Chin. Ann. of Math.
3 (4) 1982

SOME NONEXISTENCE THEOREMS ON STABLE HARMONIC MAPPINGS

PAN YANGLIAN
(Institute of Mathematics, Fudan University)

Dedicated to Professor Su Bu-chin on the Occasion of his 80th Birthday and his 50th Year of Educational Work

As well known, a harmonic mapping is a critical point of the energy integral Therefore, from the viewpoint of variational calculus, it is natural to study the stable harmonic mapping, i.e. the harmonic mapping with non-negative second variation.

Xin, Y. L.[1] proved the nonexistence of nonconstant stable harmonic mapping from Euclidean sphere S^n $(n>2)$ to any Riemannian manifold, and generalized the result to the case of $S^n\times S^m$ $(n>2,\ m>2)$. All these theorems generalized the results of Eells, J., Sampson J. H. and Smith, R. T.

Noting the fact that S^n is a 1-codimensional submanifold in Euclidean space E^{n+1}, in this paper we study the stable harmonic mapping from a compact submanifold with codimension p in Euclidean space E^{n+p} to any Riemannian manifold. We obtain several nonexistence theorems which generalize the results in[1].

1. Basic formulas Let M and N be Riemannian manifolds with dimensions n and m respectively. M is compact without boundary. And ∇, ∇' represent the Riemannian connections of M, N respectively.

Suppose that ϕ: $M\to N$ is a differentiable mapping, with $\phi_*:TM\to TN$ as its induced mapping, where TM, TN are the tangent bundles of M, N respectively. As known to all, the induced bundle $E=\phi^{-1}TN\to M$ possesses the induced Riemannian connection as follows

$$\tilde{\nabla}_X S=\nabla'_{\phi_* X}S, \tag{1.1}$$

where $X\in TM$, $S\in\Gamma(E)$.

$E(\phi)$, the energy of the mapping ϕ, is defined by the formula

$$E(\phi)=\int_M e(\phi)*1=\frac{1}{2}\int_M\langle\phi_*e_i,\ \phi_*e_i\rangle_N*1, \tag{1.2}$$

where $e(\phi)$ is the energy density, $\{e_i\}$ $(i,\ j,\ k,\ \cdots,\ =1,\ \cdots,\ n)$ an orthonormal basis in M, $\langle,\rangle_N$ the Riemannian metric in N and $*1$ the volume form of M. We follow

Manuscript received May 6, 1981.

the summation convention throughout this paper.

Set

$$\tau(\phi)=\tilde{\nabla}_{e_i}\phi_* e_i-\phi_*\nabla_{e_i}e_i, \tag{1.3}$$

ϕ is called a harmonic mapping if $\tau(\phi)=0$.

For any vector field V along ϕ, we have a 1-parametric family of mappings ϕ_t: $M\to N$, $\phi_t(p)=\exp_{\phi(p)}(tV(p))$. The first variational formula for the correpending energy functional $E(\phi)$ is

$$\frac{d}{dt}E(\phi_t)\big|_{t=0}=-\int_M\langle V,\ \tau(\phi)\rangle_N *1. \tag{1.4}$$

Hence, the harmonic mapping is the critical point of the energy functional.

As known to all, the second variation of the energy functional is given as follows[3]

$$\frac{d^2}{dt^2}E(\phi_t)\big|_{t=0}=-\int_M\langle V,\ \tilde{\nabla}*\tilde{\nabla}V+R^N(\phi_*e_i,\ V)\phi_*e_i\rangle_N *1, \tag{1.5}$$

where $\tilde{\nabla}*\tilde{\nabla}$ is the trace Laplacian with respect to $\tilde{\nabla}$, and R^N is the curvature operator of N.

Therefore, the index form for harmonic mapping is

$$I(V,\ W)=\int_M\langle-\tilde{\nabla}*\tilde{\nabla}V-R^N(\phi_*e_i,\ V)\phi_*e_i,\ W\rangle_N *1. \tag{1.6}$$

The harmonic mapping ϕ is called stable, if $I(V,\ V)\geqslant 0$ for any vector field V along ϕ.

Consider ϕ_* as $\phi^{-1}TN$ valued 1-form $d\phi$ i.e.

$$d\phi(X)=\phi_*X. \tag{1.7}$$

When mapping ϕ is harmonic, it follows from Weitzenböck formula that

$$-\tilde{\nabla}*\tilde{\nabla}\,d\phi+S=0, \tag{1.8}$$

where

$$S(v)=-R^N(\phi_*e_i,\ \phi_*v)\phi_*e_i+\phi_*(\mathrm{Ric}^M(v)),\ v\in TM. \tag{1.9}$$

Thus

$$R^N(\phi_*e_i,\ \phi_*v)\phi_*e_i=-(\tilde{\nabla}*\tilde{\nabla}\,d\phi)(v)+\phi_*(\mathrm{Ric}^M(v)). \tag{1.10}$$

Substitutng (1.10) into (1.6), we have

$$I(\phi_*v,\ \phi_*w)=\int_M\langle-\tilde{\nabla}*\tilde{\nabla}\phi_*v+(\tilde{\nabla}*\tilde{\nabla}\,d\phi)(v)-\phi_*(\mathrm{Ric}^M(v)),\ \phi_*w\rangle_N *1. \tag{1.11}$$

On the other hand, for any fixed point P, choose $\{e_i\}$ such that $\nabla_{e_i}e_j|_P=0$. Then

$$\tilde{\nabla}*\tilde{\nabla}\,d\phi=\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}d\phi.$$

Therefore

$$\begin{aligned}-\tilde{\nabla}*\tilde{\nabla}\phi_*v+(\tilde{\nabla}*\tilde{\nabla}\,d\phi)(v)&=-\tilde{\nabla}*\tilde{\nabla}\phi_*v+(\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}d\phi)(v)\\&=-\tilde{\nabla}*\tilde{\nabla}\phi_*v+\tilde{\nabla}_{e_i}((\tilde{\nabla}_{e_i}d\phi)v)-(\tilde{\nabla}_{e_i}d\phi)(\nabla_{e_i}v)\\&=d\phi(\nabla_{e_i}\nabla_{e_i}v)-2\tilde{\nabla}_{e_i}(d\phi(\nabla_{e_i}v)).\end{aligned} \tag{1.12}$$

Hence

$$I(\phi_*v,\ \phi_*w)=\int_M\langle d\phi(\nabla_{e_i}\nabla_{e_i}v)-2\tilde{\nabla}_{e_i}(d\phi(\nabla_{e_i}v))-\phi_*(\mathrm{Ric}^M(v)),\ \phi_*w\rangle_N *1. \tag{1.13}$$

2. The main theorems Suppose that M is a p-codimensional compact submanifold without boundary in Euclidean space E^{n+p} and $\{e_\alpha\}(\alpha, \beta, \gamma, \cdots=n+1, \cdots, n+p)$ is an orthonormal basis in the normal bundle of M. Choose an orthonormal basis of M such that $\nabla_{e_i}e_j|_P=0$ is satisfied.

Denote the second fundamental form of M by h

$$h(X, Y)=h^\alpha(X, Y)e_\alpha, \; h^\alpha(e_i, e_j)=h^\alpha_{ij}. \tag{2.1}$$

Gauss-Codazzi equations are

$$\overline{\nabla}_X Y=\nabla_X Y+h^\alpha(X, Y)e_\alpha, \tag{2.2}$$

$$\overline{\nabla}_X e_\alpha=-h^\alpha(X, e_i)e_i+\nabla^\perp_X e_\alpha, \; \nabla^\perp_{e_i}e_\alpha=\mu_{i\alpha\beta}e_\beta, \; \mu_{i\alpha\beta}+\mu_{i\beta\alpha}=0, \tag{2.3}$$

where $\overline{\nabla}$ denotes the connection in E^{n+p} and $\nabla^\perp$ the connection in the normal bundle of M.

Set $L=\{a|_M: a$ is a constant vector in $E^{n+p}\}$. If $v\in L$, then $v=a-\langle a, e_\alpha\rangle e_\alpha$, where $\langle\ ,\ \rangle$ is the inner product in E^{n+p}. Using (2.2) and (2.3), we have

$$\nabla_{e_i}v=\overline{\nabla}_{e_i}v-h(e_i, v)=\langle a, e_\alpha\rangle h^\alpha_{ij}e_j, \tag{2.4}$$

$$\nabla_{e_i}(\nabla_{e_i}v)=-\langle a, e_k\rangle h^\alpha_{ik}h^\alpha_{ij}e_j+\langle a, e_\beta\rangle\mu_{i\alpha\beta}h^\alpha_{ij}e_j+\langle a, e_\alpha\rangle(\nabla_{e_i}h^\alpha_{ij})e_j, \tag{2.5}$$

$$\begin{aligned}\tilde{\nabla}_{e_i}(d\phi(\nabla_{e_i}v))=&-\langle a, e_k\rangle h^\alpha_{ik}h^\alpha_{ij}\phi_*e_j+\langle a, e_\beta\rangle\mu_{i\alpha\beta}h^\alpha_{ij}\phi_*e_j+\langle a, e_\alpha\rangle(\tilde{\nabla}_{e_i}h^\alpha_{ij})\phi_*e_j\\&+\langle a, e_\alpha\rangle h^\alpha_{ij}\tilde{\nabla}_{\phi_*e_i}\phi_*e_j.\end{aligned} \tag{2.6}$$

Thus

$$\begin{aligned}I(\phi_*v, \phi_*v)=\int_M\langle&\langle a, e_k\rangle h^\alpha_{ik}h^\alpha_{ij}\phi_*e_j-\langle a, e_\beta\rangle\mu_{i\alpha\beta}h^\alpha_{ij}\phi_*e_j-\langle a, e_\alpha\rangle(\nabla_{e_i}h^\alpha_{ij})\phi_*e_j\\&-2\langle a, e_\alpha\rangle h^\alpha_{ij}\tilde{\nabla}_{\phi_*e_i}\phi_*e_j-\phi_*(\mathrm{Ric}^M(v)), \langle a, e_i\rangle\phi_*e_i\rangle_N*1.\end{aligned} \tag{2.7}$$

From Gauss formula it follows that

$$\mathrm{Ric}^M(v)=\langle a, e_i\rangle(h^\alpha_{ll}h^\alpha_{ij}-h^\alpha_{ik}h^\alpha_{kj})e_j. \tag{2.8}$$

Substituting (2.8) into (2.7), we have

$$\begin{aligned}I(\phi_*v, \phi_*v)=\int_M\langle&2\langle a, e_k\rangle h^\alpha_{ik}h^\alpha_{ij}\phi_*e_j-\langle a, e_i\rangle h^\alpha_{ll}h^\alpha_{ij}\phi_*e_j-\langle a, e_\beta\rangle\mu_{i\alpha\beta}h^\alpha_{ij}\phi_*e_j\\&-\langle a, e_\alpha\rangle(\nabla_{e_i}h^\alpha_{ij})\phi_*e_j-2\langle a, e_\alpha\rangle h^\alpha_{ij}\tilde{\nabla}_{\phi_*e_i}\phi_*e_j, \langle a, e_i\rangle\phi_*e_i\rangle_N*1.\end{aligned} \tag{2.9}$$

Denote $I(\phi_*v, \phi_*w)=\int_M F(\phi_*v, \phi_*w)*1$, $v, w\in L$. Then $\mathrm{tr}I=\int_M \mathrm{tr}F*1$. Because trace is independent of the choice of orthonormal basis, we can pointwise take $\{e_i, e_\alpha\}$ as the basis of L to compute $\mathrm{tr}F$. Thus

$$\mathrm{tr}I=\int_M(2h^\alpha_{ik}h^\alpha_{ij}-h^\alpha_{ll}h^\alpha_{ij})\langle\phi_*e_j, \phi_*e_k\rangle_N*1. \tag{2.10}$$

Theorem 1. *Let $M^n\to E^{n+p}$ be a compact submanifold without boundary in Euclidean space E^{n+p}. If there exists a negative constant B such that*

$$2\langle h(e_i, e_k), h(e_i, e_j)\rangle-\langle h(e_i, e_i), h(e_k, e_j)\rangle\leqslant B\delta_{jk}, \tag{2.11}$$

then there is no nonconstant stable harmonic mapping from M^n to any Riemannian manifold.

Proof By (2.10), we see that

$$\operatorname{tr} I \leqslant B \int_M \langle \phi_* e_j, \phi_* e_j \rangle_N *1 = 2BE(\phi) \leqslant 0,$$

and the equality holds if and only if $E(\phi)=0$, i. e. ϕ is a constant mapping.

Theorem 2. *Let M^n be a closed convex hypersurface in Euclidean space E^{n+1}. If each principal curvature of M^n is less than half the sum of all principal curvatures, then there is no nonconstant stable harmonic mapping from M^n to any Riemannian manifold.*

Proof Let $\lambda_1, \cdots, \lambda_n$ be n principal curvatures, and $e_1, \cdots, e_n$ eigenvectors of h. Set $H=\sum_{i=1}^{n}\lambda_i$. From (2.10), we obtain

$$\operatorname{tr} I = \int_M (2\lambda_k^2 - H\lambda_k)\langle \phi_* e_k, \phi^* e_k \rangle_N *1 = \int_M (2\lambda_k - H)\lambda_k \langle \phi_* e_k, \phi_* e_k \rangle_N *1 \leqslant BE(\phi),$$

where B is a negative constant, and the equality holds if and only if $E(\phi)=0$, i. e. ϕ is a constant mapping.

Remark. Theorem 2 obviously includes Xin's result as a special case.

Applying Theorem 1, we obtain

Theorem 3. *If* $\min(n_1, \cdots, n_q) > 2$, *then there is no nonconstant stable harmonic mapping from the product of Euclidean spheres $S^{n_1} \times \cdots \times S^{n_q}$ to any Riemannian manifold.*

Proof Noting that $S^{n_1} \times \cdots \times S^{n_q}$ can be imbedded canonically into $E^{n_1+1} \times \cdots \times E^{n_q+1}$, in this case, we may take $B = 2 - \min(n_1, \cdots, n_q)$ in (2.11).

The author would like to thank Professor Hu Hesheng for her encouragement and help.

References

[1] Xin, Y. L., Some results on stable harmonic maps, *Duke Math. J.*, **47**:3 (1980).
[2] Smith, R. T., The second variation formula for harmonic mappings, *Proc. Amer. Math. Soc.* **47** (1975).
[3] Eells, J. & Sampson J. H., Harmonic mappings of Riemannian manifolds, *Amer. J. Math.*, **86** (1964).

稳定调和映照的不存在性定理

潘 养 廉
(复旦大学数学研究所)

摘 要

本文研究了欧氏空间中紧致子流形到任何黎曼流形的稳定调和映照．得到了第二变分的有关表达式，从而证明了若干稳定调和映照的不存在性定理．特别是证明了一类凸闭超曲面到任何黎曼流形的稳定非常值调和映照的不存在性，推广了[1]中的结果．