{39}$$

and contract (38) by g^{jk}, we obtain

$$R=2(n-1)\varDelta,\qquad R_{jk}=\varDelta g_{jk}+(n-2)d_{jk}. \tag{40}$$

From (37) we have

$$R^h_{jkl}=g_{jk}d^h_l+\delta^h_l d_{jk}-d^h_k g_{jl}-\delta^h_k d_{jl} \tag{41}$$

and

$$R_{ih}R^h_{jkl}=\varDelta R_{ijkl}+(n-2)(d_{il}d_{jk}-d_{ik}d_{jl}+g_{jk}D_{il}-g_{jl}D_{ik}), \tag{42}$$

$$R^q_{klm}R^k_{qij}=4(d_{im}d_{lj}-d_{il}d_{mj})+2g_{lj}D_{im}+2g_{im}D_{lj}-2g_{il}D_{mj}-2g_{mj}D_{il}, \tag{43}$$

$$\begin{aligned}R^p_{ilq}R^q_{jpm}=&2g_{il}D_{mj}+2g_{mj}D_{il}-2d_{jl}d_{im}-(n-4)d_{il}d_{mj}-g_{im}D_{jl}\\&-g_{jl}D_{im}-\varDelta(g_{il}d_{jm}+d_{il}g_{jm})-Dg_{il}g_{jm}.\end{aligned} \tag{44}$$

Since $Q_{jlim}=P_{jlim}|_{k_0=0}$, we have from (9)

$$Q_{jlim}=R_{jl}R_{im}-R_{ip}R^p_{jlm}+R_{hl}R^h_{mij}+R^p_{jlq}R^q_{ipm}-\frac{1}{2}R^q_{klm}R^k_{qij}. \tag{45}$$

Substituting (40)—(44) into (45), we find

$$\begin{aligned}Q_{jlim}=&\varDelta^2 g_{jl}g_{im}+(n-3)\varDelta(g_{jl}d_{im}+g_{im}d_{jl})\\&+(n-1)g_{im}D_{jl}-(n-3)g_{jl}D_{im}+(n-2)(n-3)d_{jl}d_{im}-Dg_{jl}g_{im}.\end{aligned} \tag{46}$$

Substituting (46) into (30) and put.

$$\lambda_{ij}\equiv d_{ij}-\frac{\varDelta}{n}g_{ij},\qquad \mu_{ij}\equiv D_{ij}-\frac{D}{n}g_{ij}, \tag{47}$$

we obtain

$$\begin{aligned}P_{jlim}=&(n-1)(n-2)g_{im}g_{jl}K_0^2+\Big[(n-3)g_{jl}\lambda_{im}+(n-1)g_{im}\lambda_{jl}\\&+\frac{4(n-1)}{n}\varDelta g_{im}g_{jl}\Big](n-2)K_0+(n-2)(n-3)\lambda_{jl}\lambda_{im}\\&+(n-1)g_{im}\mu_{jl}-(n-3)g_{jl}\mu_{im}+\frac{2(n-1)(n-3)}{n}\varDelta(g_{jl}\lambda_{im}+g_{im}\lambda_{jl})\\&+(n-2)\Big(\frac{4n-3}{n^2}\varDelta^2-\frac{D}{n}\Big)g_{im}g_{jl}.\end{aligned} \tag{48}$$

Multiplying (48) by g^{im} and summing for i, m, we have

$$g^{im}P_{jlim}=n(n-1)(n-2)g_{jl}K_0^2+(n\lambda_{jl}+4\Delta g_{jl})(n-1)(n-2)K_0$$
$$+n(n-1)\mu_{jl}+2(n-1)(n-3)\Delta\lambda_{jl}+(n-2)\left(\frac{4n-3}{n}\Delta^2-D\right)g_{jl}. \quad (49)$$

Multiplying (48) by g^{jl} and summing for j, l we obtain

$$g^{jl}P_{jlim}=n(n-1)(n-2)g_{im}K_0^2+[n(n-3)\lambda_{im}+4(n-1)\Delta g_{im}](n-2)K_0$$
$$-n(n-3)\mu_{im}+2(n-1)(n-3)\Delta\lambda_{im}+(n-2)\left(\frac{4n-3}{n}\Delta^2-D\right)g_{im}. \quad (50)$$

Moreover, we have

$$g^{jl}g^{im}P_{jlim}=n^2(n-1)(n-2)K_0^2+4n(n-1)(n-2)\Delta K_0$$
$$+(n-2)[(4n-3)\Delta^2-nD]. \quad (51)$$

From (26) it follows that

$$P_{jlpq}P_{abim}-P_{abpq}P_{jlim}=0.$$

Multiplying by $g^{pq}g^{ab}$ and summing for p, q, a, b, by means of (48)—(51) we find

$$(g^{pq}P_{jlpq})(g^{ab}P_{abim})-(g^{ab}g^{pq}P_{abpq})P_{jlim}$$
$$=-n(n-3)\{[n\Delta^2-(n-2)^2D]\lambda_{im}+(n-1)[n(n-2)K_0$$
$$+2(n-3)\Delta]\mu_{im}\}\lambda_{jl}+n(n-1)(n-3)\{[n(n-2)K_0$$
$$+2(n-1)\Delta]\lambda_{im}-n\mu_{im}\}\mu_{jl}=0. \quad (52)$$

Therefore there exists a function ρ which is independent of the indices i and m such that

$$\mu_{jl}=\rho\lambda_{jl}. \quad (53)$$

When $\lambda_{jl}=d_{jl}-\frac{\Delta}{n}g_{jl}=0$, M^n is a space of constant curvature, as follows easily from (38). In this case $\mu_{jl}=0$ and the equation (53) or (52) reduces to an identity. In the general case, substituting (53) into (52), we have

$$\rho^2-\frac{4}{n}\Delta\rho+\frac{1}{n-1}\left[\Delta^2-\frac{(n-2)^2}{n}D\right]=0. \quad (54)$$

(53) and (54) are the necessary conditions for a conformally flat M^n which admits isometric imbedding into a space S^{n+1} of constant curvature. The equation obtained by eliminating ρ from (53) and (54) depends only upon the metric tensor g_{ij} and hence is an intrinsic necessary condition.

Eliminating μ_{ij} and D from (53), (54) and (48), we obtain

$$P_{jlim}=\frac{n-3}{n-2}\left\{(n-2)\lambda_{jl}+\left[(n-2)K_0+\frac{2(n-1)}{n}\Delta-\rho\right]g_{jl}\right\}$$
$$\cdot\left\{(n-2)\lambda_{im}+\frac{n-1}{n-3}\left[(n-2)K_0+\frac{2(n-3)}{n}\Delta+\rho\right]g_{im}\right\}. \quad (55)$$

By means of (2), (5)—(7) and (55), we have

$$2P_{hphk}T_k^p-P_{hkhk}T=-\frac{(n-3)^2}{n-2}\{(n-2)K_0+2(\Delta-\rho)\}$$
$$\cdot\left\{(n-2)\lambda_{hm}+\frac{n-1}{n-3}\left[(n-2)K_0+\frac{2(n-3)}{n}\Delta+\rho\right]g_{hk}\right\}^2. \quad (56)$$

In deriving (56) we have used the following relations

$$\lambda^j_m = d^j_m - \frac{\Delta}{n}\delta^j_m,$$

$$D_{im} = d_{ij}d^j_m = \lambda_{ij}\lambda^j_m + \frac{2\Delta}{n}\lambda_{im} + \frac{\Delta^2}{n^2}g_{im}, \tag{57}$$

$$\lambda_{ij}\lambda^j_m = \left(\rho - \frac{2\Delta}{n}\right)\lambda_{im} + \frac{n-1}{(n-2)^2}\left(\rho - \frac{2\Delta}{n}\right)^2 g_{im}.$$

Substituting (55), (56) and (2) into (15), we obtain

$$-(n-2)\{R_{jlim} - K_0(g_{ji}g_{lm} - g_{jm}g_{li})\}\{(n-2)K_0 + 2(\Delta - \rho)\}$$
$$= (n-2)^2(\lambda_{ji}\lambda_{lm} - \lambda_{jm}\lambda_{li})$$
$$+ (n-2)\left[(n-2)K_0 + \frac{2(n-1)}{n}\Delta - \rho\right](g_{lm}\lambda_{ji} + g_{ji}\lambda_{lm} - g_{li}\lambda_{jm} - g_{jm}\lambda_{li})$$
$$+ \left[(n-2)K_0 + \frac{2(n-1)}{n}\Delta - \rho\right]^2 (g_{ji}g_{lm} - g_{jm}g_{li}). \tag{58}$$

By means of (33), (34) and $C_{jlim} = 0$ the equation (58) or (15) becomes

$$2(n-2)(\Delta - \rho)R_{jlim} = (n-2)^2(\lambda_{jm}\lambda_{li} - \lambda_{ji}\lambda_{lm})$$
$$+ \left[\frac{2(n-1)}{n}\Delta - \rho\right]^2 (g_{jm}g_{li} - g_{ji}g_{lm})$$
$$+ (n-2)\left[\frac{2(n-1)}{n}\Delta - \rho\right](g_{jm}\lambda_{li} + g_{li}\lambda_{jm} - g_{ji}\lambda_{lm} - g_{lm}\lambda_{ji}). \tag{59}$$

Eliminating $g_{jm}\lambda_{li} + g_{li}\lambda_{jm} - g_{ji}\lambda_{lm} - g_{lm}\lambda_{ji}$ from (59) and the following equation

$$R_{jlim} = g_{jm}\lambda_{li} + g_{li}\lambda_{jm} - g_{ji}\lambda_{lm} - g_{lm}\lambda_{ji} + \frac{2\Delta}{n}(g_{jm}g_{li} - g_{ji}g_{lm}), \tag{37'}$$

which is equivalent to (37), We can write (59) as follows

$$(n-2)\left(\frac{2}{n}\Delta - \rho\right)R_{jlim} = (n-2)^2(\lambda_{jm}\lambda_{li} - \lambda_{ji}\lambda_{lm})$$
$$+ \left[\frac{2(n-1)}{n}\Delta - \rho\right]\left(\frac{2}{n}\Delta - \rho\right)(g_{jm}g_{li} - g_{ji}g_{lm}). \tag{59'}$$

By means of (47) and (54) we can also write (59') either in the form

$$(n-2)(\rho - \Delta)R_{jlim} = (n-2)^2(d_{ji}d_{lm} - d_{jm}d_{li}) + (\rho - \Delta)^2(g_{ji}g_{lm} - g_{jm}g_{li}) \tag{59''}$$

or

$$R_{jlim} = \frac{1}{2(n-2)(\rho - \Delta)}\{(R_{ji} - \rho g_{ji})(R_{lm} - \rho g_{lm}) - (R_{jm} - \rho g_{jm})(R_{li} - \rho g_{li})\} \tag{59'''}$$

or

$$(n-2)(\rho - 2\Delta)R_{jlim} = R_{ji}R_{lm} - R_{jm}R_{li} + \rho(\rho - 2\Delta)(g_{ji}g_{lm} - g_{jm}g_{li}). \tag{59''''}$$

In other words, when $C_{jlim} = 0$ the condition (15) is independent of K_0, we have

Theorem 4. *If a conformally flat M^n $(n \geq 4)$ admits isometric imbedding into a space S^{n+1} of constant curvature and if the rank of the matrix $(R_{ijkl} - a(g_{ik}g_{jl} - g_{il}g_{jk}))$ is ≥ 4, $R + n(n-1)a \neq 0$, where a is a constant, then M^n admits also isometric imbedding into the space S^{n+1} of constant curvature a. This is a characteristic property of the conformally flat manifold.*

By means of the foregoing results, we have also the following

Theorem 5. *If the rank of the matrix* (T_{ijkl}) *is* $\geqslant 4$ *and* $T \neq 0$, *then* (37) *and* (59″) *are the characteristic intrinsic conditions for a conformally flat Riemannian manifold* $M^n (n \geqslant 4)$ *to admit isometric imbedding into a space* S^{n+1} *of constant curvature* K_0.

It is remarkable that the conditions (37) and (59″) are independent of K_0.

It is easily seen that the condition (54) is algebraically consequence of (37) and (59″).

From (54), it follows that

Corollary 1. *A necessary condition that a conformally flat Riemannian space* $M^n (n \geqslant 4)$ *is of class one is that*

$$nD \geqslant \Delta^2. \tag{60}$$

From (53) we obtain

Corollary 2. *A necessary condition that a conformally flat* $M^n (n \geqslant 4)$ *is of class one is that*

$$\begin{vmatrix} \lambda_{jl} & \mu_{jl} \\ \lambda_{im} & \mu_{im} \end{vmatrix} = 0. \tag{61}$$

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黎曼流形 M^n 在常曲率空间 S^{n+1} 的局部等距嵌入

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摘要

本文求得黎曼流形 M^n 能够作为常曲率空间超曲面的内蕴充要条件，并举出这些条件的若干应用。设常曲率空间 S^{n+1} 的线素是 $ds^2=eg_{\alpha\beta}dy^\alpha dy^\beta$ $(e=\pm1)$，即 $g_{\alpha\beta}dy^\alpha dy^\beta$ 不一定是正定的，$n+1$ 维的 S^{n+1} 的曲率是 K_0，记为 $S^{n+1}(K_0)$. M^n 是 n 维的黎曼流形，g_{ij} 是 M^n 等距嵌入于 S^{n+1} 中所诱导的黎曼尺度，R_{ijkl} 是 M^n 的黎曼曲率张量，记

$$T_{ijkl}\equiv R_{ijkl}-K_0(g_{ik}g_{jl}-g_{il}g_{jk}),$$
$$P_{jlim}\equiv T_{jl}T_{im}-T_{ip}T^p_{jlm}+T_{pl}T^p_{mij}+T^p_{jlq}T^q_{ipm}-\frac{1}{2}T^q_{klm}T^k_{qij},$$

式内

$$T_{li}=g^{jm}T_{jlim},\quad T^p_{jlm}=g^{pk}T_{kjlm},\quad T=g^{li}T_{li}.$$

经过冗长的计算可以证明下列诸定理.

定理 1 设黎曼流形 M^n 的矩阵 (T_{ijkl}) 的秩 $\geqslant4$, $T\neq0$, 则 M^n 可等距嵌入于一个 $S^{n+1}(K_0)$ 的充要条件是

$$(2P_{hphk}T^p_k-P_{hkhk}T)T_{abcd}=P_{achk}P_{bdhk}-P_{adhk}P_{bchk},$$

$a, b, c, d=1, \cdots, n$; 任意固定一组指标 h, k 使上式两边不恒等于 0.

定理 2 设黎曼流形 $M^n(n\geqslant4)$ 可等距嵌入于 $S^{n+1}(K_0)$ 和 $S^{n+1}(K_1)$, $K_1\neq K_0$, 则 M^n 是共形平坦的.

定理 3 常曲率 a 的黎曼流形 $M^n(n\geqslant4)$ 可等距嵌入于一个 $S^{n+1}(K_0)$, $K_0\neq a$, K_0 是任意常数.

但必须指出如 $e=1$, 即 S^{n+1} 的基本二次形式 $g_{\alpha\beta}dy^\alpha dy^\beta$ 是正定的，则必须有 $K_0<a$, M^n 才可以等距嵌入于 $S^{n+1}(K_0)$.

定理 4 共形平坦流形 $M^n(n\geqslant4)$ 如可等距嵌入于 $S^{n+1}(K_1)$, 只要 (T_{ijkl}) 的秩 $\geqslant4$, $T\neq0$, 则 M^n 也可等距嵌入于 $S^{n+1}(K_0)$. 这是共形平坦流形的特有性质.

定理 5 可等距嵌入于 $S^{n+1}(K_0)$ 的共形平坦黎曼流形 $M^n(n\geqslant4)$ 除 (T_{ijkl}) 的秩 $\geqslant4$, $T\neq0$ 外，这种流形的黎曼尺度以 R_{ijkl} 能表示成下面的形状作为特征.

$$R_{ijkl}=g_{il}d_{jk}+g_{jk}d_{il}-g_{ik}d_{jl}-g_{jl}d_{ik},$$
$$(n-2)(\rho-\Delta)R_{jlim}=(n-2)^2(d_{ji}d_{lm}-d_{jm}d_{li})+(\rho-\Delta)^2(g_{ji}g_{lm}-g_{jm}g_{li}).$$

值得注意的是这条件与 K_0 无关. 常曲率黎曼流形 $S^n(n\geqslant4)$ 满足上述条件，故 M^n 可视为常曲率黎曼流形的推广. 我们将在另文研究这种流形的尺度的标准形式.