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LIMIT BEHAVIORS OF SOLUTIONS FOR SOME PARABOLIC EQUATIONS OF HIGHER ORDER AND THEIR APPLICATIONS TO THE OPTIMAL CONTROL

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Dedicated to Professor Su Bu-chin on the Occasion of his 80th Birthday and his 50th Year of Educational Work

Introduction and principal results

Let Ω be a bounded open set in $\mathbf{R}^n$ Containing the origin with a smooth boundary Γ. In this paper we restrict ourselves to the case $n=2$ or 3, which is more important in applications.

Let A be a second order self-abjoint elliptic operator with variable coefficients

$$A\varphi=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial\varphi}{\partial x_j}\right)+c(x)\varphi \tag{1}$$

with $a_{ij}(x)$ and $c(x)$ suitably smooth

$$a_{ij}(x)=a_{ji}(x),\ i,\ j=1,\ \cdots,\ n,\ x\in\Omega \tag{2}$$

and there exists a constant $\alpha>0$ such that

$$\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\geqslant\alpha|\xi|^2\quad\forall\xi\in\mathbf{R}^n,\ x\in\Omega. \tag{3}$$

In the first part of this paper we shall prove the following results for limit behaviors of solutions for some parabolic equations of higher order:

I. 1. For any $v\in L^2(0,\ T)$, consider the following initial boundary value problem

$$\text{(I)}\qquad\begin{cases}\dfrac{\partial y}{\partial t}+Ay=v(t)\delta(x) & \text{in } Q=\Omega\times(0,\ T),\\ y=0 & \text{on } \Sigma=\Gamma\times(0,\ T),\\ y(x,\ 0)=0 & \text{in } \Omega,\end{cases}$$

where $\delta(x)$ is the Dirac mass at the origin. By transposition (cf. [1, 2]), problem (I) admits a unique weak solution $y=y(t;\ v)\in L^2(Q)$.

For any $\varepsilon>0$ fixed, consider the following approximation of problem (I)

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$$(\mathrm{I})_\varepsilon \quad \begin{cases} \dfrac{\partial y_\varepsilon}{\partial t}+\varepsilon A^2 y_\varepsilon+Ay_\varepsilon=v_\varepsilon(t)\delta(x), & \text{in } Q, \\ y_\varepsilon=Ay_\varepsilon=0, & \text{on } \Sigma, \\ y_\varepsilon(x, 0)=0, & \text{in } \Omega. \end{cases}$$

It is well known (cf. [3]) that for any $v_\varepsilon(t)\in L^2(0, T)$, problem $(\mathrm{I})_\varepsilon$ possesses a unique solution $y_\varepsilon=y_\varepsilon(t; v_\varepsilon)$

$$y_\varepsilon\in L^2(0, T; H^2(\Omega)\cap H_0^1(\Omega)), \ \frac{\partial y_\varepsilon}{\partial t}\in L^2(0, T; H^{-2}(\Omega)) \tag{4}$$

and

$$v_\varepsilon(t)\to\left\{y_\varepsilon, \frac{\partial y_\varepsilon}{\partial t}\right\}$$

is a continuous linear mapping from $L^2(0, T)$ to

$$\{L^2(0, T; H^2(\Omega))\times L^2(0, T; H^{-2}(\Omega))\}.$$

Hence

$$y_\varepsilon(T; v_\varepsilon)\in L^2(\Omega) \tag{5}$$

and

$$L_\varepsilon: v_\varepsilon(t)\to y_\varepsilon(T; v_\varepsilon) \tag{6}$$

is a continuous linear mapping from $L^2(0, T)$ to $L^2(\Omega)$.

In I. § 1—2 we shall prove

Theorem 1 (resp. **Theorem 1 bis**) *As* $\varepsilon\to 0$, *if*

$$v_\varepsilon(t)\to v(t) \tag{7}$$

in $L^2[0, T]$ *weakly (resp. strongly), then*

$$y_\varepsilon(t; v_\varepsilon)\to y(t; v) \tag{8}$$

in $L^2(Q)$ *weakly (resp. strongly).*

Corollary 1. 1. *In Theorem* 1 (*resp. Theorem* 1 *bis*) *if we suppose further that for* $\varepsilon>0$ *small enough*

$$\{y_\varepsilon(T; v_\varepsilon)\} \text{ belongs to a weakly (resp. strongly) compact subset of } L^2(\Omega), \tag{9}$$

then

$$y_\varepsilon(T; v_\varepsilon)\to y(T; v) \text{ in } L^2(\Omega) \text{ weakly (resp. strongly), as } \varepsilon\to 0 \tag{10}$$

and

$$v\in\mathscr{U}, \tag{11}$$

where $\mathscr{U}$ *is the function space (cf.* [2], [7—9])

$$\mathscr{U}=\{v\,|\,v\in L^2(0, T), \ y(T; v)\in L^2(\Omega)\} \tag{12}$$

provided with the graph norm.

I. 2. For problems (I) and $(\mathrm{I})_\varepsilon$, we consider the corresponding adjoint problems as follows

$$(\mathrm{II}) \quad \begin{cases} -\dfrac{\partial \varphi}{\partial t}+A\varphi=\psi & \text{in } Q, \\ \varphi=0 & \text{on } \Sigma, \\ \varphi(x, T)=0 & \text{in } \Omega; \end{cases}$$

$$\text{(II)}_\varepsilon \qquad \begin{cases} -\dfrac{\partial \varphi_\varepsilon}{\partial t}+\varepsilon A^2\varphi_\varepsilon+A\varphi_\varepsilon=\psi_\varepsilon & \text{in } Q, \\ \varphi_\varepsilon=A\varphi_\varepsilon=0 & \text{on } \Sigma, \\ \varphi_\varepsilon(x, T)=0 & \text{in } \Omega. \end{cases}$$

Let $\varphi(t; \psi)$ and $\varphi_\varepsilon(t; \psi_\varepsilon)$ denote the solutions of problem (II) and (II)_ε respectively, in I. § 1—2 we shall prove

Theorem 2 (resp. Theorem 2 bis). *As* $\varepsilon\to 0$, *if*

$$\psi_\varepsilon\to\psi \text{ in } L^2(Q) \text{ strongly (resp. weakly)}, \tag{13}$$

then

$$\varphi_\varepsilon(t; \psi_\varepsilon)\to\varphi(t; \psi) \text{ in } H^{2,1}(Q) \text{ strongly (resp. weakly)}, \tag{14}$$

where

$$H^{2,1}(Q)=\left\{\varphi \,|\, \varphi\in L^2(0, T; H^2(\Omega)), \frac{\partial\varphi}{\partial t}\in L^2(Q)\right\}; \tag{15}$$

In particular

$$\varphi_\varepsilon(0, t; \psi_\varepsilon)\to\varphi(0, t; \psi) \text{ in } L^2(0, T) \text{ strongly (resp. weakly)}. \tag{16}$$

I. 3. For any $S(x)\in L^2(\Omega)$, the following problem

$$\text{(III)} \qquad \begin{cases} -\dfrac{\partial p}{\partial t}+Ap=0 & \text{in } Q, \\ p=0 & \text{on } \Sigma, \\ p(x, T)=S(x) & \text{in } \Omega \end{cases}$$

admits a unique solution $p=p(t; S)$ (cf. [3])

$$p\in L^2(0, T; H_0^1(\Omega)), \ \frac{\partial p}{\partial t}\in L^2(0, T; H^{-1}(\Omega)). \tag{17}$$

For any $\varepsilon>0$ fixed, construct the following approximation of problem (III)

$$\text{(III)}_\varepsilon \qquad \begin{cases} -\dfrac{\partial p_\varepsilon}{\partial t}+\varepsilon A^2 p_\varepsilon+Ap_\varepsilon=0 & \text{in } Q, \\ p_\varepsilon=Ap_\varepsilon=0 & \text{on } \Sigma, \\ p_\varepsilon(x, T)=S_\varepsilon(x) & \text{in } \Omega. \end{cases}$$

It is well known (cf. [3]) that for any $S_\varepsilon(x)\in L^2(\Omega)$, problem (III)_ε admits a unique solution $p_\varepsilon=p_\varepsilon(t; S_\varepsilon)$

$$p_\varepsilon\in L^2(0, T; H^2(\Omega)\cap H_0^1(\Omega)), \ \frac{\partial p_\varepsilon}{\partial t}\in L^2(0, T; H^{-2}(\Omega)) \tag{18}$$

and

$$S_\varepsilon(x)\to\left\{p_\varepsilon, \frac{\partial p_\varepsilon}{\partial t}\right\}$$

in a continuous linear mapping from $L^2(\Omega)$ to

$$\{L^2(0, T; H^2(\Omega))\times L^2(0, T; H^{-2}(\Omega))\}.$$

Hence

$$p_\varepsilon(0, t; S_\varepsilon)\in L^2(0, T) \tag{19}$$

and

$$L_\varepsilon^*: S_\varepsilon(x)\to p_\varepsilon(0, t; S_\varepsilon) \tag{20}$$

is a continuous linear mapping from $L^2(\Omega)$ to $L^2(0, T)$.

In I. § 3—4 we shall prove

Theorem 3 (**resp. Theorem 3 bis**). *As* $\varepsilon\to 0$, *if*

$$S_\varepsilon(x)\to S(x) \text{ in } L^2(\Omega) \text{ weakly (resp. strongly)}, \tag{21}$$

then

$$p_\varepsilon(t;\ S_\varepsilon)\to p(t;\ S) \text{ in } L^2(0,\ T;\ H_0^1(\Omega)) \text{ weakly (resp. strongly)}. \tag{22}$$

Corollary 3.1. *In Theorem 3 (resp. Theorem 3 bis), if we suppose further that for* $\varepsilon>0$ *small enough*

$$\{p_\varepsilon(0,\ t;\ S_\varepsilon)\} \text{ belongs to a weakly (resp. strongly) compact subset of } L^2(0,\ T), \tag{23}$$

then

$$p_\varepsilon(0,\ t;\ S_\varepsilon)\to p(0,\ t;\ S) \text{ in } L^2(0,\ T) \text{ weakly (resp. strongly), as } \varepsilon\to 0 \tag{24}$$

and

$$S\in\mathscr{U}^*, \tag{25}$$

where $\mathscr{U}^*$ *is the function space (cf.* [2, 7, 9])

$$\mathscr{U}^*=\{S\,|\,S\in L^2(\Omega),\ p(0,\ t;\ S)\in L^2(0,\ T)\} \tag{26}$$

provided with the graph norm.

In the second part of this paper we shall use the preceding results in the first part to study the following various problems of optimal control:

II. 1. By means of Problem (I) we define the cost function

$$J(v)=N\int_0^T v^2\,dt+\int_\Omega |y(T;\ v)-Z_d|^2dx,\ \ \forall v\in\mathscr{U}, \tag{27}$$

where Z_d is given in $L^2(\Omega)$, N is given >0 and $\mathscr{U}$ is defined by (12). It is easy to see (cf. [5]) that there exists a uniqne element $u_0=u_0(t)\in\mathscr{U}$ such that

$$J(u_0)=\inf_{v\in\mathscr{U}} J(v). \tag{28}$$

The corresponding optimality system is the following

$$\text{(IV)}\quad\begin{cases}\dfrac{\partial y}{\partial t}+Ay=-\dfrac{1}{N}p(0,\ t)\delta(x), \ -\dfrac{\partial p}{\partial t}+Ap=0 & \text{in } Q,\\ y=0,\ p=0 & \text{on } \Sigma,\\ y(x,\ 0)=0,\ p(x,\ T)=y(x,\ T)-Z_d & \text{in } \Omega\end{cases}$$

and the optimal control u_0 is given by

$$u_0=-\frac{1}{N}p(0,\ t). \tag{29}$$

Similarly, by means of problem $(\mathrm{I})_\varepsilon$, we can define the cost function

$$J_\varepsilon(v)=N\int_0^T v^2\,dt+\int_\Omega |y_\varepsilon(T;\ v)-Z_d|^2dx,\ \ \forall v\in L^2(0,\ T). \tag{30}$$

Noticing (6), it is easily seen that there exists a unique element $u_\varepsilon=u_\varepsilon(t)\in L^2(0,\ T)$ such that

$$J_\varepsilon(u_\varepsilon)=\inf_{v\in L^2(0,T)} J_\varepsilon(v). \tag{31}$$

The corresponding optimality system is the following

$$
(\mathrm{IV})_\varepsilon \quad \begin{cases} \dfrac{\partial y_\varepsilon}{\partial t}+\varepsilon A^2 y_\varepsilon+Ay_\varepsilon=-\dfrac{1}{N}p_\varepsilon(0,\ t)\delta(x), \\ -\dfrac{\partial p_\varepsilon}{\partial t}+\varepsilon A^2 p_\varepsilon+Ap_\varepsilon=0 & \text{in } Q, \\ y_\varepsilon=Ay_\varepsilon=0,\ p_\varepsilon=Ap_\varepsilon=0 & \text{on } \Sigma, \\ y_\varepsilon(x,\ 0)=0,\ p_\varepsilon(x,\ T)=y_\varepsilon(x,\ T)-Z_d & \text{in } \Omega \end{cases}
$$

and the optimal control u_ε is given by

$$
u_\varepsilon=-\frac{1}{N}p_\varepsilon(0,\ t). \tag{32}
$$

In II. § 1 we shall prove

Theorem 4 *As $\varepsilon\to 0$, we have*

(i) $J_\varepsilon(u_\varepsilon)\to J(u_0)$; (33)

(ii) $u_\varepsilon\to u_0$ *in* $L^2(0,\ T)$ *strongly.* (34)

Moreover, for the solutions $\{y_\varepsilon,\ p_\varepsilon\}$ *and* $\{y,\ p\}$ *of the optimality systems* $(\mathrm{IV})_\varepsilon$ *and* (IV) *respectively, we have. as* $\varepsilon\to 0$

$$
y_\varepsilon\to y \text{ in } L^2(Q) \text{ strongly}, \tag{35}
$$

$$
y_\varepsilon(T)\to y(T) \text{ in } L^2(\Omega) \text{ strongly}, \tag{36}
$$

$$
p_\varepsilon\to p \text{ in } L^2(0,\ T;\ H_0^1(\Omega)) \text{ strongly}, \tag{37}
$$

$$
p_\varepsilon(0,\ t)\to p(0,\ t) \text{ in } L^2(0,\ T) \text{ strongly}. \tag{38}
$$

II. 2. By duality, we can define the cost function by means of problem (III) as follows

$$
M(S)=N\int_\Omega S^2\,dx+\int_0^T (p(0,\ t;\ S)-Z_d(t))^2\,dt,\ \forall S\in\mathscr{U}^*, \tag{39}
$$

where $Z_d(t)$ is given in $L^2(0,\ T)$, N is given>0 and $\mathscr{U}^*$ is defined by (26). There exists a unique element $q_0=q_0(x)\in\mathscr{U}^*$ such that

$$
M(q_0)=\inf_{S\in\mathscr{U}^*} M(S). \tag{40}
$$

The corresponding optimality system is the following

$$
(\mathrm{V}) \quad \begin{cases} -\dfrac{\partial p}{\partial t}+Ap=0,\ \dfrac{\partial y}{\partial t}+Ay=(p(0,\ t)-Z_d(t))\delta(x) & \text{in } Q, \\ p=0,\ y=0 & \text{on } \Sigma, \\ p(x,\ T)=-\dfrac{1}{N}y(x,\ T),\ y(x,\ 0)=0 & \text{in } \Omega \end{cases}
$$

and the optimal control q_0 is given by

$$
q_0=-\frac{1}{N}y(x,\ T). \tag{41}
$$

Besides, by means of problem $(\mathrm{III})_\varepsilon$ we can also define the cost function

$$
M_\varepsilon(S)=N\int_\Omega S^2(x)\,dx+\int_0^T (p_\varepsilon(0,\ t;\ S)-Z_d(t))^2\,dt,\ \forall S\in L^2(\Omega). \tag{42}
$$

Noticing (20), it is easy to see that there exists a unique element $q_\varepsilon=q_\varepsilon(x)\in L^2(\Omega)$ such that

$$M_\varepsilon(q_\varepsilon)=\inf_{S\in L^2(\Omega)} M_\varepsilon(S). \tag{43}$$

The corresponding optimality system is the following

$$(\mathrm{V})_\varepsilon \quad \begin{cases} -\dfrac{\partial p_\varepsilon}{\partial t}+\varepsilon A^2 p_\varepsilon+Ap_\varepsilon=0, \\ \dfrac{\partial y_\varepsilon}{\partial t}+\varepsilon A^2 y_\varepsilon+Ay_\varepsilon=(p_\varepsilon(0,\ t)-Z_d(t))\delta(x) & \text{in } Q, \\ p_\varepsilon=Ap_\varepsilon=0,\ y_\varepsilon=Ay_\varepsilon=0 & \text{on } \Sigma, \\ p_\varepsilon(x,\ T)=-\dfrac{1}{N}y_\varepsilon(x,\ T),\ y_\varepsilon(x,\ 0)=0 & \text{in } \Omega \end{cases}$$

and the optimal control q_ε is given by

$$q_\varepsilon=-\frac{1}{N}y_\varepsilon(x,\ T). \tag{44}$$

The following theorem can be proved in a similar way as in the proof of Theorem 4.

Theorem 5 *As $\varepsilon\to 0$, we have*

(i) $M_\varepsilon(q_\varepsilon)\to M(q_0)$, (45)

(ii) $q_\varepsilon\to q_0$ *in* $L^2(\Omega)$ *strongly.* (46)

Moreover, for the solutions $\{p_\varepsilon,\ y_\varepsilon\}$ and $\{p,\ y\}$ of the optimality systems $(\mathrm{V})_\varepsilon$ *and* (V) *respectively, as $\varepsilon\to 0$, we have the same results* (35)—(38).

II. 3. Instead of (27) we can also define the cost function by means of (I) as follows

$$J(v)=N\int_0^T v^2\,dt+\int_Q |y(t;\ v)-Z_d|^2\,dx\,dt,\ \forall v\in L^2(0,\ T), \tag{47}$$

where Z_d is given in $L^2(Q)$. There exists a unique element $u_0=u_0(t)\in L^2(0,\ T)$ such that

$$J(u_0)=\inf_{v\in L^2(0,T)} J(v). \tag{48}$$

The corresponding optimality system is the following

$$(\mathrm{VI}) \quad \begin{cases} \dfrac{\partial y}{\partial t}+Ay=-\dfrac{1}{N}\varphi(0,\ t)\delta(x),\ -\dfrac{\partial \varphi}{\partial t}+A\varphi=y-Z_d & \text{in } Q, \\ y=0,\ \varphi=0 & \text{on } \Sigma, \\ y(x,\ 0)=0,\ \varphi(x,\ T)=0 & \text{in } \Omega \end{cases}$$

and the optimal control u_0 is given by

$$u_0=-\frac{1}{N}\varphi(0,\ t). \tag{49}$$

Similarly, by means of $(\mathrm{I})_\varepsilon$ we can define the cost function

$$J_\varepsilon(v)=N\int_0^T v^2\,dt+\int_Q |y_\varepsilon(t;\ v)-Z_d|^2 dx\,dt,\ \forall v\in L^2(0,\ T). \tag{50}$$

There exists a unique element $u_\varepsilon=u_\varepsilon(t)\in L^2(0,\ T)$ such that

$$J_\varepsilon(u_\varepsilon)=\inf_{v\in L^2(0,T)} J_\varepsilon(v). \tag{51}$$

The corresponding optimality system is the following

$$(\text{VI})_\varepsilon \quad \begin{cases} \dfrac{\partial y_\varepsilon}{\partial t}+\varepsilon A^2 y_\varepsilon+Ay_\varepsilon=-\dfrac{1}{N}\varphi_\varepsilon(0, t)\delta(x), & \\ -\dfrac{\partial \varphi_\varepsilon}{\partial t}+\varepsilon A^2\varphi_\varepsilon+A\varphi_\varepsilon=y_\varepsilon-Z_d & \text{in } Q, \\ y_\varepsilon=Ay_\varepsilon=0,\ \varphi_\varepsilon=A\varphi_\varepsilon=0 & \text{on } \Sigma, \\ y_\varepsilon(x, 0)=0,\ \varphi_\varepsilon(x, T)=0 & \text{in } \Omega \end{cases}$$

and the optimal control u_ε is given by

$$u_\varepsilon=-\frac{1}{N}\varphi_\varepsilon(0, t). \tag{52}$$

In II. § 2, we shall prove

Theorem 6 *As $\varepsilon\to0$, We have*

(i) $J_\varepsilon(u_\varepsilon)\to J(u_0)$, (53)

(ii) $u_\varepsilon\to u_0$ *in* $L^2(0, T)$ *strongly.* (54)

Moreover, for the solutions $\{y_\varepsilon, \varphi_\varepsilon\}$ *and* $\{y, \varphi\}$ *of the optimality systems* $(\text{VI})_\varepsilon$ *and* (VI) *respectively, we have, as* $\varepsilon\to0$

$$y_\varepsilon\to y \text{ in } L^2(Q) \text{ strongly}, \tag{55}$$

$$\varphi_\varepsilon\to\varphi \text{ in } H^{2,1}(Q) \text{ strongly}, \tag{56}$$

$$\varphi_\varepsilon(0, t)\to\varphi(0, t) \text{ in } L^2(0, T) \text{ strongly}. \tag{57}$$

II. 4. By duality, we can define the following cost functions by means of problems (II) and $(\text{II})_\varepsilon$ respectively

$$M(\psi)=N\int_Q\psi^2\,dx\,dt+\int_0^T(\varphi(0, t;\ \psi)-Z_d)^2dt,\ \forall\psi\in L^2(Q) \tag{58}$$

and

$$M_\varepsilon(\psi)=N\int_Q\psi^2\,dx\,dt+\int_0^T(\varphi_\varepsilon(0, t;\ \psi)-Z_d)^2\,dt,\ \forall\psi\in L^2(Q). \tag{59}$$

We can obtain the similar results as in Theorem 6. The detail is omitted here.

In what follows, the letter C always denotes certain constants independent of ε.

I. Limit behaviors of solutions for some parabolic equations of higher order

§ 1. Proof of Theorems 1 and 2.

By transposition, the solution $y=y(t;\ v)$ of (I) is defined by the following Greens formula

$$\int_Q y\psi\,dx\,dt=\int_0^T\varphi(0, t)v(t)dt,\ \forall\psi\in L^2(Q), \tag{I. 1. 1}$$

where $\varphi=\varphi(t;\ \psi)$ is the solution of (II).

Moreover, the solution $y_\varepsilon=y_\varepsilon(t;\ v_\varepsilon)$ of $(\text{I})_\varepsilon$ satisfies the similar formula

$$\int_Q y_\varepsilon\psi_\varepsilon\,dx\,dt=\int_0^T\varphi_\varepsilon(0, t)v_\varepsilon(t)dt,\ \forall\psi_\varepsilon\in L^2(Q), \tag{I. 1. 2}$$

where $\varphi_\varepsilon=\varphi_\varepsilon(t;\ \psi_\varepsilon)$ is the solution of $(\text{II})_\varepsilon$.

According to (I. 1. 1) and (I. 1. 2), Theorem 1 is a direct consequence of Theorem 2 by duality (cf. [4]).

Proof of Theorem 2:

Lemma 1 (cf. [3]). *For any $\psi\in L^2(Q)$, the solution φ of (II) satisfies*

$$\varphi\in H^{2,1}(Q) \tag{I. 1. 3}$$

and

$$\|\varphi(t;\ \psi)\|_{H^{2,1}(Q)}\leqslant C\|\psi\|_{L^2(Q)}. \tag{I. 1. 4}$$

In particular

$$\|\varphi(0,\ t;\ \psi)\|_{L^2(0,T)}\leqslant C\|\psi\|_{L^2(Q)}. \tag{I. 1. 5}$$

Lemma 2. *For any $\varepsilon>0$ fixed, $\forall\psi_\varepsilon\in L^2(Q)$, the solution φ_ε of* (II)$_\varepsilon$ *satisfies*

$$\varphi_\varepsilon\in L^2(0,\ T;\ H^4(\Omega)),\ \frac{\partial\varphi_\varepsilon}{\partial t}\in L^2(Q) \tag{I. 1. 6}$$

and

$$\|\varphi_\varepsilon(t;\ \psi_\varepsilon)\|_{H^{2,1}(Q)}\leqslant C\|\psi_\varepsilon\|_{L^2(Q)}. \tag{I. 1. 7}$$

In particular

$$\|\varphi_\varepsilon(0,\ t;\ \psi_\varepsilon)\|_{L^2(0,T)}\leqslant C\|\psi_\varepsilon\|_{L^2(Q)}. \tag{I. 1. 8}$$

Proof Multiplying the equation

$$-\frac{\partial\varphi_\varepsilon}{\partial t}+\varepsilon A^2\varphi_\varepsilon+A\varphi_\varepsilon=\psi_\varepsilon$$

by φ_ε and $\frac{\partial\varphi_\varepsilon}{\partial t}$ respectively and integrating by parts, Lemma 2 follows from the classical theorem of regularity of solutions for linear elliptic equations.

From Lemmas 1 and 2 we obtain

Lemma 3. *It is sufficient to prove Theorem 2 for $\psi_\varepsilon\equiv\psi$ independent of ε and $\psi\in\mathscr{D}(Q)$, where $\mathscr{D}(Q)$ is the space of infinitely differentiable functions with compact support.*

Hence, it remains only to prove

Lemma 4. *Suppose that φ_ε is the solution of*

$$\begin{cases} -\frac{\partial\varphi_\varepsilon}{\partial t}+\varepsilon A^2\varphi_\varepsilon+A\varphi_\varepsilon=\psi & \text{in } Q,\\ \varphi_\varepsilon=A\varphi_\varepsilon=0 & \text{on } \Sigma,\\ \varphi_\varepsilon(x,\ T)=0 & \text{in } \Omega,\end{cases} \tag{I. 1. 9}$$

where $\psi\in\mathscr{D}(Q)$, then as $\varepsilon\to0$, we have

$$\varphi_\varepsilon\to\varphi \text{ in } H^{2,1}(Q) \text{ strongly}. \tag{I. 1. 10}$$

In particular

$$\varphi_\varepsilon(0,\ t)\to\varphi(0,\ t) \text{ in } L^2(0,\ T) \text{ strongly}. \tag{I. 1. 11}$$

Proof Since $\psi\in\mathscr{D}(Q)$, φ and φ_ε are all regular. According to Iemma 2, we have

$$\|\varphi_\varepsilon\|_{H^{2,1}(Q)}\leqslant C, \tag{I. 1. 12}$$

$$\|\varphi_\varepsilon(0,\ t)\|_{L^2(0,T)}\leqslant C. \tag{I. 1. 13}$$

By regularity, we have also

$$\left\|\frac{\partial\varphi_\varepsilon}{\partial t}\right\|_{H^{2,1}(Q)}\leqslant C,\ \left\|\frac{\partial^2\varphi_\varepsilon}{\partial t^2}\right\|_{H^{2,1}(Q)}\leqslant C, \tag{I. 1. 14}$$

$$\left\| \frac{\partial \varphi_\varepsilon}{\partial t}(0,\ t) \right\|_{L^2(0,T)} \leqslant C. \tag{I. 1. 15}$$

Hence

$$\|\varphi_\varepsilon\|_{H^2(Q)} \leqslant C, \tag{I. 1. 16}$$

$$\|\varphi_\varepsilon(0,\ t)\|_{H^1(0,T)} \leqslant C. \tag{I. 1. 17}$$

Moreover, since $A\varphi_\varepsilon$ is also the solution of (I. 1. 9) in which ψ is replaced by $A\psi$, we have

$$\|A\varphi_\varepsilon\|_{H^{2,1}(Q)} \leqslant C, \quad \left\| \frac{\partial A\varphi_\varepsilon}{\partial t} \right\|_{H^{2,1}(Q)} \leqslant C, \tag{I. 1. 18}$$

hence

$$\|A\varphi_\varepsilon\|_{H^1(Q)} \leqslant C. \tag{I. 1. 19}$$

According to (I. 1. 16)、(I. 1. 17) and (I. 1. 19), by compactness from $\{\varphi_\varepsilon\}$ we can extract a subsequence $\{\varphi_\eta\}$ such that

$$\varphi_\eta \to \Phi \text{ in } H^1(Q) \text{ strongly}, \tag{I. 1. 20}$$

$$\varphi_\eta(0,\ t) \to \Phi(0,\ t) \text{ in } L^2(0,\ T) \text{ strongly}, \tag{I. 1. 21}$$

$$A\varphi_\eta \to A\Phi \text{ in } L^2(Q) \text{ strongly}. \tag{I. 1. 22}$$

Then, according to the classical theorem of reqularity of solutions for linear elliptic equations, it follows from (I. 1. 20) and (I. 1. 22) that

$$\varphi_\eta \to \Phi \text{ in } H^{2,1}(Q) \text{ strongly}. \tag{I. 1. 23}$$

Hence, it remains only to prove

$$\Phi = \varphi. \tag{I. 1. 24}$$

Passing to the limit in (I. 1. 9), from (I. 1. 23) we get

$$\begin{cases} -\dfrac{\partial \Phi}{\partial t} + A\Phi = \psi & \text{in } Q, \\ \Phi = 0 & \text{on } \Sigma \\ \Phi(x,\ T) = 0 & \text{in } \Omega, \end{cases}$$

so (I. 1. 24) holds.

Thus, Theorem 2 (hence Theorem 1) is proved.

§ 2. Proof of Theorems 1 bis and 2 bis.

Lemma 5. *Theorem 1 bis is a direct consequence of Theorem 2 bis.*

Proof This Lemma can be obtained by duality. Here we give a direct proof. Taking $\psi = y$ and $\psi_\varepsilon = y$ in (I. 1. 1) and (I. 1. 2) respectively, we get

$$\int_Q y^2\, dx\, dt = \int_0^T \varphi(0,\ t;\ y)\, v(t)\, dt, \tag{I. 2. 1}$$

$$\int_Q y_\varepsilon y\, dx\, dt = \int_0^T \varphi_\varepsilon(0,\ t;\ y)\, v_\varepsilon(t)\, dt. \tag{I. 2. 2}$$

Using Theorem 2 we have

$$\int_Q y_\varepsilon y\, dx\, dt \to \int_Q y^2\, dx\, dt, \text{ as } \varepsilon \to 0. \tag{I. 2. 3}$$

Now taking $\psi = y$ and $\psi_\varepsilon = y_\varepsilon$ in (I. 1. 1) and (I. 1. 2) respectively, we get (I. 2. 1) and

$$\int_Q y_\varepsilon^2 \, dx\, dt = \int_0^T \varphi_\varepsilon(0, t; y_\varepsilon) v_\varepsilon(t) dt. \tag{I. 2. 4}$$

Hence under the hypotheses of Theorem 1 bis, if Theorem 2 bis holds, then using Theorem 1 we have

$$\int_Q y_\varepsilon^2 \, dx\, dt \to \int_Q y^2 \, dx\, dt, \text{ as } \varepsilon \to 0. \tag{I. 2. 5}$$

Thus, Theorem 1 bis follows from (I. 2. 3) and (I. 2. 5).

Proof of Theorem 2 bis:

According to Theorem 2, it is easy to see that as $\varepsilon \to 0$, if

$$\psi_\varepsilon \to 0 \text{ in } L^2(Q) \text{ weakly,} \tag{I. 2. 6}$$

then

$$\varphi_\varepsilon(t; \psi_\varepsilon) \to 0 \text{ in } H^{2,1}(Q) \text{ weakly.} \tag{I. 2. 7}$$

By Lemma 2, from $\{\psi_\varepsilon\}$ we can extract a subsequence $\{\psi_\eta\}$ such that

$$\varphi_\eta \to \Phi \text{ in } L^2(0, T; H^2(\Omega)) \text{ weakly,}$$

$$\frac{\partial \varphi_\eta}{\partial t} \to \frac{\partial \Phi}{\partial t} \text{ in } L^2(Q) \text{ weakly,}$$

$$A\varphi_\eta \to A\Phi \text{ in } L^2(Q) \text{ weakly,}$$

$$A^2\varphi_\eta \to A^2\Phi \text{ in } L^2(0, T; H^{-2}(\Omega)) \text{ weakly.}$$

Thus, passing to the limit in $(\mathrm{II})_\eta$, we get

$$\begin{cases} -\dfrac{\partial \Phi}{\partial t} + A\Phi = 0 & \text{in } Q, \\ \Phi = 0 & \text{on } \Sigma, \\ \Phi(x, T) = 0 & \text{in } \Omega, \end{cases}$$

hence $\Phi = 0$, namely (I. 2. 7) holds.

Theorem 2 bis (hence Theorem 1 bis) is proved.

Proof of corollary 1. 1: Since $y_\varepsilon(t; v_\varepsilon)$ converges to $y(t; v)$ in $L^2(Q)$ as $\varepsilon \to 0$, it follows from $(\mathrm{I})_\varepsilon$ that

$$\frac{\partial y_\varepsilon}{\partial t}(t; v_\varepsilon) \to \frac{\partial y}{\partial t}(t; v), \text{ in } L^2(0, T; H^{-k}(\Omega)) \ (k>0, \text{ suitable integer}),$$

hence $\quad y_\varepsilon(T; v_\varepsilon) \to y(T; v)$, in $H^{-s}(\Omega)$ $(s>0$, suitable integer),

from this the corollary follows easily.

§ 3. Proof of Theorem 3

It is easy to verify the following lemmas.

Lemma 6. *For any* $S(x) \in L^2(\Omega)$, *the solution* p *of* (III) *satisfies*

$$\|p(t; S)\|_{L^2(0,T;\, H_0^1(\Omega))} \leqslant C\|S\|_{L^2(\Omega)}. \tag{I. 3. 1}$$

Lemma 7. *For any* $S_\varepsilon(x) \in L^2(\Omega)$, *the solution* p_ε *of* $(III)_\varepsilon$ *satisfies*

$$\|p_\varepsilon(t; S_\varepsilon)\|_{L^2(0,T;\, H_0^1(\Omega))} \leqslant C\|S_\varepsilon\|_{L^2(\Omega)}. \tag{I. 3. 2}$$

Proof of Theosem 3 By Iemma 7 we can extract from $\{S_\varepsilon\}$ a subsequence $\{S_\eta\}$ such that

$$p_\eta(t;\ S_\eta)\to P \text{ in } L^2(0,\ T;\ H_0^1(\Omega)) \text{ weakly,} \tag{I. 3. 3}$$

then

$$\begin{cases} Ap_\eta\to AP, \text{ in } L^2(0,\ T;\ H^{-1}(\Omega)) \text{ weakly,} \\ A^2p_\eta\to A^2P, \text{ in } L^2(0,\ T;\ H^{-3}(\Omega)) \text{ weakly,} \\ \dfrac{\partial p_\eta}{\partial t}\to\dfrac{\partial P}{\partial t}, \text{ in } L^2(0,\ T;\ H^{-3}(\Omega)) \text{ weakly.} \end{cases} \tag{I. 3. 4}$$

Passing to the limit in $(\mathrm{III})_\eta$ we get

$$\begin{cases} -\dfrac{\partial P}{\partial t}+AP=0 & \text{in } Q, \\ P=0 & \text{on } \Sigma, \\ P(x,\ T)=S(x) & \text{in } \Omega, \end{cases}$$

hence

$$P=p(t;\ S). \tag{I. 3. 5}$$

From this, Theorem 3 follows easily.

Proof of Corollary 3. 1 It is well known [cf. [2、7、9]) that for any $S(x)\in L^2(\Omega)$ we have

$$p(0,\ t;\ S)\in\mathscr{U}', \tag{I. 3. 6}$$

and

$$\int_0^T p(0,\ t;\ S)v(t)\,dt=\int_\Omega y(T;\ v)S(x)\,dx,\ \forall v\in\mathscr{U}, \tag{I. 3. 7}$$

where $\mathscr{U}'$ denotes the dual of $\mathscr{U}$ (when $L^2(0,\ T)$ is identified with its dual), $y(T;\ v)$ is given by (I) and in the left hand side of (I. 3. 7) the integral denotes the duality between $\mathscr{U}'$ and $\mathscr{U}$.

Besides, from $(\mathrm{I})_\varepsilon$ and $(\mathrm{III})_\varepsilon$ we can also get

$$\int_0^T p_\varepsilon(0,\ t;\ S_\varepsilon)v(t)\,dt=\int_\Omega y_\varepsilon(T;\ v)S_\varepsilon(x)\,dx,\ \forall v\in L^2(0,\ T). \tag{I. 3. 8}$$

In order to prove Corollary 3.1 we shall use the following Lemma 8 whose proof will be given at the end of this section.

Lemma 8. *For any $v(t)\in\mathscr{D}(0,\ T)$ fixed, we have*

$$y_\varepsilon(T;\ v)\to y(T;\ v) \textit{ in } L^2(\Omega) \textit{ strongly, as } \varepsilon\to 0, \tag{I. 3. 9}$$

where y_ε and y are defined by $(\mathrm{I})_\varepsilon$ and (I) respectively.

Suppose (if necessary, extract a subsequence)

$$p_\varepsilon(0,\ t;\ S_\varepsilon)\to d_0(t) \text{ in } L^2(0,\ T) \text{ weakly (resp. strongly),} \tag{I. 3. 10}$$

for any $v(t)\in\mathscr{D}(0,\ T)$ fixed, passing to the limit in (I. 3. 8), by Lemma 8 it follows from (21) and (I. 3. 10) that

$$\int_0^T d_0(t)v(t)\,dt=\int_\Omega y(T;\ v)S(x)\,dx,\ \forall v\in\mathscr{D}(0,\ T), \tag{I. 3. 11}$$

hence, according to (I. 3. 7) we get

$$d_0(t)=p(0,\ t;\ S) \text{ in } \mathscr{D}'(0,\ T). \tag{I. 3. 12}$$

Noticing (cf. [2、7、8])

$$\mathscr{D}(0,\ T)\subset\mathscr{U}\subset L^2(0,\ T)\subset\mathscr{U}'\subset\mathscr{D}'(0,\ T), \tag{I. 3. 13}$$

(24) and (25) follow from (I. 3. 12). Corollary 3.1 is proved.

Proof of Lemma 8 It follows from Theorem 1 bis that

$$y_\varepsilon(t;\ v)\to y(t;\ v) \text{ in } L^2(Q) \text{ strongly, as } \varepsilon\to 0. \tag{I. 3. 14}$$

Since $v(t)\in\mathscr{D}(0,\ T)$, by regularity we have also

$$\frac{\partial y_\varepsilon}{\partial t}(t;\ v)\to\frac{\partial y}{\partial t}(t;\ v) \text{ in } L^2(Q) \text{ strongly, as } \varepsilon\to 0. \tag{I. 3. 15}$$

Hence (I. 3. 9) follows from (I. 3. 14) and (I. 3. 15).

§ 4. Proof of Theorem 3 bis

From Lemmas 6 and 7, we get

Lemma 9. *It is sufficient to prove Theorem 3 bis for $S_\varepsilon\equiv S$ independent of ε and $S\in\mathscr{D}(\Omega)$.*

Hence, it remains only to prove

Lemma 10. *Let p_ε be the solution of*

$$\begin{cases} -\dfrac{\partial p_\varepsilon}{\partial t}+\varepsilon A^2 p_\varepsilon+Ap_\varepsilon=0, & \text{in } Q,\\ p_\varepsilon=Ap_\varepsilon=0, & \text{on } \Sigma,\\ p_\varepsilon(x,\ T)=S(x), & \text{in } \Omega \end{cases} \tag{I. 4. 1}$$

in which $S(x)\in\mathscr{D}(\Omega)$, then as $\varepsilon\to 0$, we have

$$p_\varepsilon\to p(t;\ S) \text{ in } L^2(0,\ T;\ H_0^1(\Omega)) \text{ strongly.} \tag{I. 4. 2}$$

Proof From Lemma 7 we have

$$\|p_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))}\leqslant C. \tag{I. 4. 3}$$

Since Ap_ε is also the solution of (I. 4. 1) in which S is seplaced by AS, we have

$$\|Ap_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))}\leqslant C. \tag{I. 4. 4}$$

Then, according to the classical theorem of regularity of solutions for linear elliptic equations, we get

$$\|p_\varepsilon\|_{L^2(0,T;H^3(\Omega))}\leqslant C. \tag{I. 4. 5}$$

By regularity, we have also

$$\left\|\frac{\partial p_\varepsilon}{\partial t}\right\|_{L^2(0,T;H^2(\Omega))}\leqslant C. \tag{I. 4. 6}$$

Hence, it follows from (I. 4. 5) and (I. 4. 6) that p_ε belongs to a strongly compact subset of $L^2(0,\ T;\ H_0^1(\Omega))$, then Lemma 10 holds.

Theorem 3 bis is proved.

II. Applications to the optimal control

§ 1. Proof of Theorem 4

1. Since

$$\begin{aligned} 0\leqslant J_\varepsilon(u_\varepsilon) &= N\int_0^T u_\varepsilon^2\,dt+\int_\Omega |y_\varepsilon(T;\ u_\varepsilon)-Z_d|^2\,dx\\ &= \inf_{v\in L^2(0,T)} J_\varepsilon(v)\leqslant J_\varepsilon(0)=C_0, \end{aligned} \tag{II. 1. 1}$$

where

$$C_0=\int_\Omega Z_d^2 dx. \tag{II. 1. 2}$$

we have

$$\|u_\varepsilon(t)\|_{L^2(0,T)}\leqslant C, \tag{II. 1. 3}$$

$$\|y_\varepsilon(T;\ u_\varepsilon)\|_{L^2(\Omega)}\leqslant C. \tag{II. 1. 4}$$

Hence, we can extract from $\{u_\varepsilon\}$ a subsequence $\{u_\eta\}$ such that, as $\eta\to 0$,

$$u_\eta(t)\to w_0 \text{ in } L^2(0,\ T) \text{ weakly}, \tag{II. 1. 5}$$

$$y_\eta(T;\ u_\eta)\to Y \text{ in } L^2(\Omega) \text{ weakly}. \tag{II. 1. 6}$$

By Corollary 1.1 we get

$$y_\eta(T;\ u_\eta)\to y(T;\ w_0) \text{ in } L^2(\Omega) \text{ weakly}, \tag{II. 1. 7}$$

and

$$w_0\in \mathscr{U}. \tag{II. 1. 8}$$

According to the inferior semi-continuity for the weak topology, we can pass to the limit in (II. 1. 1) and obtain that

$$\varliminf_{\eta\to 0} J_\eta(u_\eta)\geqslant N\int_0^T w_0^2 dt+\int_\Omega |y(T;\ w_0)-Z_d|^2 dx=J(w_0)\geqslant J(u_0), \tag{II. 1. 9}$$

hence

$$\varliminf_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon)\geqslant J(u_0). \tag{II. 1. 10}$$

2. By means of the duality, we are going to prove

$$\varlimsup_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon)\leqslant J(u_0). \tag{II. 1. 11}$$

Set

$$F(v)=N\int_0^T v^2 dt,\ \forall v\in \mathscr{U}, \tag{II. 1. 12}$$

$$G(q)=\int_\Omega |q-Z_d|^2 dx,\ \forall q\in L^2(\Omega), \tag{II. 1. 13}$$

we introduce the corresponding conjugate functionals

$$F^*(w)=\begin{cases}\dfrac{1}{4N}\displaystyle\int_0^T w^2 dt, & \text{if } w\in L^2(0,\ T),\\ +\infty, & \text{if } w\in \mathscr{U}'\backslash L^2(0,\ T),\end{cases} \tag{II. 1. 14}$$

$$G^*(S)=\int_\Omega\left(\frac{S^2}{4}+SZ_d\right)dx,\ \ \forall S\in L^2(\Omega), \tag{II. 1. 15}$$

where $\mathscr{U}'$ is the dual of $\mathscr{U}$.

Let L be the continuous linear mapping from $\mathscr{U}$ to $L^2(\Omega)$, defined by

$$L:\ v\to y(T;\ v), \tag{II. 1. 16}$$

where $y(T;\ v)$ is given by problem (I). The corresponding adjoint mapping L^* can be defined by

$$L^*:\ S\to p(0,\ t;\ S), \tag{II. 1. 17}$$

where p is the solution of problem (III). It is well known that L^* is a continuous linear mapping from $L^2(\Omega)$ to $\mathscr{U}'$ (cf. [2, 7, 9]).

According to Rockafellar's duality Theorem (cf. [6]), we have

$$J(u_0)=\inf_{v\in\mathscr{U}} J(v)=\inf_{v\in\mathscr{U}}(F(v)+G(Lv))$$
$$=-\inf_{S\in L^2(\Omega)}(F^*(L^*S)+G^*(-S))=-\inf_{S\in\mathscr{U}^*} M(S)+C_0, \tag{II. 1. 18}$$

where

$$M(S)=\frac{1}{4N}\int_0^T p(0,\ t;\ S)^2 dt+\int_\Omega\left(\frac{S}{2}-Z_d\right)^2 dx, \tag{II. 1. 19}$$

$\mathscr{U}^*$ is defined by (26) and C_0 is given by (I. 1. 2).

Obviously, there exists a unique element $q_0\in\mathscr{U}^*$ such that

$$M(q_0)=\inf_{S\in\mathscr{U}^*} M(S). \tag{II. 1. 20}$$

Hence

$$J(u_0)=-M(q_0)+C_0. \tag{II. 1. 21}$$

In a similar way, for any $\varepsilon>0$ fixed, set

$$\tilde{F}(v)=N\int_0^T v^2 dt,\ \forall v\in L^2(0,\ T) \tag{II. 1. 22}$$

and

$$\tilde{F}^*(w)=\frac{1}{4N}\int_0^T w^2 dt,\ \ \forall w\in L^2(0,\ T), \tag{II. 1. 23}$$

according to Rockafellar's duality Theorem we have

$$J_\varepsilon(u_\varepsilon)=\inf_{v\in L^2(0,T)} J_\varepsilon(v)=\inf_{v\in L^2(0,T)}(\tilde{F}(v)+G(L_\varepsilon v))$$
$$=-\inf_{S\in L^2(\Omega)}(\tilde{F}^*(L_\varepsilon^* S)+G^*(-S))=-\inf_{S\in L^2(\Omega)} M_\varepsilon(S)+C_0, \tag{II. 1. 24}$$

where L_ε and L_ε^* are defined by (6) and (20) respectively

$$M_\varepsilon(S)=\frac{1}{4N}\int_0^T p_\varepsilon(0,\ t;\ S)^2 dt+\int_\Omega\left(\frac{S}{2}-Z_d\right)^2 dx \tag{II. 1. 25}$$

and C_0 is given by (II. 1. 2). Besides, in (II. 1. 25), $p_\varepsilon(0,\ t;\ S)$ is given by problem $(\text{III})_\varepsilon$ with $S_\varepsilon\equiv S$.

There exists a unique element $q_\varepsilon\in L^2(\Omega)$ such that

$$M_\varepsilon(q_\varepsilon)=\inf_{S\in L^2(\Omega)} M_\varepsilon(S), \tag{II. 1. 26}$$

hence

$$J_\varepsilon(u_\varepsilon)=-M_\varepsilon(q_\varepsilon)+C_0. \tag{II. 1. 27}$$

Thus, in order to get (II. 1. 11) it is sufficient to prove

$$\underline{\lim}_{\varepsilon\to 0} M_\varepsilon(q_\varepsilon)\geqslant M(q_0). \tag{II. 1. 28}$$

3. Proof of (II.1.28)

Since

$$0\leqslant M_\varepsilon(q_\varepsilon)=\frac{1}{4N}\int_0^T p_\varepsilon(0,\ t;\ q_\varepsilon)^2 dt+\int_\Omega\left(\frac{q_\varepsilon}{2}-Z_d\right)^2 dx$$
$$=\inf_{S\in L^2(\Omega)} M_\varepsilon(S)\leqslant M_\varepsilon(0)=C_0, \tag{II. 1. 29}$$

we have

$$\|q_\varepsilon\|_{L^2(\Omega)}\leqslant C, \tag{II. 1. 30}$$

$$\|p_\varepsilon(0,\ t;\ q_\varepsilon)\|_{L^2(0,T)}\leqslant C. \tag{II. 1. 31}$$

Hence, we can extract from $\{q_\varepsilon\}$ a subsequence $\{q_\eta\}$ such that as $\eta\to 0$,

$$q_\eta\to S_0 \text{ in } L^2(\Omega) \text{ weakly}, \tag{II. 1. 32}$$

$$p_\eta(0,\ t;\ q_\eta)\to d_0 \text{ in } L^2(0,\ T) \text{ weakly.} \tag{II. 1. 33}$$

By Corollary 3.1, we have

$$d_0=p(0,\ t;\ S_0) \tag{II. 1. 34}$$

and

$$S_0\in \mathscr{U}^*, \tag{II. 1. 35}$$

hence

$$p_\eta(0,\ t;\ q_\eta)\to p(0,\ t;\ S_0) \text{ in } L^2(0,\ T) \text{ weakly.} \tag{II. 1. 36}$$

According to the inferior semi-continuity for the weak topology, we can pass to the limit in (II. 1. 29) and obtain that

$$\begin{aligned}\varliminf_{\eta\to 0} M_\eta(q_\eta) &\geqslant \frac{1}{4N}\int_0^T p(0,\ t;\ S_0)^2\,dt+\int_\Omega\Big(\frac{S_0}{2}-Z_d\Big)^2 dx\\ &=M(S_0)\geqslant M(q_0),\end{aligned} \tag{II. 1. 37}$$

hence (II.1.28) follows immediately.

4. From (II. 1. 5)、(II. 1. 9)—(II. 1. 11) we obtain (33) and

$$u_\varepsilon\to u_0 \text{ in } L^2(0,\ T) \text{ weakly,} \tag{II. 1. 38}$$

hence it follows from Corollary 1.1 and (II. 1. 4) that

$$y_\varepsilon(T)\to y(T) \text{ in } L^2(\Omega) \text{ weakly.} \tag{II. 1. 39}$$

Thus, according to the inferior semi-continuity for the weak topology, it follows from (33) that

$$\|u_\varepsilon\|^2_{L^2(0,T)}\to \|u_0\|^2_{L^2(0,T)}, \tag{II. 1. 40}$$

$$\|y_\varepsilon(T)\|^2_{L^2(\Omega)}\to \|y(T)\|^2_{L^2(\Omega)}, \tag{II. 1. 41}$$

hence (34) and (36) hold. Moreover, from Theorem 1 bis and (29)、(32) we can get (35) and (38). Finally, (37) follows from Theorem 3 bis.

§ 2. Proof of Theorem 6

The proof is similar to the proof of Theorem 4, but we must set

$$F(v)=N\int_0^T v^2\,dt,\ \forall v\in L^2(0,\ T), \tag{II. 2. 1}$$

$$G(q)=\int_Q |q-Z_d|^2\,dx\,dt,\ \forall q\in L^2(Q), \tag{II. 2. 2}$$

$$F^*(w)=\frac{1}{4N}\int_0^T w^2\,dt,\ \forall w\in L^2(0,\ T), \tag{II. 2. 3}$$

$$G^*(\psi)=\int_Q\Big(\frac{\psi^2}{4}+\psi Z_d\Big)dx\,dt,\ \forall\psi\in L^2(Q) \tag{II. 2. 4}$$

and

$$L:\ v\to y(t;\ v), \tag{II. 2. 5}$$

$$L^*:\ \psi\to\varphi(0,\ t;\ \psi), \tag{II. 2. 6}$$

$$L_\varepsilon:\ v\to y_\varepsilon(t;\ v), \tag{II. 2. 7}$$

$$L^*_\varepsilon:\ \psi\to\varphi_\varepsilon(0,\ t;\ \psi). \tag{II. 2. 8}$$

Moreover, instead of using Theorems 3 and 3 bis, we must use Theorems 2 and 2 bis. The detail is omitted here.

III. Remarks

The previous results remain still valid if the boundary condition on Σ is changed by any one of the following boundary conditions:

For the original problem:

1° $$\frac{\partial y}{\partial \nu_A}+\lambda(x)y=0; \tag{III. 1}$$

or

2° $y|_\Gamma=c(t)$ (unknown function of t) and

$$\int_\Gamma\left(\frac{\partial y}{\partial \nu_A}+\lambda(x)y\right)dS=0 \text{ for a. e. } t\in(0,\ T); \tag{III. 2}$$

or

3° $\left.\frac{\partial y}{\partial \nu_A}\right|_\Gamma=c(t)$ (unknown function of t) and

$$\int_\Gamma\left(y+\lambda(x)\frac{\partial y}{\partial \nu_A}\right)dS=0 \text{ for a. e. } t\in(0,\ T). \tag{III. 3}$$

For the approximate problem:

1° $$\frac{\partial y_s}{\partial \nu_A}+\lambda(x)y_s=\frac{\partial Ay_s}{\partial \nu_A}+\lambda(x)Ay_s=0; \tag{III. 1)'$$

or

2° $y_s|_\Gamma=c_s(t)$, $Ay_s|_\Gamma=d_s(t)$ (unknown functions of t) and

$$\int_\Gamma\left(\frac{\partial y_s}{\partial \nu_A}+\lambda(x)y_s\right)dS=\int_\Gamma\left(\frac{\partial Ay_s}{\partial \nu_A}+\lambda(x)Ay_s\right)dS=0 \tag{III. 2)'$$

for a. e. $t\in(0,\ T)$;

or

3° $\left.\frac{\partial y_s}{\partial \nu_A}\right|_\Gamma=c_s(t)$, $\left.\frac{\partial Ay_s}{\partial \nu_A}\right|_\Gamma=d_s(t)$ (unknown functions of t) and

$$\int_\Gamma\left(y_s+\lambda(x)\frac{\partial y_s}{\partial \nu_A}\right)dS=\int_\Gamma\left(Ay_s+\lambda(x)\frac{\partial Ay_s}{\partial \nu_A}\right)dS=0 \tag{III. 3)'$$

for a. e. $t\in(0,\ T)$.

Where $\lambda(x)\geqslant0$ is a smooth function on Γ

$$\frac{\partial y}{\partial \nu_A}=\sum_{i,j=1}^{n}a_{ij}\nu_i\frac{\partial y}{\partial x_j}, \tag{III. 4}$$

in which $\boldsymbol{\nu}=(\nu_1,\ \cdots,\ \nu_n)$ is the unit normal oriented towards the exterior of Ω.

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某些高阶抛物型方程的解的极限性态及其在最优控制中的应用

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摘 要

在本文中，对某些高阶抛物型方程以及某些由抛物型方程所支配的系统的最优控制问题，我们证明了其解的一些极限性态。