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ON A CLASS OF MIXED PARTIAL DIFFERENTIAL EQUATIONS OF HIGHER ORDER

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Dedicated to Professor Su Bu-chin on the Occasion of his 80th Birthday and his 50th Year of Educational Work

I. Introduction

To the author's knowledge the existing results for mixed partial differential equations involve equations of 2nd order only. The theory of mixed equations of higher order is to be developed.

Busemann, A.[1], starting from the wave equation in three variables, derived a special mixed equation of second order in two variables. This equation has been applied to gasdynamics extensively[2,3]. Hua, L. K. also obtained the same equation from differential geometry and discussed various boundary value problrms[4]. Using the theory of positive symmetric systems[5,6], Gu, C. H. considered more general equations in n variables and obtained a large class of well-posed boundary value problems[7]. The method can be used to treat much more general equations of second order, including some quasilinear equations[8,9,10]. Some new phenomena have been found. Hong, J. X. considered in detail the equations whose degenerate surface is characteristic[11]. On the basis of the appaoach in [8, 9], Sun, L. X. obtained some results on a class of equations with non-characteristic degenerate surface[12]. The results stated above mainly concerned the existence of C^r solutions with $r\geqslant 2$, whereas many other papers on mixed equations in several variables considered only the existence of weak solutions or strong solutions[13].

The resluts in [7] can also be obtained through the properties of the wave equation without using the theory of symmetric positive systems. In the present paper we extend this approach to the equations of higher order, solve two kinds of boundary problems and consider the existence and uniqueness of C^∞ solutions. The results in [7, 14] have been completely generalized to the cases of higher order. As a continuation of the present work, Hong, J. X. obtained some further results on mixed equations of higher order. However, his work cannot cover the results obtained here,

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since the related hyperbolic equations in the present paper may have multiple characteristics.

II. A class of mixed equations of higher order

Let $(y, t)=(y_1, \cdots, y_n, t)$ be the coordinates of the point in R^{n+1} and $(\xi, \tau)=(\xi_1, \cdots, \xi_n, \tau)$ be their dual variables. Thus

$$\xi\cdot y+\tau t=0 \tag{1}$$

is an n-dimensional subspace of R^{n+1}, provided $(\xi, \tau)\neq(0, \tau)$. Let

$$P(\xi, \tau)=\sum_{j=0}^{m} P_{m-j}(\xi)\tau^j \tag{2}$$

be a homogeneous hyperbolic polynomial of m-th degree. Here $P_{m-j}(\xi)$ are homogeneous polynomials of degree $m-j$ with real and constant coefficients. Moreover, (2) is hyperbolic with respect to $(\xi, \tau)=(0, 1)$, i. e., as an equation of τ

$$P(\xi, \tau)=0 \tag{3}$$

admits real roots only. No loss of generality, we suppose that $P(0, 1)=1$. Corresponding to (2) we have the hyperbolic equation

$$P(\partial_y, \partial_t)u(y, t)=F(y, t). \tag{4}$$

In particular, if u is a homogeneous function of degree $a+1$ defined on the half space $t>0$ of R^{n+1}

$$u=t^{a+1}\varphi\left(\frac{y}{t}\right),\ t>0, \tag{5}$$

then $F(y, t)$ must be a homogeneous function of degree $a+1-m$

$$F=t^{a+1-m}f\left(\frac{y}{t}\right). \tag{6}$$

Let $\left(\frac{y_1}{t}, \cdots, \frac{y_n}{t}\right)=(x_1, \cdots, x_n)$. From (4), (5) and (6) we obtain a partial differential equation

$$L(x, \partial_x, a)\varphi=f. \tag{7}$$

Here

$$\begin{aligned} L(x, \partial_x, a)&=\sum_{j=0}^{m} P_{m-j}(\partial_x)\prod_{k=1}^{j}(a+2-k-x\cdot\partial_x)\\ &=\sum_{j=0}^{m}\prod_{k=j}^{1}(a+1-m+k-x\cdot\partial_x)P_{m-j}(\partial_x) \end{aligned} \tag{8}$$

with

$$x\cdot\partial_x=\sum_{i=1}^{n}x_i\frac{\partial}{\partial x_i}.$$

In particular, for the case of

$$P(\xi, \tau)=\xi_1^2+\cdots+\xi_n^2-\tau^2,$$

we obtain the equation considered in [7, 8].

When (3) is satisfied by $(\xi, \tau)\neq(0, \tau)$, (1) is just the characteristic plane of (4). The envelop of the characteristic planes is the characteristic cone Λ, having the origin O as the vertex. When (3) has multiple roots for some $\xi\neq 0$, the structure of Λ may be very complicate[16].

Consider R^n as the plane $t=1$ in R^{n+1} and denote the intersection $\Lambda\cap R^n$ by Σ. Then Σ is the envelop of the family of the planes

$$\xi\cdot x+\tau=0 \quad (P(\xi,\ \tau)=0). \tag{9}$$

Further Σ is the degenerate surface of (7). The number of the real roots of the characteristic equation for the equation (7) is changed when the point x moves across Σ through a regular point of Σ.

For simplicity we denote $L(x,\ \partial_x,\ a)$ by $L(a)$.

Lemma 1. *The characteristic directions of the differential operator $L(a)$ at a point x_0 are the normal directions of planes (9), passing through x_0.*

Proof The vector $\xi\neq 0$ is a characteristic direction of $L(a)$ at x_0, if the characteristic equation for $L(a)$

$$Q(x_0,\ \xi)=\sum_{j=0}^{m}P_{m-j}(\xi)(-\xi\cdot x_0)^j=P(\xi,\ -\xi\cdot x_0)=0 \tag{10}$$

is satisfied. The fact that the plane $\xi\cdot x+\tau=0$ passes through x_0 is equivalent to $\tau=-\xi\cdot x_0$. The conclusion of Lemma 1 follows from (3) and (10) immediately.

Lemma 2. *The principal part of $L(a)$, $Q(x,\ \partial_x)$ is hyperbolic outside the convex hull of Σ.*

Proof The condition of hyperbolicity of $Q(x,\ \partial_x)$ at x_0 is that there exists a nonnull vector $-\eta$ such that

$$Q(x_0,\ \xi-\tau\eta)=P(\xi-\tau\eta,\ -\xi\cdot x+\tau(\eta\cdot x_0))=0, \tag{11}$$

as an equation with an unknown τ, admits only real roots([17], Theorem 5.5.3). Let

$$\Gamma=\{(-\eta,\ \lambda)\,|\,P(-\eta,\ \sigma)>0,\ \forall\sigma\geqslant\lambda\} \tag{12}$$

be a cone in the dual space of R^{n+1}. Evidently for each $-\eta$ we have $(-\eta,\ \lambda)\in\Gamma$ if λ is suffeciently large. It is known that Γ is a convex cone ([17], Theorem 5.5.6) and that

$$P(\xi-\tau\eta,\ \mu+\tau\lambda)=0, \tag{13}$$

as an equation for τ, admits only real roots for each non-null $(\xi,\ \mu)$. Suppose that $x_0\in R^n$ and there is a vector $(-\eta,\ \lambda)$ such tha $(-\eta,\ \eta\cdot x_0)\in\Gamma$.

Comparing (11) with (13), we see that (11) admits only real roots and hence $Q(x,\ \partial_x)$ is hyperblic with respect to η. Here we require that there is a plane $-\eta\cdot x+\lambda=0$ in R^n or a plane $-\eta\cdot y+\lambda\tau=0$ in R^{n+1} such that $(-\eta,\ \lambda)\in\Gamma$ with $\lambda=\eta\cdot x_0$. This means that $(x_0,\ 1)$ lies outside the supporting planes of the dual cone of Γ or x_0 lies outside the convex hull of Σ.

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Lemma 3. *The formal adjoint of $L(a)$ is*

$$L^*(a)=(-1)^m L(-a-3+m-n). \tag{14}$$

Proof From (8) by diret calcuation we obtain

$$L^*(a)=\sum_{j=0}^{m}P^*_{m-j}(\partial_x)\prod_{k=1}^{j}(a+1-m+k+n+x\cdot\partial_x)$$
$$=(-1)^m\sum_{j=0}^{m}P_{m-j}(\partial_x)\prod_{k=1}^{j}(-a-3+m-n+2-k-x\cdot\partial_x)$$
$$=(-1)^mL(-a-3+m-n).$$

Let Ω be a bounded region, containing the convex hull of Σ. Suppose that $\partial\Omega$ is smooth and space-like, i. e., $(-n,\ n\cdot x_0)\in\Gamma$, where n is the normal of $\partial\Omega$.

We consider the following two kinds of boundary value problems.

Problem T_1. To find the solution to equation (7) in Ω such that the boundary value conditions

$$\varphi|_{\partial\Omega}=n\cdot\partial_x\varphi|_{\partial\Omega}=\cdots=(n\cdot\partial_x)^{m-1}\varphi|_{\partial\Omega}=0 \tag{15}$$

are satisfied.

Problem T_2. To find the solution to equation (7) in Ω without any given boundary condition.

III. Analytical Lemmas

Let $H_s(R^n)$ be the functional space with the norm

$$\|\varphi\|_s^2=\int(1+|\xi|^2)^s|\hat\varphi(\xi)|^2d\xi, \tag{16}$$

where $\hat\varphi(\xi)$ is the Fourier transform of φ. Moreover

$$\mathring{H}_s(\Omega)=\{u\,|\,u\in H_s(R^n),\ \operatorname{supp}u\in\bar\Omega\}. \tag{17}$$
$$\mathring{H}^{\mathrm{loc}}_s(\bar R^{n+1}_+)=\{u\,|\,u\in H^{\mathrm{loc}}_s(R^{n+1}),\ \operatorname{supp}u\in\bar R^{n+1}_+\}. \tag{18}$$

Here $\bar R^{n+1}_+$ is the closed half space $t\geqslant0$ of R^{n+1} and $u\in H^{\mathrm{loc}}_s$ means that $\varphi u\in H_s(R^{n+1})$ for each $\varphi\in C_0^\infty(R^{n+1})$.

Let $K_{\bar\Omega}$ be the conical region in $\bar R^{n+1}$

$$K_{\bar\Omega}=\left\{(y,\ t)\,|\,t>0,\ \frac{y}{t}\in\bar\Omega\quad\text{or } t=0,\ y=0\right\} \tag{19}$$

and

$$\Phi_p(y,\ t)=H(t)t^p\varphi\left(\frac{y}{t}\right). \tag{20}$$

Here φ is a function defined on $\bar\Omega$ and $H(t)$ is the Heaviside function.

Lemma 4. (1) *Suppose that* $p>s-\frac{n}{2}-\frac{1}{2}$ $(s\geqslant0)$. $\varphi(x)\in\mathring{H}_s(\bar\Omega)$, *iff* $\Phi_p(y,\ t)\in\mathring{H}^{\mathrm{loc}}_s(\bar R^{n+1}_+)$ *and* $\operatorname{supp}\Phi_p\subset K_{\bar\Omega}$.

(2) *If* $p>-\frac{n}{2}-\frac{1}{2}$, $\varphi(x)\in\mathring{H}_{-s}(\Omega)$ $(s>0)$, *then*

$$\Phi_p(y,\ t)\in\mathring{H}^{\mathrm{loc}}_{-s}(\bar R^{n+1}_+) \tag{21}$$

and

$$\Phi_p(y,\ t)e^{-\varepsilon t}\in H_{-s}(\bar R^{n+1}_+)\quad(\varepsilon>0). \tag{22}$$

Proof (1) Evidently supp $\Phi_p(y, t)\subset K_{\bar\Omega}$ is equivalent to supp $\varphi(x)\subset\bar\Omega$. From

$$\partial_t^j\partial_y^\alpha\left(\varphi\left(\frac{y}{t}\right)t^p\right)=t^{p-|\alpha|-j}\prod_{k=1}^{j}(p-|\alpha|+1-k-x\cdot\partial_x)\partial_x^\alpha\varphi\Big|_{x=\frac{y}{t}}$$

it is seen that

$$\int_0^{t_0}\int_{R^n}|\partial_t^j\partial_y^\alpha\Phi_p(y, t)|^2\,dy\,dt\leqslant Ct_0^{2(p-|\alpha|-j)+n+1}\int_\Omega\sum_{|\beta|\leqslant|\alpha|+j}|\partial_x^\beta\varphi|^2\,dx$$

for each $t_0>0$, if $p>s-\frac{n}{2}-\frac{1}{2}$ and $|\alpha|+j\leqslant s$. Here C is a constant, independent of t_0. This proves the "only if" part.

Conversely, if $\Phi_p(y, t)\in\mathring{H}_s^{\mathrm{loc}}(\bar R_+^{n+1})$ $(s\geqslant0)$, for each $t_0>0$ and $|\alpha|\leqslant s$, we have

$$\infty>\int_0^{t_0}\int_{R^n}|\partial_y^\alpha\Phi_p(y, t)|^2\,dy\,dt=\frac{t_0^{2(p-|\alpha|)+n+1}}{2(p-|\alpha|)+n+1}\int_{R^n}|\partial_x^\alpha\varphi|^2\,dx.$$

Moreover, supp $\varphi(x)\subset\bar\Omega$. Consequently, $\varphi(x)\in\mathring{H}_s(\Omega)$.

(2) From (20) it is seen that

$$\Phi_p(y, t)\in H_{-s}(R_y^n)$$

for each $t>0$. Then from

$$\int_{-\infty}^{t_0}\|\Phi_p(y, t)\|_{-s}^2\,dt=\int_0^{t_0}t^{2(p+n)}dt\int_{R^n}(1+|\xi|^2)^{-s}|\hat\varphi(\xi t)|^2d\xi$$

$$\leqslant\max(1, t_0^{2s})\int_0^{t_0}t^{2p+n}\,dt\cdot\|\varphi(x)\|_{-s}^2$$

we obtain (21). (22) can be obtained by the similar way. Lemma 4 is proved.

Suppose that $P(\xi, \tau)=0$, as an equation of τ, has a root of multiplicity l for some $\xi\neq0$ and has no root of multiplicity $\geqslant l+1$ for any $\xi\neq0$. As usual, let

$$\tilde P^2=\sum_{|\alpha|\geqslant0}|P^{(\alpha)}(\xi, \tau)|^2.$$

Lemma 5. (1) *For each polynomial P mentioned above there is a constcnt C such that*

$$\tilde P^2\geqslant C(1+|\xi|^2+|\tau|^2)^{m-l},\tag{23}$$

where $(\xi, \tau)\in\mathbf{C}^{n+1}$ *and* $C>0$.

(2) *For each polynimial P mentioned above there is a constant C′ such that*

$$|P(\xi, \tau-i\varepsilon)|^2\geqslant C'(\varepsilon)(1+|\xi|^2+|\tau|^2)^{m-l},\tag{24}$$

where $(\xi, \tau)\in R^{n+1}$, $\varepsilon>0$ *and* $C'(\varepsilon)>0$.

Proof (1) For any $(\xi, \tau)\in\mathbf{C}^{n+1}$ with $|\xi|^2+|\tau|^2=1$, all l th derivatives of $P(\xi, \tau)$ cannot vanish simultaneously. Otherwise, $P(\xi, \tau)$ would admit a root with multiplicity $\geqslant l+1$, since $P(\xi, \tau)$ is homogeneous. Hence there is a constant C_1 such that

$$\tilde P^2\geqslant\sum_{|\alpha|=l}|P^{(\alpha)}(\xi, \tau)|^2\geqslant C_1(|\xi|^2+|\tau|^2)^{m-l},$$

if $|\xi|^2+\tau^2\geqslant1$. Moreover, $\sum_{|\alpha|=m}|P^{(\alpha)}|^2$ is a positive constant. Hence we have (23).

(2) By the hyperbolicity of $P(\xi, \tau)$ we have $P(\xi, \tau-i\varepsilon)\neq0$ $(\varepsilon>0)$. Moreover

$$|\tilde P(\xi, \tau-i\varepsilon)|\leqslant C(\varepsilon)|P(\xi, \tau-i\varepsilon)|$$

(see Lemma 4.1.1[17]). Using inequality (23) we obtain (24).

The Cauchy problem for the hyperbolic operator $P(\partial_y, \partial_t)$ has a fundamental solution E which satisfies

$$\widehat{e^{-\varepsilon t}E}=\frac{1}{P(\xi, \tau-i\varepsilon)}, \ \forall \varepsilon>0 \tag{25}$$

and

$$\operatorname{supp} E\subset \Gamma_0=\{(y, t)\mid -\eta\cdot y+\tau t\geqslant 0, \ (-\eta, \tau)\in \Gamma\} \tag{26}$$

([17], Theorem 4.6.3). Moreover

$$E*F\in H^{\text{loc}}_{s+m-l}(\bar{R}^{n+1}_+), \ \forall F\in \mathring{H}^{\text{loc}}_s(\bar{R}^{n+1}_+) \tag{27}$$

is the unique distribution solution of (4), if the support of the solution belongs to $\bar{R}^{n+1}_+$. Here we have utilized Lemma 5(2).

Lemma 6. *Let $\{f_h\}$ be a sequence of distributions such that $f_h\in \mathscr{D}'(R^{n+1})$, supp $f_h\subset \bar{R}^{n+1}_+$. If there is $\varepsilon>0$ such that*

$$e^{-\varepsilon t}f_h\in S'(R^{n+1}) \text{ and } e^{-\varepsilon t}f_h \xrightarrow{S'} e^{-\varepsilon t}f \quad (h\to\infty), \tag{28}$$

then

$$u_h=E*f_h \xrightarrow{\mathscr{D}'} E*f=u \quad (h\to\infty). \tag{29}$$

Here S' is the space set of all temperate distributions[17].

Proof From the properties of the convolution we have

$$e^{-\varepsilon t}u_h=e^{-\varepsilon t}E*e^{-\varepsilon t}f_h. \tag{30}$$

Considering the Fourier transforms of the both sides of (30) and using (25), we obtain

$$\widehat{e^{-\varepsilon t}u_h}=\frac{1}{P(\xi, \tau-i\varepsilon)}\widehat{e^{-\varepsilon t}f_h}. \tag{31}$$

From (24) it is seen that

$$\frac{1}{P(\xi, \tau-i\varepsilon)}\widehat{e^{-\varepsilon t}f_h} \xrightarrow{S'} \frac{1}{P(\xi, \tau-i\varepsilon)}\widehat{e^{-\varepsilon t}f}=\widehat{e^{-\varepsilon t}E*e^{-\varepsilon t}f} \quad (h\to\infty). \tag{32}$$

Since the Fourier transformation is a continuous map from S' to itself, we have

$$e^{-\varepsilon t}u_h \xrightarrow{S'} e^{-\varepsilon t}u \quad (h\to\infty).$$

Consequently, for any $\psi\in C_0^\infty(R^{n+1})$

$$\langle u_h, \psi\rangle=\langle e^{-\varepsilon t}u_h, e^{\varepsilon t}\psi\rangle\to\langle e^{-\varepsilon t}u, e^{\varepsilon t}\psi\rangle=\langle u, \psi\rangle.$$

This proves Lemm 6.

IV. Problems T_1 and T_2

For the problem T_1 we have

Theorem 1. *If $a>m+q-\frac{n}{2}-\frac{3}{2}$ $(q\geqslant 0)$ and $f\in \mathring{H}_s(\bar{\Omega})$ $(s\leqslant q)$, then in the space $\mathring{H}_{s+m-l}(\Omega)$ the problem T_1 has a unique solution which satisfies (7) in the sense of distribution. Moreover*

$$\|\varphi\|_{s+m-l}\leqslant C_s\|f\|_s \quad (C_s=\text{constant}). \tag{33}$$

Proof Firstly, suppose that $f(x)\in C^\infty(R^n)$ and $\operatorname{supp} f(x)\subset\bar{\Omega}$. From Lemma 4 we have

$$F(y,\ t)=H(t)f\left(\frac{y}{t}\right)t^{a+1-m}\in \mathring{H}_q^{\text{loc}}(\overline{R}_+^{n+1}). \tag{34}$$

Hence equation (4) admits a solution

$$u(y,\ t)=E*F(y,\ t)\in \mathring{H}^{\text{loc}}_{q+m-l}(\overline{R}_+^{n+1}). \tag{35}$$

Since $F(y,\ t)$ is a homogeneous function of degree $a+1-m$, $u(\lambda y,\ \lambda t)/\lambda^{a+1}(\lambda>0)$ is a solution of (4) too. Moreover

$$u(\lambda y,\ \lambda t)=\lambda^{a+1}u(y,\ t).$$

Hence there is a function $\varphi(x)$ such that

$$u(y,\ t)=H(t)t^{a+1}\varphi\left(\frac{y}{t}\right).$$

From Lemma 4(1) we have

$$\varphi(x)\in H_{q+m-l}(R^n),$$

since $q+m-l\geqslant 0$. Further, from the properties of the convolution we have

$$\operatorname{supp} u\subset \operatorname{supp} E+\operatorname{supp} F\subset \Gamma_0+K_{\overline{\Omega}}\subset K_{\overline{\Omega}},$$

since $\overline{\Omega}$ contains the convex hull of Σ and $\partial\Omega$ is spacelike. Hence $\varphi(x)\in H_{q+m-l}(\Omega)$ Now we prove that $\varphi(x)$ satisfies (7) in the sense of distribution. In fact, for any $\psi(t)\in C_0^\infty(R_+^1)$ and $V(x)\in C_0^\infty(R^n)$

$$\left\langle Pu,\ \psi(t)V\left(\frac{y}{t}\right)\right\rangle=\langle f(x),\ V(x)\rangle\langle t^{a+1-m+n},\ \psi(t)\rangle$$

holds. On the other hand

$$\left\langle Pu,\ \psi(t)V\left(\frac{y}{t}\right)\right\rangle=\left\langle t^{a+1-m}L(a)\varphi\left(\frac{y}{t}\right),\ \psi(t)V\left(\frac{y}{t}\right)\right\rangle$$

$$=\langle L(a)\varphi,\ V\rangle\langle t^{a+1-m+n},\ \psi(t)\rangle.$$

Choose $\psi(t)$ such that $\langle t^{a+1-m+n},\ \psi(t)\rangle\neq 0$, we have

$$\langle f(x),\ V(x)\rangle=\langle L(a)\varphi(x),\ V(x)\rangle \tag{36}$$

and hence (7) holds in the sense of distribution.

Let $f\in \mathring{H}_s(\Omega)$ $(s\leqslant q)$. There exists a sequence $\{f_h\}$ such that $f_h\in C_0^\infty(\Omega)$ and $f_h\to f$ in $\mathring{H}_s(\Omega)$ as $h\to\infty$. From Lemma 4 we have

$$e^{-st}F_h(y,\ t)\xrightarrow{H_s} e^{-st}F(y,\ t). \tag{37}$$

Since $f_h\in C_0^\infty(\Omega)$, there is $\varphi_h\in \mathring{H}_{q+m-l}(\Omega)$ such that $L(a)\varphi_h=f_h$. Let $u_h=E*F_h=\varphi_h\left(\frac{y}{t}\right)t^{a+1}$. From (36) and Lemma 6 it is seen that

$$u_h\xrightarrow{\mathscr{D}'(R^{n+1})} u=E*F(y,\ t). \tag{38}$$

Hence $\left\langle u_h,\ V\left(\frac{y}{t}\right)\psi(t)\right\rangle=\langle\varphi_h(x),\ V(x)\rangle\langle t^{a+1+n},\ \psi(t)\rangle\to\left\langle u,\ V\left(\frac{y}{t}\right)\psi(t)\right\rangle.$

Consequently $\langle\varphi_h(x),\ V(x)\rangle\to\left\langle u,\ \psi(t)V\left(\frac{y}{t}\right)\right\rangle\Big/\langle t^{a+1+n},\ \psi(t)\rangle$

holds for any $V\in C_0^\infty(R^n)$. Hence there is an element $\varphi(x)\in\mathscr{D}'(R^n)$ defined by

$$\langle\varphi(x),\ V(x)\rangle=\left\langle u,\ \psi(t)V\left(\frac{y}{t}\right)\right\rangle\Big/\langle t^{a+1+n},\ \psi(t)\rangle.$$

It is easily seen that

$$\varphi_h(x)\xrightarrow{\mathscr{D}'}\varphi(x),\ \operatorname{supp}\varphi\subset\overline{\Omega}$$

and

$$L(a)\varphi=\lim_{h\to\infty}L(a)\varphi_h=f.$$

This proves the existence of the solution.

Suppose $\varphi\in\mathring{H}_{s+m-l}(\overline{\Omega})$ is a solution of (7). Then $u=H(t)\varphi\left(\frac{y}{t}\right)t^{a+1}\in\mathscr{D}'(R^{n+1})$ satisfies (4) in the sense of distribuion. From the uniqueness of the solution of Cauchy problem for (4) follows the uniqueness of the solution of problem T_1. Hence $L^{-1}(a)$ is a linear operator from $\mathring{H}_s(\Omega)$ to $\mathring{H}_{s+m-l}(\Omega)$.

Finally, we shall prove the boundedness of $L^{-1}(a)$ by using the closed graph theorem. Suppose that $\{f_h\}$ is a sequence in $\mathring{H}_s(\Omega)$ such that $f_h\to f$ in $\mathring{H}_s(\Omega)$ and $L^{-1}(a)f_h\to\varphi'$ in $\mathring{H}_{s+m-l}(\Omega)$ as $h\to\infty$. Let $\varphi=L^{-1}(a)f$. Repeating the above argument we see that

$$L^{-1}(a)f_h\xrightarrow{\mathscr{D}'}L^{-1}(a)f=\varphi. \tag{39}$$

Hence $\varphi'=\varphi$, i. e., $\{(f,\ L^{-1}(a)f\mid f\in\mathring{H}_s(\Omega)$ is a closed set in $\mathring{H}_s(\Omega)\times\mathring{H}_{s+m-l}(\Omega)$. From the closed graph theorem it follows that $L^{-1}(a)$ is bounded. The proof is completed.

Theorem 2. *If $a<-\frac{n}{2}-\frac{3}{2}-l$ and $f\in H_s(\Omega)$ with $s\geqslant l$, then the problem T_2 has unique solution in H_{s+m-l}.*

Proof From Lemma 3 we see that $L^*(a)=(-1)^mL(a^*)$ with

$$a^*=-a-3+m-n>m+l-\frac{n}{2}-\frac{3}{2}. \tag{40}$$

If $\psi\in H_m(\Omega)$, $L(a)\psi=0$, then the Green formula gives

$$(L(a^*)\varphi,\ \psi)=0$$

for all $\varphi\in\mathring{H}_m(\Omega)$. If $\sigma\in C_0^\infty(\Omega)$, then from Theorem 1 we see that there is a function $\varphi\in\mathring{H}_{2m-l}(\Omega)$ such that $L(a^*)\varphi=\sigma\psi$. Then

$$(\sigma\psi,\ \psi)=0$$

holds for all $\sigma\in C_0^\infty(\Omega)$. Hence $\psi=0$. This proves the uniqueness of the solution.

Let $f\in H_s(\Omega)$. For all $\psi\in C_0^\infty(\Omega)$ we have

$$|(f,\ \psi)|\leqslant\|f\|_s\|\psi\|_{-s}\leqslant c\|f\|_s\|L^*(a)\psi\|_{-s-m+l}. \tag{41}$$

Thus $(\psi,\ f)$ is a bounded linear functional defined on the subspace $\{L^*(a)\psi\mid\psi\in C_0^\infty(\Omega)\}$ of the space $\mathring{H}_{-s+l-m}$. From the Hahn-Banach Theorem it can be extended to a bounded linear functional on $\mathring{H}_{-s+l-m}$. According to the properties of $\mathring{H}_{-s+l-m}$, this functional can be expressed by an element $\varphi\in H_{s+m-l}$, i. e., for any $\psi\in C_0^\infty(\Omega)$

$$(f,\ \psi)=(\varphi,\ L^*(a)\psi)=(L(a)\varphi,\ \psi). \tag{42}$$

This means that φ is the solution in $H_{s+m-l}(\Omega)$. The proof is completed.

Using Theorem 1 and 2, we obtain

Corollary 1. *If* $s\geqslant l+\left[\frac{n}{2}\right]+1$, $a>m+s-\frac{n}{2}-\frac{3}{2}$ *and* $f\in H_s$, *then the problem* T_1 *admits a* $C^m(\bar{\Omega})$ *solution uniquely.*

Corollary 2. *If* $s\geqslant l+\left[\frac{n}{2}\right]+1$, $a<-\frac{n}{2}-\frac{3}{2}-l$ and $f\in H_s$, *then the problem* T_2 *admits a* $C^m(\bar{\Omega})$ *solution uniquely.*

Remark 1. By the fundamental solution E the solution to problem T_1 can be expressed as

$$\varphi(x)=\left(E*t^{a+1-m}f\left(\frac{y}{t}\right)H(t)\right)\Big|_{t=1,\ y=x}. \tag{43}$$

Remark 2. If we consider classical solutions, then the boundary conditions in problem T_1 may be non-homogeneous.

V. C^∞ Solutions

Let $f\in C^\infty(\bar{\Omega})$. we consider the $C^\infty(\bar{\Omega})$ solutions of (7). From Theorem 2 it follows that (7) admits only one $C^\infty(\bar{\Omega})$ solution, if $a<-\frac{n}{2}-\frac{3}{2}-l$. We next discuss the case $a\geqslant-\frac{n}{2}-\frac{3}{2}-l$. For convenience we use the comma as the notation for partial differentiation, e. g.

$$\varphi_{,i_1\cdots i_s}=\frac{\partial^s\varphi}{\partial x_{i_1}\cdots\partial x_{i_s}} \tag{44}$$

$$\varphi_{i_1\cdots i_{s-1},i_s}=\frac{\partial}{\partial x_{i_s}}\varphi_{i_1\cdots i_{s-1}}.$$

Lemma 7. *If* φ *satisfies* (7), *then*

$$L(a-1)\varphi_{,i}=f_{,i}=(L(a)\varphi)_{,i}. \tag{45}$$

Proof Let $\Phi(y,t)=t^{a+1}\varphi\left(\frac{y}{t}\right)$ and let Φ satisfy

$$P(\partial_y,\partial_t)\Phi=t^{a+1-m}f\left(\frac{y}{t}\right).$$

Differentiating with respect to y_i, we obtain

$$P(\partial_y,\partial_t)\left[t^a\varphi_{,i}\left(\frac{y}{t}\right)\right]=t^{a-m}f_{,i}\left(\frac{y}{t}\right).$$

(45) follows from the definition of $L(a-1)$.

Lemma 8. *If we write* $L(a)$ *in the form*

$$L(a)=\sum_{|\alpha|\leqslant m}a_\alpha(x)\partial^\alpha, \tag{46}$$

then

$$a_0(a)=a_{0\cdots 0}(a)=(a+1)\cdot a\cdots(a-m+2). \tag{47}$$

Proof (47) is a direct consequence of (8).

Suppose that φ is a $C^\infty(\bar{\Omega})$ solution of (7). Then the s-th derivatives $\varphi_{,i_1\cdots i_s}$ satisfy

$$L(a-s)\varphi_{,i_1\cdots i_s}=f_{,i_1\cdots i_s}. \tag{48}$$

Conversely, choose s such that $a-s<-\frac{n}{2}-\frac{3}{2}-l$. Then (48) admits one and only one solution $\varphi_{i_1\cdots i_s}\in C^\infty(\bar{\Omega})$. Since $f_{,i_1\cdots i_s}$ are symmetric with respect to the lower indices, the uniqueness of the solution gives that $\varphi_{i_1\cdots i_s}$ are also symmetric with respect to the lower indices. Differentiating (45), we obtain

$$L(a-s-1)\varphi_{i_1\cdots i_s,i_{s+1}}=f_{,i_1\cdots i_{s+1}}. \tag{49}$$

From the above argument, we have

$$\varphi_{i_1\cdots i_s,i_{s+1}}=\varphi_{i_1\cdots i_{s-1}i_{s+1},i_s}. \tag{50}$$

Because $\bar{\Omega}$ is simply connected, there exists a system of functions $\varphi_{i_1\cdots i_{s-1}}$ such that

$$\varphi_{i_1\cdots i_{s-1},i_s}=\varphi_{i_1\cdots i_s}. \tag{51}$$

$\varphi_{i_1\cdots i_{s-1}}$ are determined by (50) except for additional constants. We may choose a system of $\varphi_{i_1\cdots i_{s-1}}$ which are symmetric with respect to the lower indices. From (48) and (45) we obtain

$$(L(a-s+1)\varphi_{i_1\cdots i_{s-1}})_{,i_s}=L(a-s)\varphi_{i_1\cdots i_{s-1},i_s}=f_{,i_1\cdots i_s}$$

and hence

$$L(a-s+1)\varphi_{i_1\cdots i_{s-1}}-f_{,i_1\cdots i_{s-1}}=c_{i_1\cdots i_{s-1}}. \tag{52}$$

Here $c_{i_1\cdots i_{s-1}}$ are constants which are symmetric with respect to the indices. If $a_0(a-s+1)\neq 0$, we can change $\varphi_{i_1\cdots i_{s-1}}$ by adding a system of constants $d_{i_1\cdots i_{s-1}}$ which are determined uniquely such that $\varphi_{i_1\cdots i_{s-1}}$ satisfy the equations

$$L(a-s+1)\varphi_{i_1\cdots i_{s-1}}=f_{,i_1\cdots i_{s-1}}. \tag{53}$$

Using the same procedure successively, we see that if $a+1$ is not a non-negative integer the $C^\infty(\bar{\Omega})$ solution of (7) exists uniquely for any $f\in C^\infty(\bar{\Omega})$.

Now let $a+1$ be a non-negative integer. For example, let $a=-1$. The above procedure gives a system of $C^\infty(\bar{\Omega})$ functions φ_{i_1}, satisfying $L(a-1)\varphi_{i_1}=f_{,i_1}$ and $\varphi_{i_1,i_2}=\varphi_{i_2,i_1}$. Hence there is a $C^\infty(\bar{\Omega})$ function φ such that $L(a)\varphi-f=C$. φ is uniquely determined up to an additional constant. However, the additional constant does not effect the value of the constant C, for $a_0(a)=a_0(-1)=0$. Define the equivalent class of $C^\infty(\bar{\Omega})$ functions $[f(x)]_0$ as

$$[f(x)]_0=\{f(x)+c\,|\,c\in R^1\}.$$

In each class there is one and only one $f(x)$ such that (7) has a $C^\infty(\bar{\Omega})$ solution. If $\varphi(x)$ is a solution, then every function in the class $[\varphi]_0$ is also a solution.

More generally, let $a=-1+k$, where k is an integer and $0\leqslant k\leqslant m-1$. Define an equivalent class of functions

$$[f(x)]_k=\{\tilde{f}(x)\,|\,\tilde{f}(x)\in C^\infty(\bar{\Omega}),\ \tilde{f}(x)-f(x) \text{ is a polynomial of degree } s\leqslant k\}.$$

From the above procedure and

$$a_0(-1)=a_0(0)=\cdots=a_0(-1+k)=0,$$

it is seen that in each class $[f(x)]_k$, there is one and only one function such that (7) admits a $C^\infty(\overline{\Omega})$ solution. Moreover, if φ is a $C^\infty(\overline{\Omega})$ solution, then all functions in $[\varphi]_k$ are solutions.

Let $a=m-2+p$, where p is a positive integer. Using the same argument, we obtain the following results: Let

$$[f(x)]_{p,m}=\{\tilde{f}(x)\,|\,\tilde{f}(x)\in C^\infty(\overline{\Omega}),$$
$$\tilde{f}_{,i_1\cdots i_p}-f_{,i_1\cdots i_p} \text{ are polynomials of degree } s\leqslant m-1\}.$$

In each class $[f(x)]_{p,m}$ there is one and only one function such that (7) admits a $C^\infty(\overline{\Omega})$ solution. Moreover, if $\varphi(x)$ is a $C^\infty(\overline{\Omega})$ solution of (7), then for each set of polynomials $k_{i_1\cdots i_p}$ which are symmetric with respect to $i_1\cdots i_p$ and of degree $<m$, there exists one and only one solution $\tilde{\varphi}(x)$ such that

$$\tilde{\varphi}_{,i_1\cdots i_p}=\varphi_{,i_1\cdots i_p}+k_{i_1\cdots i_p}.$$

The sets $[f(x)]_k$ and $[f(x)]_{p,m}$ have dimension

$$1+C_1^n+\cdots+C_k^{n+k-1}$$

and

$$C_p^{n+p-1}+C_{p+1}^{n+p}+\cdots+C_{p+m-1}^{n+p+m-2}$$

respectively. Thus we have obtained

Theorem 3. *Let $f(x)\in C^\infty(\overline{\Omega})$. If $a+2$ is not a positive integer, then (7) admits one and only one $C^\infty(\overline{\Omega})$ solution. If $a=-1+k$ ($k=0, 1, \cdots, m-1$), then in the class $[f(x)]_k$ there is one and only one function such that (7) admits solutions. Further, if $\varphi(x)$ is a solution, then the whole set of solutions is $[\varphi(x)]_k$. If $a=m-2+p$, then in the class $[f(x)]_{p,m}$ there is one and only one function such that (7) admits $C^\infty(\overline{\Omega})$ solutions. The solutions are determined except for a set of polynomials $k_{i_1\cdots i_p}(x)$ which are symmetric with respect to their indices and of degree $s<m$.*

Remark. In IV we have seen that the problem T_1 admits a solution. However. even for the case $f\in C^\infty(\overline{\Omega})$, the problem T_1 does not admit a $C^\infty(\overline{\Omega})$ solution in general. As pointed in[14], for equation of mixed type the boundary conditions for solution in $C^\infty(\overline{\Omega})$ and for the solution in $C^r(\overline{\Omega})$ ($m\leqslant r<+\infty$) may be quite different.

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一类高阶混合型偏微分方程

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摘 要

就作者所知，高阶(阶数超过 2)的混合型偏微分方程还是一个未曾讨论过的领域. 本文的目的在于讨论一类高阶的混合型方程.

设 $P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n}\right)$ 是齐 m 阶的实常系数的偏微分算子关于 $t=0$ 是双曲的. 定义 R_n 上的微分算子 $L\left(x, \frac{\partial}{\partial x}, a\right)$ 使得

$$P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}\right)\left(t^{a+1}u\left(\frac{y_1}{t}, \cdots, \frac{y_n}{t}\right)\right)=t^{a+1-m}\left[L\left(x, \frac{\partial}{\partial x}, a\right)u(x_1, \cdots, x_n)\right]_{x_i=\frac{y_i}{t}}.$$

这样定义起来的算子 $L\left(x, \frac{\partial}{\partial x}, a\right)$ 是依赖于一个参数 a 的 m 阶混合型算子. 在一个有界闭区域之外，L 为双曲型的. 记 $\bar{\Omega}$ 为一有界闭区域，其边界 $\partial\Omega$ 为充分光滑，落在双曲域之中，又 L 关于 $\partial\Omega$ 为双曲的.

我们考虑了两类的边值问题，它们的提法和参数 a 的数值有关. 主要结果是:

我们考虑了区域 Ω 上算子 $L\left(x, \frac{\partial}{\partial x}, a\right)$ 的两类边值问题，证明了这两类边值问题的适定性，且得到了古典解，同时也讨论了 C^∞ 解的存在性和唯一性.

本文是[7, 14]在高阶混合型方程情形时的推广.