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ON THE STATIC SOLUTIONS OF MASSIVE YANG-MILLS EQUATIONS

HU HESHENG (H. S. HU)
(Institute of Mathematics, Fudan University)

Dedicated to Professor Su Bu-chin on the Occasion of his 80th Birthday and his 50th Year of Educational Work

§ 1. Introduction

Usually, a pure Yang-Mills field over Minkowski spacetime $R^{1,n-1}$ is considered as a field of massless particles. Its action integral is[1]

$$L=\int -\frac{1}{4}(f_{\lambda\mu}, f^{\lambda\mu})d^n x. \tag{1}$$

Here $f_{\lambda\mu}$ is the strength of gauge field with a compact gauge group G and (,) denotes the cartan's inner product of the Lie algebra g of G. However, many particles in nature are not massless and hence it is a problem of general interest to consider the massive Yang-Mills fields. It has been proved in [2] and [3], from different points of view, that the following gauge invariant functional

$$L_m=\int\left[-\frac{1}{4}(f_{\lambda\mu}, f^{\lambda\mu})-\frac{m^2}{2}(b_\lambda-\omega_\lambda, b^\lambda-\omega^\lambda)\right]d^n x \tag{2}$$

may be considered as the action integral of the massive Yang-Mills field. Here b_λ is the gauge potential, ω_λ is defined by

$$\omega_\lambda=U^{-1}\partial_\lambda U \tag{3}$$

and U is a G-valued function which is a section of the product bundle $R^{n-1}\wedge G$.

One may think that the choice of U is a choice of gauge and that the gauge is a reference system of measuring the generalized phase of a gauge field. Let U be the variational variables as well as b_λ, the Euler equations of the action integral (1) and (2) are the massless Yang-Mills equations and massive Yang-Mills equations respectively.

We mentioned that the massive Yang-Mills field is also attractive for its relationship with harmonic mapping. The functional (2) is nothing else than the coupling of the pureYang-Mills functional and the following action integral of the harmonic maps from $R^{1,n-1}$ to the gauge group G

$$S(U)=\int(\omega_\lambda, \omega^\lambda)d^n x. \tag{4}$$

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There are guite a lot of papers devoted to the solutions to Yang-Mills equations. One problem of considerable interest is whether there exists any static solution to the Yang-Mills equations such that it has finite energy and no singularities. Recently the following facts concerning the nonexistence of the global solutions are discovered.

(a) If $n\neq 5$, the pure Yang-Mills equations on an n-dimensional spacetime $R^{1,n-1}$ do not admit any static solution which has (i) finite energy (ii) no singularities and (iii) the field strength approaching to zero sufficiently fast at infinity. (Deser, S.[5])

Thus, for $n=4$, i. e. on the real spacetime, there does not exist such solution. For $n=5$, solutions do exist[6,7], since the instantons in 4-dimensional Euclidean space may be regarded as static solutions in 5-dimensional spacetime.

(b) In an n-dimensional spacetime with $n\neq 4$, the massive Yang-Mills equations with real mass do not admit any static solution which has (i) finite energy (ii) no singularities and (iii) the field strength and potential approaching to zero sufficiently fast at infinity (Hu, H. S. [8])

Comparing these two results, we discovered that there is a "discontinuity" as $m\to 0$ in 5-dimensional spacetime, i. e. for $n=5$ and $m\neq 0$, no such solution, but when $m=0$ such solutions do exist. Deser, S. and Isham, C. J. in a recent paper[9] wrote that this is the first explicit example which make us recognize that there exists a classical "discontinuity". In their paper, the results are extended to the gauge field with "soft" mass, i. e. Yang-Mills-Higgs-Kibble field. For $n=5$, the "discontinuity" holds in general.

In the present paper, we will show that in the results (a) and (b) not only condition (iii) can be removed, but also the finite energy condition (ii) can be weakened. In other words, when the total energy within the sphere of radius r approaches to infinity quite slowly as $r\to\infty$, the above nonexistence theorem holds true also. In the proof we use a certain technique used in[10] with some improvement. Since finite energy and infinite energy is essentially different in physics, this new discovery may be of interest in physics.

The method of proving the main theorem of the present paper is utilizable for more general case. For example, in the case of the Yang-Mills-Higgs-Kibble field, the results for "soft" mass is improved similarly.

§ 2. Massive Yang-Mills fields

By choosing the Lorentz gauge, the gauge invariant functional becomes

$$L=-\int\left\{\frac{1}{4}(f_{\lambda\mu},\ f^{\lambda\mu})+\frac{m^2}{2}(b_\lambda,\ b^\lambda)\right\}d^{n-1}x\quad (\lambda,\ \mu=0,\ 1,\ \cdots,\ n-1). \tag{5}$$

Here the field strength is

$$f_{\lambda\mu}=b_{\lambda,\mu}-b_{\mu,\lambda}-[b_\lambda,\ b_\mu]\quad \left(b_{\lambda,\mu}=\frac{\partial b_\lambda}{\partial x^\mu}\right) \tag{6}$$

and the metric of spacetime is

$$ds^2=\eta_{\lambda\mu}\,dx^\lambda\,dx^\mu=-dx^{0^2}+dx^{1^2}+\cdots+dx^{n-1^2}. \tag{7}$$

The massive Yang-Mills equation become

$$J_\alpha-m^2b_\alpha=0, \tag{8}$$

$$\eta^{\lambda\mu}b_{\lambda,\mu}=0. \tag{9}$$

Here

$$J_\alpha=\eta^{\lambda\mu}f_{\alpha\lambda|\mu}=\eta^{\lambda\mu}(f_{\alpha\lambda,\mu}+[b_\mu,\ f_{\alpha\lambda}]). \tag{10}$$

moreover, it is interesting to note that (9) is a consequence of (8), so we only need to consider equation (8).

We always assume the gauge group G is a compact group. Under the Lorentz gauge the energy monentum tensor

$$T_{\alpha\beta}=(f_{\alpha\nu},\ f^\nu_\beta)-\frac{1}{4}\eta_{\alpha\beta}(f_{\mu\nu},\ f^{\mu\nu})+m^2(b_\alpha,\ b^\alpha)-\frac{m^2}{2}\eta_{\alpha\beta}(b_\lambda,\ b^\lambda) \tag{11}$$

and we have the conservation law

$$\frac{\partial T^\beta_\alpha}{\partial x^\alpha}=0. \tag{12}$$

Especially, the energy density is

$$T_{00}=\frac{1}{2}\left[(f_{0i},\ f_{0i})+\frac{1}{2}(f_{ij},\ f_{ij})\right]+\frac{m^2}{2}(b_0,\ b_0)+\frac{m^2}{2}(b_i,\ b_i)$$
$$(i,\ j=1,\ 2,\ \cdots,\ n-1) \tag{13}$$

and

$$T_{ii}=T^\alpha_\alpha-T^0_0=-\frac{1}{2}(n-3)(f_{0i},\ f^{0i})+\frac{1}{4}(5-n)(f_{ij},\ f^{ij})$$
$$+\frac{m^2}{2}(3-n)(b_\lambda,\ b^\lambda)+m^2(b_0,\ b_0). \tag{14}$$

For a static gauge field, b_λ is independent of x^0.

The total energy of the field is

$$\int_{R^{n-1}}T_{00}d^{n-1}x, \tag{15}$$

where R^{n-1} is $x^0=$const. In the previous works, one often assume that the total energy is finite, now the condition is weakened to

$$\int\frac{T_{00}}{\psi(r)}\,d^{n-1}x<\infty, \tag{16}$$

where $\psi(r)$ is a positive, unbounded, continuous function of r satisfying

$$\int_R^\infty\frac{dr}{r\psi(r)}=\infty\quad (R>0). \tag{17}$$

If $\psi(r)=1$, the energy is finite. But the energy may be infinite, for example, in the case $\psi(r)=O(\log r)$ (as $r\to\infty$). Hence when (16) holds true, the total enegy may be either finite or infinite. In this paper the energy is called "slowly divergent energy" if $\int T_{00}d^{n-1}x=\infty$ and (16) holds.

§ 3. Nonexistence of the static solution

In the following we give the precise statement and the proof of the main theorem concerning the massive Yang-Mills equations.

Theorem. *In an n-dimensional spacetime $R^{1,n-1}$ with $n\neq 4$, the compact group Yang-Mills field with real mass does not possess any non-trivial static solution which is free of singularities and has finite or "slowly divergent" energy.*

Proof From the expression (10) for J_0 and the static condition $b_{\alpha,0}=0$, we have

$$\begin{aligned}(b_0,\ J_0) &= (b_0,\ f_{0i|i}) = (b_0,\ f_{0i})_{,i} - (b_{0,i},\ f_{0i}) + (b_0,\ [b_i,\ f_{0i}]) \\ &= (b_0,\ f_{0i})_{,i} - (b_{0,i},\ f_{0i}) - ([b_i,\ b_0],\ f_{0i}) \\ &= (b_0,\ f_{0i})_{,i} + (f_{0i},\ f_{0i}). \end{aligned} \tag{18}$$

Consider the integral

$$0=\int_0^\infty \omega(r)dr\int_{|x|<r} (J_0-m^2b_0,\ b_0)d^{n-1}x, \tag{19}$$

where $|x|=\{(x^1)^2+\cdots+(x^{n-1})^2\}^{\frac{1}{2}}$ and $\omega(r)$ will be defined later. Using (18), we have

$$0=\int_0^\infty \omega(r)dr\int_{|x|<r} K\,d^{n-1}x+\int_0^\infty \omega(r)dr\int_{|x|=r} (b_0,\ f_{0i})\frac{x^i}{r}dS, \tag{20}$$

where

$$K=-(f_{0i},\ f_{0i})-m^2(b_0,\ b_0)\leqslant 0, \tag{21}$$

The equality in (21) holds if and only if $f_{0i}=b_0=0$. If K does not equal zero identically, then there exists a constant $R>0$ and a positive constant ε such that

$$\int_{|x|\leqslant R_1} K d^{n-1}x<-\varepsilon \quad (R_1\leqslant R). \tag{22}$$

Choose

$$\omega(r)=\begin{cases}0, & r<R,\\ \dfrac{1}{r\psi(r)}, & R\leqslant r\leqslant R_1,\\ 0, & r>R_1,\end{cases} \tag{23}$$

where $\psi(r)$ is positive unbounded, continuous functon of r satisfying

$$\int_R^\infty \frac{dr}{r\psi(r)}=\infty. \tag{24}$$

Then, from (21), we have

$$\begin{aligned}0&<-\varepsilon\int_R^{R_1}\frac{dr}{r\psi(r)}+\int_R^{R_1}\frac{dr}{r\psi(r)}\int_{|x|=r}\{(b_0,\ b_0)+(f_{0i},\ f_{0i})\}dS\\ &<-\varepsilon\int_R^{R_1}\frac{dr}{r\psi(r)}+\frac{1}{R}\int_{|x|<R_1}\frac{(b_0,\ b_0)+(f_{0i},\ f_{0i})}{\psi(r)}d^{n-1}x.\end{aligned} \tag{25}$$

Choose R_1 sufficiently large, it is easily seen that the right side should be negative. This is a contradiction. Consequently, we should have $K=0$ identically, i. e.

$$b_0=0,\quad f_{0i}=0. \tag{26}$$

Thus, we have

$$T_{ii}=\frac{1}{4}(5-n)(f_{ij},\ f_{ij})+\frac{m^2}{2}(3-n)(b_i,\ b_i) \tag{27}$$

and (12) is reduced to

$$T_{ij,i}=0. \tag{28}$$

Consider the integral

$$0=\int_0^\infty \omega(r)dr\int x^j T_{ij,i}d^{n-1}x=\int_0^\infty \omega(r)dr\int_{|x|<r}\{(x^jT_{ij})_{,i}-T_{ii}\}d^{n-1}x$$

$$=\int_0^\infty \omega(r)dr\int_{|x|=r}(x^jT_{ij})\frac{x^i}{r}-\int_0^\infty \omega(r)dr\int_{|x|<r}T_{ii}d^{n-1}x. \tag{29}$$

It is easily seen that there exists a constant A such that

$$|T_{ij}|\leqslant AT_{00}. \tag{30}$$

Moreover, from (27), we have

(a) If $n\geqslant 5$, then $T_{ii}\leqslant 0$

and the equality holds only when $b_i=0$.

(b) If $n\leqslant 3$, then $T_{ii}\geqslant 0$

and the equality holds only when $b_i=0$.

In either case, if $T_{ii}\not\equiv 0$, we have $T_{ii}<0$ (or $T_{ii}>0$) in some region. Hence there exist two constant $R>0$ and $\varepsilon>0$ such that

$$\int_{|x|<R_1}T_{ii}d^{n-1}x<-\varepsilon\quad(\text{or}>\varepsilon)\quad(R_1\geqslant R). \tag{31}$$

Choosing the same $\omega(r)$ as in (23), for the case (a), we have

$$0<-\varepsilon\int_R^{R_1}\frac{dr}{r\psi(r)}+A\int_0^\infty\int\frac{T_{00}}{\psi(r)}dS\,dr. \tag{32}$$

By the assumption that the energy is finite or "slowly divergent", we can choose R_1 sufficiently large, and it is easily seen that the right side of equation (32) should be negative. This gives a contradiction again. Consequently, we should have

$$T_{ii}=0.$$

For the case (b), the situation is quite similar. Consequently

(i) when $n\neq 3,\ 4,\ 5$, we have $f_{ij}=0,\ b_i=0$;

(ii) when $n=5$, we have $b_i=0$, hence $f_{ij}=0$;

(iii) when $n=3$, from the field equation (8) and $f_{ij}=0$ we have $b_i=0$.

In other words, when $n\neq 4$, the solution should be a trivial one. Thus the Theorem is proved completely.

Remarks.

1. For the massless case $m=0$. Deser's Theorem is also improved similarly.

2. Consider the Yang-Mills-Higgs-Kibble field (the gauge field with "soft" mass)

$$I=\int\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{m^2}{2}b_\mu b^\mu-\frac{1}{4}\nabla_\mu\phi\nabla^\mu\phi-V(\phi)\right), \tag{33}$$

where ϕ is a scalar invariant and $V(\phi)$ is the potential. By using the same method, the result of [9] can be improved and extended to the case of "slowly divergent" energy and the classical "discontinuity" holds also for $n=5$.

In the following section, we shall specialize that the condition for the energy in our theorem cannot be omitted, because for any dimensional spacetime in massive and massless Yang-Mills field we can find static regular solutions with energy diverges sufficiently fast. In the meanwhile we obtain all the static solutions of strictly spherically symmetric gauge field.

§ 4. Static solutions of strictly spherically symmetric gauge field

Suppose the gauge potential of stricty spherically symmetric static gauge fields are in the canonical form[11]

$$b_i(x)=\phi(r)x_i, \quad b_0=\sigma(r), \tag{34}$$

where $\phi(r)$, $\sigma(r)$ are g-valued functions, depending only on r. In order to solve the massive Yang-Mills equations, we substitute (34) in (9) and we have

$$r\phi'(r)+(n-1)\phi(r)=0. \tag{35}$$

Hence we obtain

$$\phi(r)=\frac{\text{const}}{r^{n-1}}. \tag{36}$$

The requirement of regularity at the origin implies that $\phi(r)=0$. From (8) we obtain

$$\Delta\sigma(r)-m^2\sigma(r)=0 \tag{37}$$

or

$$\frac{d^2\sigma}{dr^2}+\frac{(n-2)}{r}\frac{d\sigma}{dr}-m^2\sigma=0. \tag{38}$$

Let

$$mr=R, \quad \sigma=R^{-p}q(R) \quad \left(p=\frac{n-3}{2}\right). \tag{39}$$

We obtain

$$q''+\frac{q'}{R}-\left(1+\frac{p^2}{R^2}\right)q=0, \tag{40}$$

this is the modified Bessel equation[12]. It is known that this equation admits the following solutions which are everywhere regular.

$$q=q_0I_p(R), \tag{41}$$

where q_0 is an element of g, $I_p(R)$ is the Bessel function with purely imaginary argument. Hence the equation (8) posses the following everywhere regular solutions

$$b_i=0, \quad b_0=q_0(mr)^{-\frac{n-3}{2}}I_p(mr). \tag{42}$$

Since when $r\to\infty$

$$I_p(mr)\sim\frac{e^{n+r}}{(2\pi mr)^{\frac{1}{2}}}, \tag{43}$$

the energy of such solutions is infinite and is not "slowly divergent".

From the above discussion we conclude that for any n, the condition of energy in the theorem cannot be omitted. That is to say, the massive Yang-Mills equation admits infinite many static regular solution whose energy is infinite and divergent sufficienty fast.

At the conclusion of the present paper we give two open problems.

1. In the case $n=4$, does there exist a static regular solution of massive (or massless i. e. pure) Yang-Mills equation with finite energy or "slowly divergent" energy?

2. Are there static solutions of the massless Yang-Mills equations in R^{4+1} with $b_0 \neq 0$? This problem arises due to the fact that the instantons of R^4 are static solutions in R^{4+1} with $b_0=0$ and the solutions in R^{4+1} with $b_0 \neq 0$ may be consider as some solutions of a certain coupled Yang-Mills equations in 4-dimensional Euclidean space.

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关于具有质量的 Yang-Mills 方程的静态解

胡 和 生

(复旦大学数学研究所)

摘 要

关于 Yang-Mills 方程的静态解, Deser, S[5]. 证明了: 当 $n\neq5$ 时, 无质量的紧致群 Yang-Mills 方程不存在满足条件(i)无奇性(ii)能量有限(iii)当 $r\to\infty$ 时, 场强 $f_{\lambda\mu}\to0$ 足够快的静态解. 又已知当 $n=5$ 时, 正则静态解确实存在[6].

对于具实质量的紧致群 Yang-Mills 方程, 作者[8] 证明了: 当 $n\neq4$ 时, 不存在满足条件(i)无奇性(ii)能量有限(iii)当 $r\to\infty$ 时规范势 b_λ 与场强 $f_{\lambda\mu}\to0$ 足够快的静态解. 从而发现在 $n=5$, 当质量 $m\to0$ 时, 对 Yang-Mills 方程的可解性问题而言, 在性质上有一种"不连续性". 物理学家认为这是存在着经典的不连续性的第一个明确的例子, 并对包括 Higgs 场的情况作了推广的研究[9].

本文进一步证明了上述两个结果中不仅条件(iii)可以取消, 而且条件(ii)也可减弱. 即能量为无限, 但当以 r 为半径的球体的总能量趋于无限相当慢时定理仍成立. 这时经典的"不连续性"也仍成立. 由于能量有限与能量无限在物理上有根本的不同, 所揭示的现象是值得注意的.

文中又证明, 如果取消总能量趋于无限相当慢这个条件, 定理的结论就不成立.

这里的证明方法, 可用于更一般的情况. 例如包括 Higgs 场的情况, 从而[9]中的结果也得到改进.