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SINGULAR INTEGRALS IN SEVERAL COMPLEX VARIABLES (I)——HENKIN INTEGRALS OF STRICTLY PSEUDOCONVEX DOMAIN

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Dedicated to Professor Su Bu-chin on the Occasion of his 80th Birthday and his 50th Year of Educational Work

§ 0. Introduction

0.1. Cauchy-Fantappiè kernels

In the case of one complex variable, the Cauchy kernel gives rise to singular integrals on boundaries of arbitrary smooth domains. Cauchy kernel defines an analytic function $\mathbf{H}u$ on plane domain Ω by

$$\mathbf{H}u(w)=\int_{z\in b\Omega} H(w,\ z)u(z)d\sigma_z,\ w\in\Omega,$$

where $H(w, z)$ is the Cauchy kernel $[2\pi i(z-w)]^{-1}$, $d\sigma_z$ is Lebesgue measure on $b\Omega$. There is the famous Plemelj formula

$$\mathbf{H}u(w)=\frac{1}{2}u(w)+\text{p. v.}\int_{z\in b\Omega} H(w,\ z)u(z)d\sigma_z,\ w\in b\Omega, \tag{0.1}$$

as w approaches $b\Omega$, where p. v.$\int_{z\in b\Omega}$ is defined by

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\substack{z\in b\Omega\\ |z-w|>\varepsilon}} H(w,\ z)u(z)d\sigma_z. \tag{0.2}$$

An important point is that the deleted neighborhood around w in (0.2), i. e. $\{z\in b\Omega,\ |z-w|<\varepsilon\}$ is symmetric. If the deleted neighborhood around w is not symmetric, then the limit might fail to exist or the number $\frac{1}{2}$ in (0.1) might have to be modified. There is a corresponding Plemelj formula too as w approaches $b\Omega$ from the exterior of Ω. According to (0.1) and the corresponding formula, we may establish the theory of singular integrals and singular integral equations of one variable. For

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example, we can obtain the famous Poincaré-Bertrand formula: If $\varphi\in \mathrm{Lip}\alpha$, $(0<\alpha\leqslant 1)$, then

$$\mathbf{H}^2\varphi=\frac{1}{4}\varphi. \tag{0.3}$$

Here $\mathbf{H}$ is the Cauchy singular integral operator defined by (0.2). Using these formulas, we can solve the singular integral equations with Cauchy kernel, Hilbert kernel and some important boundary value problems. There are many works about this topic([1, 2]).

In several complex variables, there is no perfect analogue of the Cauchy kernel, what come closest to it are certain Cauchy-Fantappié (C-F) kernels

$$K(w, z)=\frac{c_n}{g^n}\omega\wedge dz_1\wedge\cdots\wedge dz_n,\ w\in\Omega,\ z\in b\Omega, \tag{0.4}$$

where

$$c_n=(-1)^{\frac{n(n-1)}{2}}(n-1)!(2\pi i)^{-n}, \tag{0.5}$$

$$g(w, z)=\sum_{i=1}^{n}(z_i-w_i)g_i(w, z), \tag{0.6}$$

$$\omega=g_1\bar\partial g_2\wedge\cdots\wedge\bar\partial g_n+\cdots+(-1)^{n-1}g_n\wedge\bar\partial g_1\wedge\cdots\wedge\bar\partial g_{n-1}. \tag{0.7}$$

These kernels are different from the Cauchy kernel of one complex variable, they depend on the domain Ω. But they can be constructed for a wide class of Ω (smooth, bounded strictly pseudoconvex domains). The very important one of these domains is the ball. Henkin, Ramirez, Stein and Kerzman have constructed some important C-F kernels which are analytic when $w\in\Omega$. These kernels are called Henkin-Ramirez (H-R) kernel or Stein-Kerzman (S-K) kernel. It is well known that the requirement of the analyticity of the C-F kernels is very important.

0.2. Cauchy integral on the sphere

Singular integrals on the boundaries of special domains in $\mathbf{C}^n$ (such as the ball) first appeared and were studied in connection with Szegö kernel. Generally speaking, the Cauchy kernel is different from the Szegö kernel. C-F kernels are in the spirit of the Cauchy kernel and not of the Szegö kernel. However, if Ω is a ball in $\mathbf{C}^n$, the two kernels coincide. In 1965, Kung, S. and Sun, C. K.[3] studied the Cauchy integral on the sphere in $\mathbf{C}^n$. They proved: If $z=(z_1, \cdots, z_n)\in\mathbf{C}^n$, $z\bar z'<1$, $u\bar u'=1$, $f(u)\in \mathrm{Lip}\,p$ $(0<p\leqslant 1)$, then when z approaches v $(v\bar v'=1)$ along the nontangential direction, we have

$$\lim_{z\to v}\frac{1}{\omega_{2n-1}}\int_{u\bar u'=1}\frac{f(u)\dot u}{(1-z\bar u')^n}=\frac{1}{2}f(v)+\mathrm{p.v.}\frac{1}{\omega_{2n-1}}\int_{u\bar u'=1}\frac{f(u)\dot u}{(1-v\bar u')^n}, \tag{0.8}$$

where ω_{2n-1} is the volume of $u\bar u'=1$. The principal value is defined by

$$\mathrm{p.v.}\frac{1}{\omega_{2n-1}}\int_{u\bar u'=1}\frac{f(u)\dot u}{(1-v\bar u')^n}=\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\substack{u\bar u'=1\\|1-v\bar u'|>\varepsilon}}\frac{f(u)\dot u}{(1-v\bar u')^n}.$$

After that, Koranyi and Vagi[4] obtained the same results by generalized Cayley transformation, this is equivalent to the Plemelj formula of generalized upper half plane

$$D=\left\{z=(z_1, \cdots, z_n): \operatorname{Im} z_1-\sum_{k=2}^{n}|z_k|^2>0\right\}.$$

On the other hand, using the Plemelj formulas of the sphere, we may obtain the Plemelj formulas of the matrix hyperbolic space and the Lie sphere hyperbolic space[5,6]. Shi Jihuai[7] also proved the following result

$$\lim_{z\to v}\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1}\frac{f(u)\dot{u}}{(1-z\bar{u}')^n}=2^{n-2}f(v)+\text{p. v. }\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1(\mathrm{Im})}\frac{f(u)\dot{u}}{(1-v\bar{u}')^n},$$

where p. v. $\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1(\mathrm{Im})}$ is defined by $\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\substack{u\bar{u}'=1\\|\mathrm{Im}(v\bar{u}')|>\varepsilon}}$.

If $u\bar{u}'=1$, $v\bar{v}'=1$, we have the following singular kernels

$$H(v, u)=(1-v\bar{u}')^{-n} \quad \text{(Cauchy kernel)},$$
$$B(v, u)=H(v, u)+H(u, v)-1 \quad \text{(B kernel)},$$
$$h(v, u)=\frac{1}{i}(H(v, u)-H(u, v)) \quad \text{(Hilbert kernel)}.$$

By these singular kernels, we may get the following singular integral operators on sphere

$$\mathbf{H}\varphi=2\omega_{2n-1}^{-1}\int_{u\bar{u}'=1}\varphi(u)H(v, u)\dot{u},$$

$$\mathbf{B}\varphi=2\omega_{2n-1}^{-1}\int_{u\bar{u}'=1}\varphi(u)B(v, u)\dot{u},$$

$$\mathbf{h}\varphi=2\omega_{2n-1}^{-1}\int_{u\bar{u}'=1}\varphi(u)h(v, u)\dot{u},$$

where the principal value is defined by $\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\substack{u\bar{u}'=1\\|1-v\bar{u}'|>\varepsilon}}$. If $\varphi\in\operatorname{Lip}p$ $(0<p\leqslant 1)$, these singular integrals exist. From them we may also obtain the generalized Poincaré-Bertrand formula ($\mathbf{H}^2=I$), the generalized Hilbert formula of Hilbert kernel and solve the linear singular integral equations or the systems of linear singular integral equation with Cauchy kernel, B kernel and Hilbert kernel, etc[8,9]. It is worth mentioning that B kernel does not appear in one varirable. This shows the difference between one and several variables.

0.3. Main results of this paper

Pseudoconvex domains are the domains of holomorphy. For the strictly pseudoconvex domains, we can construct the analytic Henkin-Ramirez (H-R) kernel and Stein-Kerzman (S-K) kernel.

Alt[13] in 1974, Kerzman and Stein[14] in 1978 proved respectively the plemelj formula for the H-R kernel and S-K kernel. They proved: If Ω is C^∞ smooth, bounded strictly pseudoconvex domain, $u\in C^\infty(b\Omega)$, $H(w, z)$ is H-R kernel or S-K kernel

$$\mathbf{H}u(w)=\int_{z\in b\Omega} H(w, z)u(z)d\sigma_z, \quad w\in\Omega.$$

Then when w approaches $b\Omega$ along the nontangential direction

$$\mathbf{H}u(w)=\frac{1}{2}u(w)+\text{p. v.}\int_{z\in b\Omega} H(w, z)u(z)d\sigma_z \quad (w\in b\Omega) \tag{0.9}$$

holds, where p. v. $\int_{z\in b\Omega}$ is defined by $\lim_{\varepsilon\to 0}\int_{z\in b\Omega-B(w,\varepsilon)}$, and

$$B(w, \varepsilon)=\{z\in b\Omega, \ |g(z, w)|<\varepsilon\}.$$

In the section 1 of this paper, we consider that the deleted neighborhood $B(w, \varepsilon)$ around w in the boundary of a strictly pseudoconvex domain is a more general form instead of above form, a corresponding plemelj formula is obtained. This shows the essential difference between one and several variables: There is only one method to define the principal value of the Cauchy integral in the former, but there are infinite many methods in the latter, even if the deleted part is symmetry. Basing upon these results, we discuss the singular integral on the boundary of a strictly pseudoconvex domain.

In the sections 2 and 3, we discuss some special cases of $B(w, \varepsilon)$ and the theory of the corresponding singular integral equations. In the section 2, we consider the situation that the deleted neighborhood $B(w, \varepsilon)$ is an "ellipse" or a "belt" when the "ellipse" is a "disc", we obtain the results of Alt and Kerzman-Stein. We also discover that there is a method of deletion, such that the term $u(w)$ does not appear in the Plemelj formula, in other words, when w approaches the boundary from the interior, the Cauchy integral approaches a special principal value. In the section 3, we consider the situation that the deleted neighborhood is a "rectangle". All these show the variety of the definition of the principal value of the Cauchy integral.

Part results of this paper were announced in [10].

§1. Cauchy integrals on a strictly pseudoconvex domain

1.1 Henkin-Ramirez kernel and Stein-Kerzman kernel

Suppose Ω is a smooth, bounded strictly pseudoconvex domain, λ is a real function, $\lambda\in C^\infty(\overline{\Omega})$, $\lambda(z)<0$, if $z\in\Omega$; $\lambda(z)=0$, if $z\in b\Omega$; $\lambda(z)>0$, if $z\overline{\in}\Omega$; grad $\lambda(z)\neq 0$, if $z\in b\Omega$ and

$$\left(\frac{\partial^2\lambda}{\partial z_i\partial\bar{z}_i}\right)\geqslant CI, \tag{1.1.1}$$

where the constant $C>0$ is independent of $z\in\overline{\Omega}$, I is the identity matrix. For $z\in b\Omega$, $w\in\Omega$ and w near z, set

$$g_i^L(w, z)=\frac{\partial\lambda}{\partial z_i}(z)+\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2\lambda}{\partial z_i\partial z_j}(z)(w_j-z_j), \tag{1.1.2}$$

$$g^L(w, z)=\sum_{i=1}^{n}(z_i-w_i)g_i^L(w, z).$$

Extend the local $g^L(w, z)$ to a global $g(w, z)$: for w near z, set

$$g(w, z)=g^L(w, z)\phi(w, z),$$

where $\phi(w, z)$ is a locally defined function and is holomorphic in w for w near z. Furthermore $\phi(z, z)\neq 0$. The global $g_i(w, z)$ are obtained from the division problem

$$g(w, z)=\sum(z_i-w_i)g_i(w, z). \tag{1.1.3}$$

It may be proved that there are the functions $g_i(w, z)$ which are holomorphic in w such that (1.1.3) holds. Substitute such $g_i(w, z)$ into $g_i(w, z)$ which are needed in the C-F kernel (0.6) of Ω and substitute $g_i(w, z)$ of (1.1.3) into g of (0.4), we obtain the C-F kernel $H(w, z)$ of Ω, this is just the Henkin-Ramirez (H-R) kernel. As for the details of proof, see [11, 15].

Stein and kerzman[14] constructed another kernel $E+C$, called Stein-Kerzman (S-K) kernel, where $E(w, z)\,d\sigma_z$ is a C-F form (0.4), in which $g_i(w, z)$ are totally explicit, but are holomorphic only when w is close to $z\in b\Omega$, namely as w near z, $g_i(w, z)=g_i^L(w, z)$ and $C(w, z)$ is the correction term, such that $H(w, z)=E(w, z)+C(w, z)$ is holomorphic in w globally. Furthermore

$$C(w, z)\in C^\infty(U(\bar{\Omega})\times V(b\Omega)),$$

where $U(\bar{\Omega})$ and $V(b\Omega)$ are the neighborhoods of $\bar{\Omega}$ and $b\Omega$ respectively. That is to say, $C(w, z)$ is infinitely smooth even when $w=z$. $C(w, z)$ is a solution of a $\bar{\partial}$ problem. $C(w, z)$ is not a C-F form, but $E+C$ can reproduce holomorphic functions. As for the details, see [14].

It is known that for H-R kernel and S-K kernel, we both have

$$\int_{z\in b\Omega}|\mathrm{H}(w, z)|\,d\sigma_z=\infty,\quad w\in b\Omega, \tag{1.1.4}$$

where $d\sigma_z$ is the Lebesgue element of area on $b\Omega$.

Fix $w\in b\Omega$, without loss of generality, take $w=0$, then we have the following

Theorem 1.1.[12,14] *Suppose Ω is a smooth, bounded strictly pseudoconvex domain, $0\in b\Omega$, $H(w, z)$ is the H-R kernel or S-K kernel. Then near $z=0$, there is a holomorphic local change of variables, such that $b\Omega$ and $H(0, z)$ have the following forms: $z\in b\Omega$ if and only if $z_n=t+i\tau$ satisfies*

$$\tau=|\xi|^2+\varepsilon(\xi, t), \tag{1.1.5}$$

where $z=(z_1, \cdots, z_n)$, $\xi=(z_1, \cdots, z_{n-1})$, the error term ε is of third order, namely

$$|\varepsilon(\xi, t)|=O(\rho^3), \tag{1.1.6}$$

where $\rho^4=|\xi|^4+t^2$. For $H(0, z)$, we have

$$H(0, z)=\frac{\gamma_n}{(|\xi|^2+it)^n}+\phi(z), \tag{1.1.7}$$

where $\phi(z)$ is absolutely integrable in $z\in b\Omega$ and γ_n is a constant.

Obviously $(\xi, t)\in \mathbf{C}^{n-1}\times\mathbf{R}$ and $\tau=|\xi|^2$ is "Heisenberg group" surface S_{n-1}.

(1.1.5) shows that near $z=0$ the $b\Omega$ may be approximated by the "Heisenberg group" surface S_{n-1} and the $\varepsilon(\xi, t)$ of (1.1.6) is the error of approximation.

It is easy to know[12,13]

$$\int_{\rho<1} |(|\xi|^2+it)|^{-n} d\mu=\infty, \tag{1.1.8}$$

$$\int_{0<a<\rho<b} |(|\xi|^2+it)^{-n} d\mu=0, \tag{1.1.9}$$

where $d\mu$ is the usual Lebesgue measure $dx_1 dy_1 \cdots dx_{n-1} dy_{n-1} dt$ $(z_j=x_j+iy_j)$. In fact, we may prove

$$\frac{1}{g^n(0, z)}=\frac{1}{(|\xi|^2+it)^n}(1+O(\rho)). \tag{1.1.10}$$

1.2. General plemelj formula

We first prove the following general Plemelj formula.

Theorem 1.2. *Suppose Ω is a smooth, bounded strictly pseudoconvex domain, $H(w, z)$ is the H-R kernel or S-K kernel, $u\in C^\infty(b\Omega)$,*

$$\mathbf{H}u(w)\equiv\int_{z\in b\Omega} H(w, z)u(z)d\sigma_z, \quad w\in\Omega \tag{1.2.1}$$

is the Cauchy integral, where $d\sigma_z$ is the Lebesgue element of area on $b\Omega$. Then $\mathbf{H}u$ is holomorphic in Ω and admits a continous extension up to $\overline{\Omega}$. When $w\in b\Omega$, define

$$\text{p. v.}\int_{z\in b\Omega} H(w, z)u(z)d\sigma_z\equiv\lim_{\substack{\varepsilon\to0\\ \varepsilon>0}}\int_{z\in b\Omega-D(w,\varepsilon)\cap b\Omega} H(w, z)u(z)d\sigma_z, \tag{1.2.2}$$

here $D(w, \varepsilon)$ is the neighborhood around w and contracts to the point w as $\varepsilon\to0$. If

$$\lim_{\varepsilon\to0}\lim_{\delta\to0}\int_{z\in D(w,\varepsilon)\cap b\Omega} H(w+\delta\nu, z)d\sigma_z=a \tag{1.2.3}$$

exists, where a is a constant and ν is the inner normal to $b\Omega$ at w, $\delta>0$. Denote the value of (1.2.2) by $\mathbf{H}_a u$ and

$$\mathbf{H}u(w)=\lim_{w_0\to w}\mathbf{H}u(w_0),$$

where w_0 approaches $w\in b\Omega$ along the nontangential direction from the interior of Ω. Then we have the Plemelj formula

$$\mathbf{H}u(w)=au(w)+\mathbf{H}_a u(w). \tag{1.2.4}$$

Obviously, this deduces to the Theorem of Alt[13] and Kerzman-Stein[14] when $D(w, \varepsilon)$ is $\{z\in b\Omega, |g(z, w)|<\varepsilon\}$. In this case, $a=\frac{1}{2}$ and Plemelj formula becomes

$$\mathbf{H}u(w)=\frac{1}{2}u(w)+\mathrm{H}_{\frac{1}{2}}u(w). \tag{1.2.5}$$

We now prove Theorem 1.2.

Keep $w\in b\Omega$ fixed, say $w=0$. Then the condition (1.2.3) becomes

$$\lim_{\varepsilon\to0}\lim_{\delta\to0}\int_{z\in D(0,\varepsilon)\cap b\Omega} H(0+\delta\nu, z)d\sigma_z=a, \tag{1.2.6}$$

where ν is the inner normal to $b\Omega$ at $w=0$. Write

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{z\in b\Omega-D(0,\varepsilon)\cap b\Omega} H(0+\delta\nu,\ z)u(z)d\sigma_z$$
$$=\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{z\in b\Omega-D(0,\varepsilon)\cap b\Omega} H(0+\delta\nu,\ z)[u(z)-u(0)]d\sigma_z$$
$$+u(0)\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{z\in b\Omega-D(0,\varepsilon)\cap b\Omega} H(0+\delta\nu,\ z)d\sigma_z,$$

since $u\in C^\infty(b\Omega)$, the above first integral exists[13], and the second integral is equal to $u(0)(1-a)$ *by* (1.2.6). Therefore, for $0\in b\Omega$, we have

$$\text{p. v.}\int_{z\in b\Omega} H(0,\ z)u(z)d\sigma_z=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{z\in b\Omega-D(0,\varepsilon)\cap b\Omega} H(0,\ z)u(z)d\sigma_z$$
$$=\int_{z\in b\Omega} H(0,\ z)[u(z)-u(0)]d\sigma_z+u(0)(1-a).$$

Hence, for $w\in b\Omega$, we also have

$$\text{p. v.}\int_{z\in b\Omega} H(w,\ z)u(z)d\sigma_z=\int_{z\in b\Omega} H(w,\ z)[u(z)-u(w)]d\sigma_z+u(w)(1-a). \tag{1.2.7}$$

Let $w_0\in\Omega$

$$\int_{z\in b\Omega} H(w_0,\ z)u(z)d\sigma_z=\int_{z\in b\Omega} H(w_0,\ z)[u(z)-u(w)]d\sigma_z$$
$$+u(w)\int_{z\in b\Omega} H(w_0,\ z)d\sigma_z=I_1+I_2.$$

When w_0 approaches w along the nontangential direction, by (1.2.7) we obtain

$$\lim_{w_0\to w} I_1=\int_{z\in b\Omega} H(w,\ z)[u(z)-u(w)]d\sigma_z$$
$$=\text{p. v.}\int_{z\in b\Omega} H(w,\ z)u(z)d\sigma_z-u(w)(1-a),$$

hence

$$\mathbf{H}u(w)=\text{p. v.}\int_{z\in b\Omega} H(w,\ z)u(z)d\sigma_z-u(w)(1-a)+u(w)$$
$$=\text{p. v.}\int_{z\in b\Omega} H(w,\ z)u(z)d\sigma_z+au(w).$$

This ends the proof.

Notice the condition (1.2.3) By Theorem 1.1, there is a holomorphic local change of variables, such that $b\Omega$ and $H(0,\ z)$ can be represented by (1.1.5) and (1.1.7) respectively near $0\in b\Omega$ and (1.2.6) becomes[13, 14]

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\left\{\int_{z\in\bar D(0,\varepsilon)}\gamma_n(|\xi|^2+\delta+it)^{-n}d\mu+\int_{z\in D(0,\varepsilon)}\phi_\delta(z)d\mu\right\},$$

where $\phi_\delta(z)$ is absolutely integrable in $z\in b\Omega$ and $\lim_{\delta\to 0}\phi_\delta(z)=\phi(z)$. Since $D(0,\ \varepsilon)$ contracts to a point as $\varepsilon\to 0$, obviously

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{z\in D(0,\varepsilon)}\phi_\delta(z)d\mu=0,$$

therefore the condition (1.2.3) becomes

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{z\in D(0,\varepsilon)}\gamma_n(|\xi|^2+\delta+it)^{-n}d\mu=a.$$

Consider generalized Cayley transformation T

$$z_1=\frac{u_1}{1+u_n},\ \cdots,\ z_{n-1}=\frac{u_{n-1}}{1+u_n},\ z_n=i\frac{1-u_n}{1+u_n}.$$

Since $z_n=t+i\tau$, hence

$$t=\frac{2\operatorname{Im}u_n}{|1+u_n|^2},\quad \tau=\frac{1-|u_n|^2}{|1+u_n|^2},$$

T transforms $|\xi|^2+it+\delta$ into

$$\frac{|u_1|^2+\cdots+|u_{n-1}|^2+2i\operatorname{Im}u_n}{|1+u_n|^2}+\delta.$$

Since T maps upper half plane $\tau>0$, $\tau=|\xi|^2$ onto $u\bar{u}'=1$, the above can also be written as

$$\frac{1+2i\operatorname{Im}u_n-|u_n|^2}{|1+u_n|^2}+\delta=\frac{(1+u_n)(1-\bar{u}_n)}{|1+u_n|^2}+\delta=\frac{1-\bar{u}_n}{1+\bar{u}_n}+\delta=\frac{(1+\delta)-(1-\delta)\bar{u}_n}{1+\bar{u}_n}$$

and $d\mu=\omega_{2n-1}^{-1}(1+\bar{u}_n)^{-n}\dot{u}$.

So the condition (1.2.6) becomes

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\frac{1}{\omega_{2n-1}}\int_{z\in D(p_n,\varepsilon)}\frac{\dot{u}}{(1+\delta)^n\left(1-\frac{1-\delta}{1+\delta}\bar{u}_n\right)^n}=\lim_{\varepsilon\to 0}\lim_{\rho\to 1}\frac{1}{\omega_{2n-1}}\int_{z\in D(p_n,\varepsilon)}\frac{\dot{u}}{(1-\rho\bar{u}_n)^n}$$

$$=\lim_{\varepsilon\to 0}\lim_{\rho\to 1}\frac{1}{\omega_{2n-1}}\int_{z\in D(p_n,\varepsilon)}\frac{\dot{u}}{(1-\rho p_n\bar{u}')^n}=a, \tag{1.2.8}$$

where $p_n=(0,\cdots,0,1)$, $D(p_n,\varepsilon)=T(D(0,\varepsilon))$ is the neighborhood around p_n, $\rho=\frac{1-\delta}{1+\delta}$.

Let us observe the $D(w,\varepsilon)$ in detail. In Theorem 1.1, when changing the variables, we set $g(z,0)=-it+\tau=-iz_n$. Hence, if the definition of $D(w,\varepsilon)$ depends on $g(z,w)$, $D(w,\varepsilon)$ may be written as $D(w,\varepsilon;\operatorname{Re}g,\operatorname{Im}g)$ it becomes $D(0,\varepsilon,\operatorname{Re}g(z,0),\operatorname{Im}g(z,0))$, i. e. $D(0,\varepsilon,\tau,-t)$ after the change of variables and becomes

$$D\left(p_n,\ \varepsilon,\ \frac{1-|u_n|^2}{|1+u_n|^2},\ \frac{-2\operatorname{Im}u_n}{|1+u_n|^2}\right)$$

under the generalized transformation of Cayley.

For example

$$D(w,\varepsilon,\operatorname{Re}g,\operatorname{Im}g)=\{\alpha^2(\operatorname{Re}g)^2+\beta^2(\operatorname{Im}g)^2\leqslant\varepsilon^2\}$$

may be reduced to

$$\{\alpha^2(1-|u_n|^2)^2+4\beta^2(\operatorname{Im}u_n)^2\leqslant\varepsilon^2\};$$

$$D(w,\varepsilon,\operatorname{Re}g,\operatorname{Im}g)=\{|\operatorname{Re}g|<\alpha\varepsilon,\ |\operatorname{Im}g|<\beta\varepsilon,\ \alpha>0,\ \beta>0\}$$

may be reduced to

$$\{1-|u_n|^2<\alpha\varepsilon,\ 2|\operatorname{Im}u_n|<\beta\varepsilon,\ \alpha>0,\ \beta>0\}.$$

In the sections 2 and 3, we shall give some concrete neighborhoods $D(w,\varepsilon)$; calculate the value of a in (1.2.8), one can discover some interesting results.

1.3. Singular integrals on $b\Omega$

When $w\in b\Omega$, $\int_{z\in b\Omega} H(w, z)u(z)d\sigma_z$ is a singular integral.

Let $z\in b\Omega$, $\varphi(z)\in \mathrm{Lip}\,p$, $0<p\leqslant 1$, v, $w\in b\Omega$, set

$$\varphi_1(w)=\int_{b\Omega(a)} H(w, z)\varphi(z)d\sigma_z,\quad \varphi_2(v)=\int_{b\Omega(b)} H(v, z)\varphi_1(z)d\sigma_z,$$

where $\int_{b\Omega(a)}$ stands for the Cauchy principal value, the deleted neighborhood is $D_a(w, \varepsilon)$, such that the corresponding value of (1.2.3) is a.

Consider the Cauchy integrals

$$f(\eta)=\int_{b\Omega} H(\eta, z)\varphi(z)d\sigma_z,\ f_1(\eta)=\int_{b\Omega} H(\eta, z)\varphi_1(z)d\sigma_z,\ \eta\in\Omega,$$

when η approaches $\zeta\in b\Omega$ along the nontangential direction, by (1.2.4)

$$f(\zeta)=\int_{b\Omega(a)} H(\zeta, z)\varphi(z)d\sigma_z+a\varphi(\zeta),\quad f_1(\zeta)=\int_{b\Omega(b)} H(\zeta, z)\varphi_1(z)d\sigma_z+b\varphi_1(\zeta).$$

By the definition of $\varphi_1(\zeta)$, $\varphi_2(\zeta)$, we have

$$\varphi_1(\zeta)=f(\zeta)-a\varphi(\zeta),\ \varphi_2(\zeta)=f_1(\zeta)-b\varphi_1(\zeta),\ \zeta\in b\Omega.$$

Substitute $\varphi_1(\zeta)$ into the expression of $f_1(\eta)$

$$f_1(\eta)=\int_{b\Omega(b)} H(\eta, z)f(z)d\sigma_z-a\int_{b\Omega(b)} H(\eta, z)\varphi(z)d\sigma_z=(1-a)f(\eta),$$

therefore

$$\varphi_2(\zeta)=(1-a)f(\zeta)-b[f(\zeta)-a\varphi(\zeta)]=(1-a-b)f(\zeta)+ab\varphi(\zeta).$$

For a, if there is a neighborhood $D_b(w, \varepsilon)$ around w, such that the corresponding value of (1.2.3) is b and $1-a-b=0$, then $\varphi_2(\zeta)=ab\varphi(\zeta)$, that is to say

$$\mathbf{H}_b\mathbf{H}_a=abI.$$

where I stands for the identity operator. In the same manner, we can also prove

$$\mathbf{H}_a\mathbf{H}_b=abI.$$

Thus we obtain the generalized Poincarè-Bertrand formula.

Theorem 1.3. *Let Ω be a smooth, bounded strictly pseudoconvex domain of $\mathbf{C}^n$, $H(w, z)$ is H-R kernel or S-K kernel, $\varphi\in C^\infty(b\Omega)$. The singular integral operator $\mathbf{H}_a$ is defined by (1.2.2). If for a, there is another definition of Cauchy principal value, such that the value of (1.2.3) is b, and $1-a-b=0$. Suppose the corresponding singular integral operator is $\mathbf{H}_b$, then*

$$\mathbf{H}_a\mathbf{H}_b=\mathbf{H}_b\mathbf{H}_a=abI, \tag{1.3.1}$$

where I stands for the identity operator.

If $ab\neq 0$, (1.3.1) may also be written as: denote $\frac{1}{a}\mathbf{H}_a\varphi=\psi$, then $\frac{1}{b}\mathbf{H}_b\psi=\varphi$; or denote $\frac{1}{b}\mathrm{H}_b\varphi=\psi$, then $\frac{1}{a}\mathrm{H}_a\psi=\varphi$.

It is easy to know by the operator theory, if λ is not a characteristic value of the singular integral operator $\mathbf{H}_a$, $\mathbf{K}$ is a continuous operator, $f\in C^\infty(b\Omega)$, then the singular integral equation on $b\Omega$

$$(-\lambda I+\mathbf{H}_a+\mathbf{K})\varphi=\psi \tag{1.3.2}$$

has exactly one solution, namely it can be normalized to a Fredholm integral equation. When $D(w, \varepsilon)=\{|g(w, z)|<\varepsilon\}$, Ω is a ball, this is just the theory of the singular integral equation on a sphere[8,9].

When $z\in b\Omega$, $w\in b\Omega$, the H-R kernel or S-K kernel $H(w, z)$ of Ω are both singular kernels. For the case of sphere, one can define B kernel and Hilbert kernel from $H(w, z)$

$$B(w, z)=H(w, z)+H(z, w)-1,$$

$$h(w, z)=\frac{1}{i}[H(w, z)-H(z, w)].$$

From these kernels, we also have

$$\mathbf{B}_a\varphi=\int_{b\Omega(a)} B(w, z)\varphi(z)d\sigma_z,$$

$$\mathbf{h}_a\varphi=\int_{b\Omega(a)} h(w, z)\varphi(z)d\sigma_z,$$

where the integrals denote the Cauchy principal value, the deleted neighborhood around w is $D_a(w, \varepsilon)$ such that the value of (1.2.3) is a.

Obviously, if $\mathbf{H}_a$ has an inverse operator, then $\mathbf{B}_a$ and $\mathbf{h}_a$ also have the inverse operators. Therefore, we can also solve the singular integral equations on $b\Omega$

$$(-\mu I+\mathbf{B}_a+\mathbf{K})\varphi=f \tag{1.3.3}$$

and

$$(-\nu I+\mathbf{h}_a+\mathbf{K})\varphi=f, \tag{1.3.4}$$

where μ, ν are not the characteristic value of $\mathbf{B}_a$ and $\mathbf{h}_a$ respectively, $\mathbf{K}$ is a continuous operator, $f\in C^\infty(b\Omega)$.

As in [8, 9], one can solve the systems of singular integral equations similar to (1.3.2), (1.3.3), (1.3.4) and when $D(w, \varepsilon)=\{|g(w, \varepsilon)|<\varepsilon\}$, Ω is a ball, this is just the theory of singular integral equations discussed in [8].

§ 2. The case that the neighborhood is an "ellipse"

2.1. Plemelj formula

In this section, we shall consider some special neighborhoods $D(w, \varepsilon)$, such that the concrete Plemelj formulas are obtained from the general Plemelj formula. We first consider the case that the neighborhood $D_e(w, \varepsilon)$ is an "ellipse":

$$\{z\in b\Omega,\ \alpha^2(\operatorname{Re} g)^2+\beta^2(\operatorname{Im} g)^2\leqslant\varepsilon^2\},$$

where $\alpha\geqslant 0$, $\beta\geqslant 0$, $\alpha+\beta\neq 0$. When $\alpha=\beta$, the "ellipse" becomes a "disc", this is just the case discussed by Alt[13] and Kerzman-Stein[14]. In this case, we shall prove that (1.2.3) becomes

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{z\in D_e(w,\varepsilon)} H(w+\delta\nu,\ z)d\sigma_z=\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1}. \tag{2.1.1}$$

That is to say, the value of a in Plemelj formula (1.2.4) is

$$a=\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1}.$$

When $\alpha=\beta$, this is just the Theorem of Alt[13] and Kerzman-Stein[14]. Therefore, this result has generalized the works of Alt and Kerzman-Stein.

Another case $\beta=0$ must be noted. In this case, $a=0$, it follows that the Plemelj formula becomes vary simple

$$\mathbf{H}u(w)=\mathbf{H}_0u(w).$$

In other words, when w_0 approaches $w\in b\Omega$ along the nontangential direction from the interior of Ω

$$\lim_{w_0\to w}\int_{z\in b\Omega} H(w,\ z)u(z)d\sigma_z=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{z\in b\Omega-D(w,\varepsilon)\cap b\Omega} H(w,\ z)u(z)d\sigma_z.$$

Now $D(w,\ z)$ is a "belt"

$$\{z\in b\Omega,\ |\mathrm{Re}\, g|<\varepsilon\}.$$

Namely, one can find a manner to define the principal value of Cauchy integral, such that this principal value is just the limit value of the Cauchy integral $\mathbf{H}u(w_0)$ when w_0 approaches $w\in b\Omega$ along the nontangential direction from the interior of Ω. It is impossible in one variable that the limit value of the Cauchy integral on $b\Omega$ can be represented by a certain principal value of the Cauchy integral on $b\Omega$.

We now discuss the conditions under which the singular integral operator $\mathbf{H}_a$ has an inverse operator. It has been pointed out in the section 1.3 that if one can find an operator $\mathbf{H}_b$, such that $1-a-b=0$, $ab\neq 0$, then $\frac{1}{b}\mathbf{H}_b$ is the inverse operator of $\frac{1}{a}\mathbf{H}_a$. Obviously, if

$$\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1}<1,$$

i. e.

$$\frac{\alpha}{\beta}>2^{\frac{n-2}{n-1}}-1, \tag{2.1.2}$$

take $$D_e(w,\ \varepsilon)=\{z\in b\Omega,\ \alpha'^2(\mathrm{Re}\, g)^2+\beta'^2(\mathrm{Im}\, g)^2\leqslant\varepsilon^2\},$$

where $\alpha'\geqslant 0$, $\beta'\geqslant 0$, $\alpha'+\beta'\neq 0$ and

$$\frac{\alpha'}{\beta'}=\left[2^{2-n}-\left(\frac{\alpha+\beta}{\beta}\right)^{1-n}\right]^{-\frac{1}{n-1}}-1,$$

if denote $$\frac{1}{2}\left(\frac{2\beta'}{\alpha'+\beta'}\right)^{n-1}=b,$$

then $\frac{1}{ab}\mathbf{H}_b$ is the inverse operator of $\mathbf{H}_a$. That is to say, under the condition of (2.1.2), $\mathbf{H}_a$ has an inverse operator. In the same way, we may set up the theory of singnlar integral equations.

2.2. The proof of (2.1.1)

To prnve (2.1.1), it is enough to prove that (1.2.8) holds by § 1.2. Namely

$$\lim_{\varepsilon\to 0}\lim_{\rho\to 1}\frac{1}{\omega_{2n-1}}\int_{u\in D(p_n,\varepsilon)}\frac{\dot{u}}{(1-\rho p_n\bar{u}')^n}=\lim_{\varepsilon\to 0}\lim_{\rho\to 1}\frac{1}{\omega_{2n-1}}\int_{u\in D(p_n,\varepsilon)}\frac{\dot{u}}{(1-\rho\bar{u}_n)^n}$$
$$=\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1},$$

where $p_n=(0,\cdots,0,1)$, $0<\rho<1$, and

$$D(p_n,\varepsilon)=\{u\bar{u}'=1,\ \alpha^2(1-|u_n|^2)^2+4\beta^2(\operatorname{Im}u_n)^2\leqslant\varepsilon^2\}.$$

Let
$$\tilde{D}(p_n,\varepsilon)=\{u\bar{u}'=1,\ \alpha^2(1-|u_n|^2)^2+4\beta^2(\operatorname{Im}u_n)^2>\varepsilon^2\}.$$

Since
$$\frac{1}{\omega_{2n-1}}\int_{u\in D(p_n,\varepsilon)}\frac{\dot{u}}{(1-\rho\bar{u}_n)^n}+\frac{1}{\omega_{2n-1}}\int_{u\in\tilde{D}(p_n,\varepsilon)}\frac{\dot{u}}{(1-\rho\bar{u}_n)^n}=1,$$

we only need to prove

$$\lim_{\varepsilon\to 0}\lim_{\rho\to 1}\frac{1}{\omega_{2n-1}}\int_{u\in\tilde{D}(p_n,\varepsilon)}\frac{\dot{u}}{(1-\rho\bar{u}_n)^n}=\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{u\in\tilde{D}(p_n,\varepsilon)}\frac{\dot{u}}{(1-\bar{u}_n)^n}$$
$$=1-\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1},\tag{2.2.1}$$

where $\alpha\geqslant 0$, $\beta\geqslant 0$, $\alpha+\beta\neq 0$.

when $\alpha=0$, $\beta>0$, (2.2.1) has been proved in[7].

We now consider the case $\alpha>0$, $\beta>0$.

As has been discussed in[3, 7], let

$$\bar{u}_n=re^{i\theta},\quad v=(u_1,\cdots,u_{n-1}),$$

tnen $\tilde{D}(p_n,\varepsilon)$ may be witten in form

$$\begin{cases} v\bar{v}'=1-r^2,\\ \alpha^2(1-r^2)^2+4\beta^2r^2\sin^2\theta>\varepsilon^2.\end{cases}$$

Set
$$c=\arcsin\frac{\sqrt{\varepsilon^2-\alpha^2(1-r^2)^2}}{2\beta r},$$

we have

$$\frac{1}{\omega_{2n-1}}\int_{u\in\tilde{D}(p_n,\varepsilon)}\frac{\dot{u}}{(1-\bar{u}_n)^n}=\frac{1}{\omega_{2n-1}}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}\dot{v}\left\{\int_{-(\pi-c)}^{-c}\frac{d\theta}{(1-re^{i\theta})^n}+\int_c^{\pi-c}\frac{d\theta}{(1-re^{i\theta})^n}\right\}$$
$$+\frac{1}{\omega_{2n-1}}\int_{v\bar{v}'\geqslant\frac{\varepsilon}{\alpha}}\dot{v}\int_{-\pi}^{\pi}\frac{d\theta}{(1-re^{i\theta})^n}=I_1+I_2.\tag{2.2.2}$$

It is known by[3, 7]

$$\int_{-(\pi-c)}^{-c}\frac{d\theta}{(1-re^{i\theta})^n}+\int_c^{\pi-c}\frac{d\theta}{(1-re^{i\theta})^n}$$
$$=2\operatorname{Im}\left\{\sum_{k=1}^{n-1}\frac{1}{k(1+re^{-ic})^k}-\sum_{k=1}^{n-1}\frac{1}{k(1-re^{ic})^k}+\log\frac{1-re^{ic}}{1+re^{-ic}}\right\}+2(\pi-2c),$$

hence
$$I_1=\frac{2}{\omega_{2n-1}}\operatorname{Im}\left\{\sum_{k=1}^{n-1}J_k-\sum_{k=1}^{n-1}H_k+J_0-H_0\right\}+\frac{2}{\omega_{2n-1}}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}(\pi-2c)\dot{v},$$

where
$$J_0=\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}\log\frac{1}{1+re^{-ic}}\dot{v},\quad J_k=\int_{vv'<\frac{\varepsilon}{\alpha}}\frac{\dot{v}}{k(1+re^{-ic})^k},\quad k=1,2,\cdots,n-1,$$

$$H_0=\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}} \log\frac{1}{1-re^{ic}}\dot{v},\quad H_k=\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}} \frac{\dot{v}}{k(1-re^{ic})^k},\quad k=1,\ 2,\ \cdots,\ n-1.$$

Since $\quad \sin c=\dfrac{\sqrt{\varepsilon^2-\alpha^2(1-r^2)^2}}{2\beta r},\quad \cos c=\dfrac{\sqrt{4\beta^2r^2-\varepsilon^2+\alpha^2(1-r^2)^2}}{2\beta r},$

so $\quad 1+re^{-ic}=1+\dfrac{\sqrt{4\beta^2(1-s^2)+\alpha^2s^4-\varepsilon^2}}{2\beta}-i\dfrac{\sqrt{\varepsilon^2-\alpha^2s^4}}{2\beta},$

here $s^2=1-r^2$. Let $v=(x_1,\ x_2,\ \cdots,\ x_{2n-2})$, adopt the spherical polar coorolinates

$$x_1=s\cos\varphi_1,\ x_2=s\sin\varphi_1\cos\varphi_2,\ \cdots,\ x_{2n-2}=s\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{2n-3},$$

then $\quad \dot{v}=s^{2n-3}\sin^{2n-4}\varphi_1\sin^{2n-5}\varphi_2\cdots\sin\varphi_{2n-4}ds d\varphi_1\cdots d\varphi_{2n-3}.$

It follows that

$$J_k=\frac{2\pi^{n-1}}{k\Gamma(n-1)}\int_0^{\sqrt{\frac{\varepsilon}{\alpha}}}\left\{1+\frac{\sqrt{4\beta^2(1-s^2)+\alpha^2s^4-\varepsilon^2}}{2\beta}-i\frac{\sqrt{\varepsilon^2-\alpha^2s^4}}{2\beta}\right\}^{-k}s^{2n-3}ds.$$

Let $\eta=\dfrac{\varepsilon}{\alpha}$, $s=\sqrt{\eta}\,t$, then J_k is equal to

$$\frac{2\pi^{n-1}(2\beta)^k}{k\Gamma(n-1)}\eta^{n-1}\int_0^1\frac{t^{2n-3}\,dt}{[2\beta+\sqrt{4\beta^2(1-\eta t^2)+\alpha^2\eta^2(t^4-1)}-i\sqrt{\alpha^2\eta^2(1-t^4)}]^k},$$

since the absolute value of the integrand is bounded, hence

$$\lim_{\varepsilon\to0}J_k=0,\ k=1,\ 2,\ \cdots,\ n-1.$$

We may prove $\lim\limits_{\varepsilon\to0}J_0=0$ in the same manner.

As we do for J_k, H_k may be written as

$$H_k=\frac{2\pi^{n-1}(2\beta)^k}{k\Gamma(n-1)}\int_0^1\frac{\eta^{n-1}t^{2n-3}\,dt}{[2\beta-\sqrt{4\beta^2(1-\eta t^2)+\alpha^2\eta^2(t^4-1)}-i\sqrt{\alpha^2\eta^2(1-t^4)}]^k}.$$

When $k<n-1$, the absolute value of the integrand is not greater than

$$\frac{M}{\beta^2t^4+\alpha^2(1-t^4)-1},$$

where M is an absolute constant, it is integrable in $[0,\ 1]$, so

$$\lim_{\varepsilon\to0}H_k=0,\ k=1,\ 2,\ \cdots,\ n-2.$$

When $k=n-1$, we have

$$H_{n-1}=\frac{2(2\beta\pi)^{n-1}}{\Gamma(n)}\int_0^1\frac{\eta^{n-1}t^{2n-3}\,dt}{[2\beta-\sqrt{4\beta^2(1-\eta t^2)+\alpha^2\eta^2(t^4-1)}-i\sqrt{\alpha^2\eta^2(1-t^4)}]^{n-1}}$$

$$=\frac{2(2\beta\pi)^{n-1}}{\Gamma(n)}\int_0^1\frac{t^{2n-3}\,dt}{[\beta t^2-i\alpha\sqrt{1-t^4}+O(\eta)]^{n-1}}.$$

Set $\gamma=\dfrac{\alpha}{\beta}$, then

$$\lim_{\varepsilon\to0}H_{n-1}=\frac{2^n\pi^{n-1}}{\Gamma(n)}\int_0^1\frac{t^{2n-3}\,dt}{(t^2-i\gamma\sqrt{1-t^4})^{n-1}}.$$

Let $t^2=\cos\theta$

$$\lim_{\varepsilon\to0} H_{n-1}=\frac{(2\pi)^{n-1}}{\Gamma(n)}\int_0^{\frac{\pi}{2}}\frac{\cos^{n-2}\theta\sin\theta\,d\theta}{(\cos\theta-i\gamma\sin\theta)^{n-1}}$$

$$=\frac{(2\pi)^{n-1}}{\Gamma(n)(1+\gamma)^{n-1}}\int_0^{\frac{\pi}{2}}\frac{(e^{2i\theta}+1)^{n-2}(e^{2i\theta}-1)}{\left(1+\frac{1-\gamma}{1+\gamma}e^{2i\theta}\right)^{n-1}}d\theta$$

$$=\frac{(2\pi)^{n-1}}{\Gamma(n)(1+\gamma)^{n-1}}\int_0^{\frac{\pi}{2}}\left\{\frac{1}{i}\sum_{\substack{p=0\\p+q\neq0}}^{n-2}\sum_{q=0}^{\infty}(-1)^qC_p^{n-2}C_q^{n+q-2}\left(\frac{1-\gamma}{1+\gamma}\right)^q\right.$$

$$\left.\times\left[e^{2i\theta(p+q+1)}-e^{2i\theta(p+q)}\right]+\frac{1}{i}(e^{2i\theta}-1)\right\}d\theta$$

$$=-\frac{(2\pi)^{n-1}}{2\Gamma(n)(1+\gamma)^{n-1}}\sum_{\substack{p=0\\p+q\neq0}}^{n-2}\sum_{p\neq0}^{\infty}(-1)^qC_p^{n-2}C_q^{n+q-2}\left(\frac{1-\gamma}{1+\gamma}\right)^q$$

$$\times\left[\frac{(-1)^{p+q+1}-1}{p+q+1}-\frac{(-1)^{p+q}-1}{p+q}\right]+\frac{(2\pi)^{n-1}}{\Gamma(n)(1+\gamma)^{n-1}}\left(1-\frac{\pi}{2i}\right).$$

It follows that

$$\lim_{\varepsilon\to0}\operatorname{Im}(H_{n-1})=\operatorname{Im}(\lim_{\varepsilon\to0}H_{n-1})=\frac{2^{n-2}\pi^n}{\Gamma(n)(1+\gamma)^{n-1}}.$$

We now calculate H_0

$$H_0=\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}\log\frac{1}{1-re^{ic}}\dot{v}=-\frac{2\pi^{n-1}}{\Gamma(n-1)}\int_0^{\sqrt{\frac{\varepsilon}{\alpha}}}s^{2n-3}\log(1-re^{ic})ds.$$

Since $\operatorname{Im}\log(1-re^{ic})=\arg(1-re^{ic})=O(1),$

hence when $n>1$

$$\operatorname{Im}(H_0)=O\left(\int_0^{\sqrt{\frac{\varepsilon}{\alpha}}}s^{2n-3}ds\right)=O(\varepsilon^{n-1})=o(1)\quad(\varepsilon\to0).$$

By the same reason, when $n>1$, we have

$$\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}(\pi-2c)\dot{v}=o(1).$$

Sum up the above results, we obtain

$$\lim_{\varepsilon\to0}I_1=-\frac{1}{2}\left(\frac{2}{1+\gamma}\right)^{n-1}=-\frac{1}{2}\left(\frac{2\beta}{\alpha+\beta}\right)^{n-1}.$$

As to I_2, by virtue of

$$\int_{-\pi}^{\pi}\frac{d\theta}{(1-re^{i\theta})^n}=2\operatorname{Im}\left\{\sum_{k=1}^{n-1}\frac{1}{k(1+r)^k}+\sum_{k=1}^{n-1}\frac{1}{k(1-r)^k}\right\}+2\pi=2\pi,$$

$$\lim_{\varepsilon\to0}I_2=\frac{2\pi}{\omega_{2n-1}}\int_{0<v\bar{v}'<1}\dot{v}=1.$$

Substitute thses results into (2.2.2), we obtain (2.2.1) immediately.

Finally, we consider the case $\alpha>0$, $\beta=0$.

(2.2.1) now becomes

$$\lim_{\varepsilon\to0}\frac{1}{\omega_{2n-1}}\int_{u\in\tilde{D}(p_n,\varepsilon)}\frac{\dot{u}}{(1-\bar{u}_n)^n}=1,$$

where $\tilde{D}(p_n,\varepsilon)=\left\{u\bar{u}'=1,\ 1-|u_n|^2>\frac{\varepsilon}{\alpha}\right\}.$

using the above transformations, one may write $\tilde{D}(p_n, \varepsilon)$ as

$$\begin{cases} vv'=1-r^2, \\ 1-r^2>\dfrac{\varepsilon}{\alpha}, \end{cases}$$

then
$$\frac{1}{\omega_{2n-1}}\int_{u\in\tilde{D}(p_n,\varepsilon)}\frac{\dot{u}}{(1-\bar{u}_n)^n}=\frac{1}{\omega_{2n-1}}\int_{\frac{\varepsilon}{\alpha}<v\bar{v}'<1}\dot{v}\int_{-\pi}^{\pi}\frac{d\theta}{(1-re^{i\theta})^n}.$$

This is just the integral I_2 discussed before, but it is known $\lim\limits_{\varepsilon\to 0} I_2=1$, so (2.2.1) holds when $\alpha>0$, $\beta=0$.

§ 3. The case that the neighborhood is a "rectangle"

3.1. Plemelj formula

Now let the neighborhood $D_R(w, \varepsilon)$ around w be a "rectangle"

$$\{z\in b\Omega,\ |\mathrm{Re}\, g|<\alpha\varepsilon,\ |\mathrm{Im}\, g|<\beta\},$$

here $\alpha>0$, $\beta>0$. In this case, we shall prove that (1.2.3) becomes

$$\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\int_{z\in D_R(w,\varepsilon)} H(w+\delta\nu,\ z)d\sigma_z=\frac{2^{n-1}}{\pi}\left\{\frac{\pi}{2}-h_n\left(\mathrm{arc\,tg}\,\frac{\beta}{\alpha}\right)\right\}, \tag{3.1.1}$$

where
$$h_n(x)=\int_0^x \cos^{n-2} t\,\frac{\sin(n-1)t}{\sin t}dt,$$

and the value of a in Plemelj formula (1.2.4) is

$$a=\frac{2^{n-1}}{\pi}\left\{\frac{\pi}{2}-h_n\left(\mathrm{arc\,tg}\,\frac{\beta}{\alpha}\right)\right\}. \tag{3.1.2}$$

In particular when $\alpha=\beta$, i. e. the rectangle becomes a square,

$$a=\frac{2^{n-1}}{\pi}\left\{\frac{\pi}{2}-h_n\left(\frac{\pi}{4}\right)\right\}.$$

When $\alpha=\infty$, $D_R(w, \varepsilon)$ becomes

$$\{z\in b\Omega,\ |\mathrm{Im}\, g|<\beta\varepsilon\},$$

this is just the case discussed in the preceding section. By virtue of $h_n(0)=0$, so $a=2^{n-2}$, this coincides with the result of the preceding section.

When $\beta=\infty$, $D_R(w, \varepsilon)$ becomes

$$\{z\in b\Omega,\ |\mathrm{Re}\, g|<\alpha\varepsilon\},$$

this is the case discussed in the preceding section too. We shall see in § 3.2

$$h_n\left(\mathrm{arc\,tg}\,\frac{\beta}{\alpha}\right)=\mathrm{arc\,tg}\,\frac{\beta}{\alpha}+\sum_{k=1}^{n-2}\frac{\alpha^k}{k(\alpha^2+\beta^2)^{k/2}}\sin\left(k\,\mathrm{arc\,tg}\,\frac{\beta}{\alpha}\right),$$

therefore, when $\beta\to\infty$ $\mathrm{arc\,tg}\,\frac{\beta}{\alpha}\to\frac{\pi}{2}$, so $h_n\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$. It follows that $a=0$, this coincides with the result of the preceding section too.

As the preceding section, for the singular integal operator $\mathbf{H}_a$, if one can find an operator $\mathbf{H}_b$, such that $1-a-b=0$, $ab\neq 0$ then $\frac{1}{b}\mathbf{H}_b$ is the inverse of $\frac{1}{a}\mathbf{H}_a$.

It is not hard to prove $h_n(x)\geqslant 0$ for arbitrary positive integer n and $x\in\left[0, \frac{\pi}{2}\right]$, and

$$\max_{0<x<\frac{\pi}{2}} h_n(x)=h_n\left(\frac{\pi}{n-1}\right).$$

If α, β satisfy

$$(1-2^{1-n})\pi-h_n\left(\frac{\pi}{n-1}\right)\leqslant h_n\left(\operatorname{arc\,tg}\frac{\beta}{\alpha}\right)\leqslant(1-2^{1-n})\pi,$$

then $\mathbf{H}_a$ (a is defined by (3.1.2)) has an inverse operator. That is to say, we only choose $D_R(w, \varepsilon)$ as

$$\{z\in b\Omega,\ |\operatorname{Re} g|<\alpha'\varepsilon,\ |\operatorname{Im} g|<\beta'\varepsilon\},$$

where $\alpha'>0$, $\beta'>0$, such that

$$h_n\left(\operatorname{arc\,tg}\frac{\beta'}{\alpha'}\right)=(1-2^{1-n})\pi-h_n\left(\operatorname{arc\,tg}\frac{\beta}{\alpha}\right).$$

Set

$$b=\frac{2^{n-1}}{\pi}\left\{\frac{\pi}{2}-h_n\left(\operatorname{arc\,tg}\frac{\beta'}{\alpha'}\right)\right\},$$

then $\frac{1}{b}\mathbf{H}_b$ is the inverse of $\frac{1}{a}\mathbf{H}_a$. It follows that under the condition (3.1.3), $\mathbf{H}_a$ is invertible. We may also discuss the theory of the singular integral equations in the same way.

3.2. The proof of (3.1.1)

To prove (3.1.1), by the argument of § 1.2, we only need to prove that (1.2.8) holds, i. e.

$$\lim_{\varepsilon\to 0}\lim_{\rho\to 1}\frac{1}{\omega_{2n-1}}\int_{u\in D_R(p_n,\varepsilon)}\frac{\dot{u}}{(1-\rho\bar{u}_n)^n}=\frac{2^{n-1}}{\pi}\left\{\frac{\pi}{2}-h_n\left(\operatorname{arc\,tg}\frac{\beta}{\alpha}\right)\right\},$$

where $p_n=(0, \cdots, 0, 1)$, $\alpha>0$, $\beta>0$

$$D_R(p_n, \varepsilon)=\{u\bar{u}'=1,\ 1-|u_n|^2<\alpha\varepsilon,\ 2|\operatorname{Im} u_n|<\beta\varepsilon\}.$$

Let

$$\tilde{D}_R(p_n, \varepsilon)=\{u\bar{u}'=1\}-D_R(p_n, \varepsilon),$$

$$M_\varepsilon=\{u\bar{u}'=1,\ 1-|u_n|^2>\alpha\varepsilon\},$$

$$N_\varepsilon=\{u\bar{u}'=1,\ 2|\operatorname{Im} u_n|>\beta\varepsilon\},$$

$$Q_\varepsilon=\{u\bar{u}'=1,\ 1-|u_n|^2>\alpha\varepsilon,\ 2|\operatorname{Im} u_n|>\beta\varepsilon\}.$$

Thus the result which needs to be proved is equivalent to

$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\tilde{D}_R(p_n,\varepsilon)}\frac{\dot{u}}{(1-\bar{u}_n)^n}=1-\frac{2^{n-1}}{\pi}\left\{\frac{\pi}{2}-h_n\left(\operatorname{arc\,tg}\frac{\beta}{\alpha}\right)\right\}. \tag{3.2.1}$$

But

$$\frac{1}{\omega_{2n-1}}\int_{\tilde{D}_R(p_n,\varepsilon)}\frac{\dot{u}}{(1-\bar{u}_n)^n}=\left\{\frac{1}{\omega_{2n-1}}\int_{M_\varepsilon}+\frac{1}{\omega_{2n-1}}\int_{N_\varepsilon}-\frac{1}{\omega_{2n-1}}\int_{Q_\varepsilon}\right\}\frac{\dot{u}}{(1-\bar{u}_n)^n}. \tag{3.2.2}$$

It is known by § 2.2

$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{M_\varepsilon}\frac{\dot{u}}{(1-\bar{u}_n)^n}=1,\quad \lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{N_\varepsilon}\frac{\dot{u}}{(1-\bar{u}_n)^n}=1-2^{n-2}. \tag{3.2.3}$$

so we only need to calculate

$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{Q_\varepsilon}\frac{\dot{u}}{(1-\bar{u}_n)^n}.$$

As the argument in § 2.2, set

$$\bar{u}_n=re^{i\theta},\ v=(u_1,\ \cdots,\ u_{n-1}),$$

then Q_ε may be written as

$$\begin{cases} v\bar{v}'=1-r^2, \\ 1-r^2>\alpha\varepsilon,\ 2r|\sin\theta|>\beta\varepsilon, \end{cases}$$

$$\frac{1}{\omega_{2n-1}}\int_{Q_\varepsilon}\frac{\dot{u}}{(1-\bar{u}_n)^n}=\frac{1}{\omega_{2n-1}}\int_{P_\varepsilon}\dot{v}\left\{\int_{-(\pi-c)}^{-c}+\int_c^{\pi-c}\right\}\frac{d\theta}{(1-re^{i\theta})^n},$$

where $c=\arcsin\frac{\beta\varepsilon}{2r}$ and

$$P_\varepsilon=\left\{v=(u_1,\ \cdots,\ u_{n-1}),\ \alpha\varepsilon<v\bar{v}'<1-\left(\frac{\beta\varepsilon}{2}\right)^2\right\}.$$

By [3, 7], the above equality may be written as

$$\frac{1}{\omega_{2n-1}}\int_{Q_\varepsilon}\frac{\dot{u}}{(1-\bar{u}_n)^n}=\frac{2}{\omega_{2n-1}}\operatorname{Im}\left\{\sum_{k=1}^{n-1}J_k-\sum_{k=1}^{n-1}H_k+J_0-H_0\right\}+\frac{1}{\omega_{2n-1}}\int_{P_\varepsilon}(\pi-2c)\dot{v},$$

here $J_0=\int_{P_\varepsilon}\log\frac{1}{1+re^{-ic}}\dot{v},\ J_k=\int_{P_\varepsilon}\frac{\dot{v}}{k(1+re^{-ic})^k},\ k=1,\ 2,\ \cdots,\ n-1,$

$$H_0=\int_{P_\varepsilon}\log\frac{1}{1-re^{ic}}\dot{v},\ H_k=\int_{P_\varepsilon}\frac{\dot{v}}{k(1-re^{ic})^k},\ k=1,\ 2,\ \cdots,\ n-1.$$

Use the spherical polar coordinates

$$\int_{P_\varepsilon}c\dot{v}=\int_{P_\varepsilon}\arcsin\frac{\beta\varepsilon}{2\sqrt{1-v\bar{v}'}}\dot{v}=\frac{2\pi^{n-1}}{\Gamma(n-1)}\int_{\sqrt{\alpha\varepsilon}}^{\sqrt{1-\left(\frac{\beta\varepsilon}{2}\right)^2}}s^{2n-3}\arcsin\frac{\beta\varepsilon}{2\sqrt{1-s^2}}ds$$

$$=\frac{2\pi^{n-1}}{\Gamma(n-1)}\left\{\int_0^1\left[1-\left(\frac{\beta\varepsilon}{2}\right)^2\right]^{n-1}\arcsin\frac{\beta\varepsilon}{2\sqrt{1-\left[1-\left(\frac{\beta\varepsilon}{2}\right)^2\right]t^2}}dt\right.$$

$$\left.-\int_0^1(\alpha\varepsilon)^{n-1}\arcsin\frac{\beta\varepsilon}{2\sqrt{1-\alpha\varepsilon t^2}}dt\right\}.$$

Since the integrands are bounded,

$$\lim_{\varepsilon\to 0}\int_{P_\varepsilon}c\dot{v}=0.$$

It follows that

$$\lim_{\varepsilon\to 0}\frac{2}{\omega_{2n-1}}\int_{P_\varepsilon}(\pi-2c)\dot{v}=1.$$

By virtue of $\qquad 1+re^{-ic}=1+\frac{1}{2}\sqrt{4r^2-\beta^2\varepsilon^2}-i\frac{1}{2}\beta\varepsilon,$

hence

$$J_k=\frac{2\pi^{n-1}}{k\Gamma(n-1)}\int_{\sqrt{\alpha\varepsilon}}^{\sqrt{1-\left(\frac{\beta\varepsilon}{2}\right)^2}}s^{2n-3}\left[1+\frac{1}{2}\sqrt{4(1-s^2)-\beta^2\varepsilon^2}-\frac{i}{2}\beta\varepsilon\right]^{-k}ds$$

$$=\frac{2\pi^{n-1}}{k\Gamma(n-1)}\int_0^1\left[1-\left(\frac{\beta\varepsilon}{2}\right)^2\right]^{n-1}t^{2n-3}\left[1+\frac{1}{2}\sqrt{4\left[1-\left(1-\frac{1}{4}\beta^2\varepsilon^2\right)t^2\right]-\beta^2\varepsilon^2}\right.$$

$$\left.-\frac{i}{2}\beta\varepsilon\right]^{-k}dt-\frac{2\pi^{n-1}}{k\Gamma(n-1)}\int_0^1(\alpha\varepsilon)^{n-1}t^{2n-3}\left[1+\frac{1}{2}\sqrt{4(1-\alpha\varepsilon t^2)-\beta^2\varepsilon^2}-\frac{i}{2}\beta\varepsilon\right]^{-k}dt,$$

the integrands of these two integrals are all bounded, so

$$\lim_{\varepsilon\to 0} J_k=\frac{2\pi^{n-1}}{k\Gamma(n-1)}\int_0^1\frac{t^{2n-3}}{(1+\sqrt{1-t^2})^k}dt,$$

it is a real number, hence

$$\lim_{\varepsilon\to 0}\mathrm{Im}(J_k)=0,\ k=1,\ 2,\ \cdots,\ n-1.$$

We may prove in the same way

$$\lim_{\varepsilon\to 0}\mathrm{Im}(J_0)=0,\ \lim_{\varepsilon\to 0}\mathrm{Im}(H_0)=0.$$

As the calculation of J_k, H_k may be written as

$$H_k=\frac{2\pi^{n-1}}{k\Gamma(n-1)}\left\{\int_0^1\left(1-\frac{1}{4}\beta^2\varepsilon^2\right)^{n-1}t^{2n-3}\left[1-\sqrt{1-\left(1-\frac{1}{4}\beta^2\varepsilon^2\right)t^2-\frac{1}{4}\beta^2\varepsilon^2}\right.\right.$$

$$\left.\left.-\frac{i}{2}\beta\varepsilon\right]^{-k}dt-\int_0^1(\alpha\varepsilon)^{n-1}t^{2n-3}\left[1-\sqrt{1-\alpha\varepsilon t^2-\frac{1}{4}\beta^2\varepsilon^2}-\frac{i}{2}\beta\varepsilon\right]^{-k}dt\right\}$$

$$=\frac{2\pi^{n-1}}{k\Gamma(n-1)}(X_k-Y_k).$$

The absolute value of the integrand of the first integral is not greater than $\frac{t^{2n-3}}{(1-\sqrt{1-t^2})^k}$, it is an integrable function on the interval $[0,\ 1]$ when $k<n-1$, so we have

$$\lim_{\varepsilon\to 0}X_k=\int_0^1\frac{t^{2n-3}}{(1-\sqrt{1-t^2})^k}dt,$$

it is a real number. The absolute value of the integrand of the second integral is $O(\varepsilon^{n-k-1})$, so

$$\lim_{\varepsilon\to 0}Y_k=0,$$

as $k<n-1$. It follows that when $k<n-1$

$$\lim_{\varepsilon\to 0}\mathrm{Im}(H_k)=0.$$

Finally, we calculate H_{n-1}

$$H_{n-1}=\frac{2\pi^{n-1}}{\Gamma(n)}(X_{n-1}-Y_{n-1}),$$

where

$$X_{n-1}=\left(1-\frac{1}{4}\beta^2\varepsilon^2\right)^{n-1}\int_0^1 t^{2n-3}\left[1-\sqrt{\left(1-\frac{1}{4}\beta^2\varepsilon^2\right)(1-t^2)}-\frac{i}{2}\beta\varepsilon\right]^{-(n-1)}dt,$$

$$Y_{n-1}=(\alpha\varepsilon)^{n-1}\int_0^1 t^{2n-3}\left[1-\sqrt{1-\alpha\varepsilon t^3-\frac{1}{4}\beta^2\varepsilon^2}-\frac{i}{2}\beta\varepsilon\right]^{-(n-1)}dt.$$

Let $1-\frac{1}{4}\beta^2\varepsilon^2=\eta^2$, we have

$$X_{n-1}=\eta^{2n-2}\int_0^1 t^{2n-3}[1-\eta\sqrt{1-t^2}-i\sqrt{1-\eta^2}]^{-(n-1)}dt,$$

by [7], we obtain

$$\lim_{\varepsilon\to 0}\mathrm{Im}(X_{n-1})=2^{n-3}\pi.$$

Since $$1-\sqrt{1-\alpha\varepsilon t^2-\frac{1}{4}\beta^2\varepsilon^2}-\frac{i}{2}\beta\varepsilon=\left(\frac{1}{2}\alpha t^2-\frac{i}{2}\beta\right)\varepsilon+O(\varepsilon^2),$$
we have
$$\lim_{\varepsilon\to 0}Y_{n-1}=2^{n-1}\int_0^1\frac{t^{2n-3}\,dt}{\left(\frac{1}{2}\alpha t^2-\frac{i}{2}\beta\right)^{n-1}}=2^{n-1}\int_0^1\frac{t^{2n-3}\,dt}{(t^2-i\gamma)^{n-1}}=2^{n-2}\int_0^1\frac{x^{n-2}\,dx}{(x-i\gamma)^{n-1}}$$
$$=2^{n-2}\left\{\log\frac{1-i\gamma}{-i\gamma}-D_n+\sum_{k=1}^{n-2}C_k^{n-2}\frac{(-1)^k}{k}\right\},$$
where $\gamma=\frac{\beta}{\alpha}$, $D_n=\sum_{k=1}^{n-2}C_k^{n-2}\frac{1}{k}\left(\frac{i\gamma}{1-i\gamma}\right)^k$ $(n\geqslant 3)$. It follows that
$$\lim_{\varepsilon\to 0}\operatorname{Im}(Y_{n-1})=2^{n-2}\left[\operatorname{arc\,tg}\frac{1}{\gamma}-\operatorname{Im}(D_n)\right]\quad(n\geqslant 3)\tag{3.2.4}$$
when $n=2$ $$\lim_{\varepsilon\to 0}\operatorname{Im}(Y_1)=2\operatorname{Im}\left(\int_0^1\frac{t\,dt}{t^2-i\gamma}\right)=\operatorname{arc\,tg}\frac{1}{\gamma}.$$
If define $D_2=0$, then (3.2.4) holds too when $n\geqslant 2$.

We now have
$$\lim_{\varepsilon\to 0}\operatorname{Im}(H_{n-1})=\frac{2^{n-2}\pi^n}{\Gamma(n)}\left\{1-\frac{2}{\pi}\operatorname{arc\,tg}\frac{1}{\gamma}+\frac{2}{\pi}\operatorname{Im}(D_n)\right\},$$
here $n\geqslant 2$. Consequently
$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{Q_\varepsilon}\frac{\dot{u}}{(1-\bar{u}_n)^n}=1-2^{n-2}\left(1-\frac{2}{\pi}\operatorname{arc\,tg}\frac{1}{\gamma}+\frac{2}{\pi}\operatorname{Im}(D_n)\right).\tag{3.2.5}$$
By (3.2.2), (3.2.3), (3.2.5), we obtain
$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{D_\varepsilon(p_n,\varepsilon)}\frac{\dot{u}}{(1-\bar{u}_n)^n}=1-\frac{2^{n-1}}{\pi}\left(\operatorname{arc\,tg}\frac{1}{\gamma}-\operatorname{Im}(D_n)\right).\tag{3.2.6}$$
It is not hard to prove
$$\operatorname{Im}(D_n)=-\operatorname{arc\,tg}\frac{\beta}{\alpha}+h_n\left(\operatorname{arc\,tg}\frac{\beta}{\alpha}\right).$$
Substitute it into (3.2.6), this completes the proof of (3.1.1).

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多复变数的奇异积分(I)——强拟凸域的 Henkin 积分

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摘 要

对于多复变数强拟凸域的 Henkin-Ramirez 核或 Stein-Kerzman 核所定义的 Cauchy 型积分，本文指出：可以有多种形式的 Plemelj 公式，甚至 Cauchy 型积分的极限值可以等于某种 Cauchy 主值，这些都显示了多复变数函数与单复变数函数本质上的不同.