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ON THE GROWTH AND THE DISTRIBUTION OF VALUES OF EXPONENTIAL SERIES CONVERGENT ONLY IN THE RIGHT HALF-PLANE

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Dedicated to Professor Su Bu-chin on the Occasion of his 80th Birthday and his 50th Year of Educational Work

Ritt, J. F. investigated, the growth of entire functions defined by exponential series and introduced the Ritt order or order (R). Mandelbrojt, S. [5] and Valiron G. [10] studied the growth and the distribution of values of such functions. Blambert, M. [3], Sunyer i Balaguer, F. [7], Tanaka, C. [8], the author[13,14] and others continued to do research work in this respect,

For analytic functions defined by exponential series convergent only in the right half-plane, the author[14] introduced the order (R) and studied some exponential series and random exponential series. In this paper we introduce the order $(R\text{-}H)$ of such functions. Applying an extension of an inequality and we study the growth and the distribution of values of such functions in some horizontal half-strips and obtain results similar to the case of seme entire functions defined by exponential series.[1)]

1. The order (R) in the right half-plane. Consider the exponential series

$$f(s)=\sum_{n=0}^{+\infty} a_n e^{-\lambda_n s}, \tag{1.1}$$

where $\{a_n\}$ is a sequence of complex numbers, $0=\lambda_0<\lambda_1<\cdots<\lambda_n\uparrow+\infty$, $s=\sigma+it$, σ and t being real variables. Suppose that

$$\overline{\lim_{n\to+\infty}}\frac{\log n}{\lambda_n}=\overline{\lim_{n\to+\infty}}\frac{\log|a_n|}{\lambda_n}=0. \tag{1.2}$$

Then the abscissa of convergence and that of absolute convergence of (1.1) is 0 and the series (1.1) defines a function $f(s)$ analytic in the right-half plane.

Let $$M(\sigma)=\sup_{-\infty<t<+\infty}|f(\sigma+it)|\,(\sigma>0).$$

The quantity

$$\rho=\overline{\lim_{\sigma\to+0}}\frac{\overset{+}{\log}\overset{+}{\log} M(\sigma)}{\log(1/\sigma)} \tag{1.3}$$

is called order (R) of $f(s)$ in $\sigma>0$. We have the following theorem:

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1) Some results in this paper have been announced in a Note[15].

Theorem 1. 1.[2)] *Suppose that* (1.2) *and*

$$\overline{\lim_{n\to+\infty}}\frac{n}{\lambda_n}=D_1<+\infty \tag{1.4}$$

are verified. Then $f(s)$ *is of order* (R) ρ *in* $\sigma>0$

$$\Leftrightarrow \overline{\lim_{n\to+\infty}}\frac{\overset{+}{\log}\overset{+}{\log}|a_n|}{\log\lambda_n}=\frac{\rho}{\rho+1}, \tag{1.5}$$

where $\frac{\rho}{\rho+1}$ *must be replaced by* (1) *in the case* $\rho=+\infty$.

Following Hiong Kin-lai [4] we introduce a proximate order (R) in the case $\rho=+\infty$. Let $\rho(u)$ $(u>0)$ be a strictly increasing positive funcition such that

1) $\lim_{u\to+\infty}\rho(u)=+\infty$,

2) $\lim_{u\to+\infty}\frac{\log U(u')}{\log U(u)}=1$,

where
$$u'=u+\frac{u}{\log U(u)},\quad U(u)=u^{\rho(u)}.$$
If

$$\overline{\lim_{\sigma\to+0}}\frac{\overset{+}{\log}\overset{+}{\log}M(\sigma)}{\log U(1/\sigma)}=1, \tag{1.6}$$

we say that $f(s)$ is of order $(R\text{–}H)$ $\rho(1/\sigma)$ in $\sigma>0$. We establish a theorem similar to Theorem 1.1.

Theorem 1. 2. *Suppose that* (1.2) *and* (1.4) *are verified. Then*

$$f(s) \text{ is of order } (R\text{–}H)\ \rho(1/\sigma)\Leftrightarrow \overline{\lim_{u\to+\infty}}\, t_n=1, \tag{1.7}$$

where
$$t_n=\begin{cases}\log\lambda_n/\log U(\lambda_n/\log|a_n|) & (|a_n|>1),\\ 0 & (|a_n|\leqslant 1).\end{cases}$$

Proof First we prove that if $\overline{\lim_{n\to+\infty}}\, t_n=1$, then

$$\overline{\lim_{\sigma\to+0}}\frac{\overset{+}{\log}\overset{+}{\log}M(\sigma)}{\log U(1/\sigma)}\leqslant 1. \tag{1.8}$$

For any $\eta>0$, there exists an integer $N>0$ such that for $n>N$ and $|a_n|>1$

$$\lambda_n<\left[U\left(\frac{\lambda_n}{\log|a_n|}\right)\right]^{1+\eta}.$$

Let $v=U(u)$ and $u=\varphi(v)$ be two reciprocally inverse functions. Then for $n>N$ and $|a_n|>1$,

$$\varphi\left(\lambda_n^{\frac{1}{1+\eta}}\right)<\frac{\lambda_n}{\log|a_n|}$$

and consequently

2) This theorem was stated in [14] without the condition (1.4). But in order to prove it we must have $\sum_{n=0}^{+\infty}e^{-\lambda_n\varepsilon\sigma}=O\left(\frac{1}{\varepsilon\sigma}\right)$ $(\varepsilon>0,\ \sigma\to+0)$ for which the condition (1.2) is not sufficient ([14], p. 102), the condition (1.4) must be added in other related theorems in [15].

$$|a_n|e^{-\lambda_n\sigma}<\exp\left[\frac{\lambda_n}{\varphi(\lambda_n^{\frac{1}{1+\eta}})}-\lambda_n\sigma\right]. \tag{1.9}$$

Evidently (1.9) holds when $|a_n|\leqslant 1$. Hence we have (1.9) for $n>N$.

Fix $\sigma>0$ and take $\bar\lambda$ such that

$$\frac{1}{\sigma}\left(1+\frac{1}{\log U(1/\sigma)}\right)=\varphi(\bar\lambda^{\frac{1}{1+\eta}}).$$

Then

$$U\left[\frac{1}{\sigma}\left(1+\frac{1}{\log U(1/\sigma)}\right)\right]=\bar\lambda^{\frac{1}{1+\eta}}$$

and consequently $\bar\lambda=[U(1/\sigma)]^{1+\eta+o(1)}$ $(\sigma\to+0)$.

By (1.9), when $\lambda_n>\bar\lambda$, and $n>N$

$$|a_n|e^{-\lambda_n\sigma}<\exp\left[\lambda_n\left(\frac{1}{\varphi(\bar\lambda^{\frac{1}{1+\eta}})}-\sigma\right)\right]=\exp\left[\frac{-\lambda_n\sigma}{1+\log U(1/\sigma)}\right]<1.$$

Let

$$m(\sigma)=\max_n|a_n|e^{-\lambda_n\sigma}$$

and

$$n(\sigma)=\max_k\{k\,|\,|a_k|e^{-\lambda_k\sigma}=\max_n|a_n|e^{-\lambda_n\sigma}\}.$$

For sufficiently small σ, we have $\lambda_{n(\sigma)}<\bar\lambda$. Since [13]

$$\log m(\sigma)=A+\int_\sigma^a\lambda_{n(x)}\,dx\quad(0<\sigma<1),$$

where a $(>\sigma)$ and A are constants, we obtain

$$\log m(\sigma)<[U(1/\sigma)]^{1+2\eta+0(1)}(\sigma\to+0).$$

On the other hand, for any $\varepsilon>0$, there exists an integer $N_1>0$ such that for any $n>N_1$,

$$\lambda_n>n/(D_1+\varepsilon).$$

Suppose that σ is fixed such that $n(\sigma)>N_1$. Then

$$|f(s)|\leqslant\sum_{\lambda_n\leqslant\bar\lambda}|a_n|e^{-A_n\sigma}+\sum_{\lambda_n>\bar\lambda}|a_n|e^{-\lambda_n\sigma}$$
$$\leqslant(D_1+\varepsilon)[U(1/\sigma)]^{1+\eta+o(1)}\exp\{[U(1/\sigma)]^{1+2\eta+o(1)}\}$$
$$+\sum_{n=1}^{+\infty}\left\{\exp\left[-\frac{\sigma}{1+\log U(1/\sigma)}\right]\right\}^{\frac{n}{D_1+\varepsilon}}\quad(\sigma\to+0).$$

Hence for sufficiently small $\sigma>0$,

$$M(\sigma)<\exp\{[U(1/\sigma)]^{1+3\eta}\}+\frac{2(D_1+\varepsilon)}{\sigma}[1+\log U(1/\sigma)]$$
$$<\exp\{[U(1/\sigma)]^{1+4\eta}\}$$

and consequently (1.8) holds.

Now we prove that if $\overline{\lim\limits_{n\to+\infty}}\,t_n=1$, then we cannot have

$$\overline{\lim_{\sigma\to+\infty}}\frac{\overset{+}{\log}\,\overset{+}{\log}\,M(\sigma)}{\log U(1/\sigma)}=c<1. \tag{1.10}$$

Suppose that (1.10) would hold. Choose $\varepsilon>0$ and that $c+2\varepsilon<1$. Then there would exist $\sigma_0(0<\sigma_0<1)$ such that for $0<\sigma<\sigma_0$

$$\log M(\sigma)<[U(1/\sigma)]^{c+\varepsilon}$$

and conseqnently for $n=0, 1, 2, \cdots,$

$$\log|a_n| - \lambda_n\sigma < [U(1/\sigma)]^{c+\varepsilon}. \tag{1.11}$$

On the other hand, there exist arbitrarily large integers n such that

$$\lambda_n > [U(\lambda_n/\log|a_n|)]^{1-\varepsilon} \quad (|a_n|>1). \tag{1.12}$$

Take such sufficiently large n and take $\sigma(0<\sigma<\sigma_0)$ such that

$$[U(1/\sigma)]^{c+\varepsilon} = \frac{\lambda_n}{(\lambda_n/\log|a_n|)\log U(\lambda_n/\log|a_n|)}. \tag{1.13}$$

Combining (1.11) and (1.13), we see that for any $\eta>0$, there are arbitrarily large n and $\sigma\in(0, \sigma_0)$ such that

$$\frac{1}{\sigma} < \frac{\lambda_n}{\log|a_n|}\left[1+\frac{1}{\log U(\lambda_n/\log|a_n|)}\right]$$

and consequently

$$U(1/\sigma) < [U(\lambda_n/\log|a_n|)]^{1+\eta} \tag{1.14}$$

By (1.13) and (1.14) we have, for those sufficiently large n for which (1.12) holds.

$$\lambda_n < [U(\lambda_n/\log|a_n|)]^{(1+\eta)(c+\varepsilon)} \frac{\lambda_n}{\log|a_n|} \log U(\lambda_n/\log|a_n|)$$

and, for the other sufficiently large n, $t_n \leqslant 1-\varepsilon$, whence $\overline{\lim\limits_{n\to+\infty}} t_n < 1$, contrary to the hypothesis. The sufficiency of the condition (1.7) is proved.

We can prove easily the necessity of this condition.

The proof is completed.

2. The order in a horizontal half-strip. Let t_0 be a real number and a be a positive number. Denote the horizontal half-strip $\{s|\sigma>0, |t-t_0|\leqslant a\}$ by $S=S(t_0, a)$. Let

$$M_S(\sigma) < \max_{|t-t_0|\leqslant a} |f(\sigma+it)| \ (\sigma>0),$$

where $f(s)$ is defined by (1.1), (1.2) being verified. Replacing $M(\sigma)$ in (1.3) and (1.6) by $M_S(\sigma)$ we obtain the definitions of the orders (R) and $(R\text{-}H)$ of $f(s)$ in $S=S(t_0, a)$. In order to study these orders we establish a lemma.

Inspired by the idea of Anderson, J. M. and Binmore, K. G. [1], we suppose that

$$\inf_{q>0} \overline{\lim_{x\to+\infty}} \frac{N((x+1)q)-N(xq)}{q} = D < +\infty, \tag{2.1}$$

and

$$\underline{\lim_{n\to+\infty}} \frac{\log(\lambda_{n+1}-\lambda_n)}{\log\lambda_n} > -\infty, \tag{2.2}$$

where $N(x)$ denotes the number of λ_n less than $x(>0)$. Evidently if

$$\lim_{n\to+\infty}(\lambda_{n+1}-\lambda_n) = +\infty, \tag{2.3}$$

then (2.1) and (2.2) hold with $D=0$. We can prove $D_1 \leqslant D$.

Lemma 2. 1. *Suppose that the series (1.1) satifies (1.2), (2.1) and (2.2). Then for any $\varepsilon>0$, for any real number t_0 and for any $\sigma>0$, we have*

$$|a_k| \leqslant A\lambda_k^B M_S(\sigma)e^{\lambda_k\sigma} \quad (k=0,\ 1,\ 2,\ \cdots), \tag{2.4}$$

where $S=S(t_0,\ \pi(D+\varepsilon))$, *and* A *and* B *are positive constants depending only on* ε *and* $\{\lambda_n\}$.

The proof of this lemma is similar to that of Theorem 7 in [1]. In virtue of (2.1) for any $\varepsilon>0$ there exist a positive real number q_0 and a positve integer M such that the number of λ_j satisfying

$$nq_0 \leqslant \lambda_j < (n+1)q_0 \quad (n>M)$$

does not exceed $(D+\varepsilon)q_0$. Suppose that

$$\lambda_{p+1}=\min\{\lambda_l \mid \lambda_l \geqslant (M+1)q_0\}.$$

For the fixed k construct $j(t)$, $J(x)$, $h_\xi(t)$, $H_\xi(x)$ as in [1]. Corresponding to $\lambda_1,\ \lambda_2, \cdots,\ \lambda_p$, there exist[2] functions $g_\xi(t)$ and $G_\xi(x)$ such that $g_\xi(t) \in L(-\infty,\ +\infty)$ and

1) $G_\xi(x)=\int_{-\infty}^{+\infty} e^{ixt} g_\xi(t)\,dt$,

2) $g_\xi(t)=0 \left(|t|>\frac{\pi}{2}\cdot\frac{p}{\lambda_k}+p_\xi\right)$,

3) for the fixed value k

$$G_\xi(\lambda_k-\lambda_j)=0 \quad (j=1,\ 2,\ \cdots,\ p),$$

4) $\lim_{\xi\to+0} G_\xi(0)=\prod_{j=0}^{p}\cos\frac{\pi}{2}\cdot\frac{\lambda_j}{\lambda_k}$,

5) $\int_{-\infty}^{+\infty}|g_\xi(t)|\,dt \leqslant 1$.

We define $L(x)=G_\xi(x)H_\xi(x)J(x)$ and $l(t)=(g_\xi * h_\xi * j)(t)$ similar to $L(x)$ and $l(t)$ in [1] and define

$$f_N(s)=\sum_{j=0}^{N} a_j e^{-\lambda_j\sigma}e^{-i\lambda_j t} \quad (\sigma>0).$$

Then

$$\int_{-\infty}^{+\infty} l(t)e^{i\lambda_k t}f_N(s)\,dt=a_k e^{-\lambda_k\sigma}L(0),$$

whence

$$a_k e^{-\lambda_k\sigma}L(0)=\int_{-\eta}^{\eta} l(t)e^{i\lambda_k t}f_N(s)\,dt,$$

where $\eta=\pi\rho\delta+\tau_k+2hp\xi+(\pi p/2\lambda_k)+p\xi$; $\rho,\ \delta,\ \tau_k,\ h$ and p being the same as in [1]. Reasoning as in [1] and taking account of (2.2) to estimate

$$\prod_j \left(\cos\frac{\pi}{2}\cdot\frac{\lambda_{k-j}}{\lambda_k}\right)\left(\cos\frac{\pi}{2}\cdot\frac{\lambda_k}{\lambda_{k+j}}\right),$$

we come to the conclusion in Lemma 2.1.[3]

If (2.3) holds, we have a more precise result.

$$|a_k| \leqslant AM_S(\sigma)e^{\lambda_k\sigma} \quad (k=0,\ 1,\ 2,\ \cdots), \tag{2.5}$$

where $S=S(t_0,\ \varepsilon)$ and A is a positive constant depending only on ε and $\{\lambda_n\}$.

The proof is similar to that of Lemma 2.1.

Now we apply Lemma 2.1 to prove the following theorem:

3) Below the formula (5.13) in [1], there is a misprint. We must have $\tau_k=2^{-1}\pi\left(hp\lambda_k^{-1}+\sum_{j=k+1}^{k+hp}\lambda_j^{-1}\right)$.

Theorem 2.1. *Suppose that the series* (1.1) *satisfies* (1.2), (2.1) and (2.2). *Then the order* (R) *or* $(R\text{–}H)$ *of* $f(s)$ *in* $\sigma>0$ *is the same as that in any half-strip* $S(t_0, a)$, *where* t_0 *is an arbitrary real number* a *is an arbitrary number* $>\pi D$.

Proof 1) Suppose that $f(s)$ is of finite order (R) $\rho>0$. Then in any half-strip $S=S(t_0, a)$ $(a>\pi D)$, $f(s)$ is of order (R) $\rho_1\leqslant\rho$. If $f(s)$ were of order (R) $\rho^*<\rho$ in a half-strip $S^*=S(t_0^*, a^*)$ $(a^*>\pi D)$, for any $\varepsilon>0$, there would exist $\sigma_0>0$ such that for $\sigma\in(0, \sigma_0)$

$$\overset{+}{\log} M_{S^*}(\sigma)<(1/\sigma)^{\rho^*+\varepsilon}.$$

By the above inequality and Lemma 2.1 we would have, for $\sigma\in(0, \sigma_0)$ and $k\in\{0, 1, 2, \cdots\}$

$$\overset{+}{\log}|a_k|\leqslant\log A+B\log\lambda_k+(1/\sigma)^{\rho^*+\varepsilon}+\lambda_k\sigma. \tag{2.6}$$

Substituting $\left(\frac{\rho^*+\varepsilon}{\lambda_k}\right)^{1/(\rho^*+1+\varepsilon)}$ for σ in (2.6), we would get

$$\varlimsup_{k\to+\infty}\frac{\overset{+}{\log}\,\overset{+}{\log}|a_k|}{\log\lambda_k}\leqslant\frac{\rho^*}{\rho^*+1}<\frac{\rho}{\rho+1},$$

contrary to (1.5). The case $\rho=0$ or $\rho=+\infty$ can be studied in the same way.

2) The case that $f(s)$ is of order $(R\text{–}H)$ $\rho(\sigma)$ in $\sigma>0$ can be studied similarly. The proof of the Theorem is then completed.

We have similar results for proximate orders (R) [14]. In particular we can prove

Theorem 2. 2. *Suppose that the series* (1.1) *satisfies* (1.2), (2.1) *and* (2.2). *Then if*

$$\varlimsup_{\sigma\to+0}\sigma\overset{+}{\log}M(\sigma)=+\infty, \text{[4)]} \tag{2.7}$$

we have

$$\varlimsup_{\sigma\to+0}\sigma\overset{+}{\log}M_S(\sigma)=+\infty \tag{2.8}$$

for any half-strip $S=S(t_0, a)$, *where* t_0 *is an arbitrary real number and* a *is an arbitrary number* $>\pi D$.

3. Picard points and Borel points. Now we study the distribution of values of the function $f(s)$ in the vicinity of $\sigma=0$. If $f(s)$ takes, in every neighborhood of a certain point s_0, every finite complexvalue infinitely many times, with one possible *point* of $f(s)$. We have the following theorem: exception, then s_0 is called a *Picard*

Theorem 3. 1. *Suppose that the series* (1.1) *satisfies* (1.2), (2.1), (2.2) *and* (2.7). *Then in any interval of length* $2\pi D$ *on* $\sigma=0$, *there exists at least a Picard point of* $f(s)$.

Proof By Theorem 2.2, for an arbitrary real number t_0 and for an arbitrary number $a>\pi D$, (2.7) holds, where $S=S(t_0, a)$. Divide the horizontal half-strip S

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4) This condition is equivalent to

$$\varlimsup_{n\to+\infty}\frac{\log|a_n|}{\sqrt{\lambda_n}}=+\infty. \text{[14]}$$

into two horizontal half-strips $S^{(1)}$ and $S^{(2)}$ of the same breadth. Then

$$\overline{\lim_{\sigma\to+0}}\ \sigma\overset{+}{\log} M_{S^{(j)}}(\sigma)=+\infty \tag{3.1}$$

must be true for at least one of the half-strips $S^{(j)}(j=1, 2)$. Denote one half-strip for which (3.1) holds by S_1.

Divide S_1 into two horizontal half-strips $S_1^{(1)}$ and $S_1^{(2)}$ and repeat the above process indefinitely.

We obtain a sequence of harizental half-strips $\{S_l\}$ in $\sigma>0$ $(l=1, 2, \cdots)$ for which we have

$$\overline{\lim_{\sigma\to+0}}\ \sigma\overset{+}{\log} M_{S_l}(\sigma)=+\infty, \tag{3.2}$$

the breadth of S_{l+1} being half of that of S_l. Hence on the line joining $i(t_0-a)$ and $i(t_0+a)$ there exists a point it^* such that for any $\eta>0$,

$$\overline{\lim_{\sigma\to+0}}\ \sigma\overset{+}{\log} M_{S^*}(\sigma)=+\infty,$$

where $S^*=S(t^*, \eta)$.

Now we can prove t^* is a Picard point of $f(s)$ as in the proof of Theorem 4.3 in [14]. Since the above reasoning is valid for any $a>\pi D$ and since a limiting point of Picard points is itself a Picard point, we can easily complete the proof of the Theorem.

We consider at present Borel (R) points. Suppose that $f(s)$ is defined by (1.1) and that (1.2) holds. Let it_1 be a point on $\sigma=0$ and η be a positive number. For any finite complex number ζ, range the points s to satisfy $f(s)=\zeta$, $|s-it_p|<\eta$, $\operatorname{Re} s=\sigma>0$ in a sequence $\{s_n(\zeta, it_1, \eta)\}$, where $\operatorname{Re} s_n(\zeta, it_1, \eta)=\sigma_n(\zeta, it_1, \eta)$ is non increasing. We have

Theorem 3. 1. *Suppose that the series* (1.1) *satisfies* (1.2), (2.1) *and* (2.2) *and that* $f(s)$ *is of order* (R) $\rho(0<\rho<+\infty)$ *in* $\sigma>0$. *Then in any interval of length* $2\pi D$ *on* $\sigma=0$, *there exists at least a Borel* (R) *point of order at least* ρ *and at most* $\rho+1$.

That is to say, in any interval of length $2\pi D$ on $\sigma=0$, there exists a point it_1 such that for any sufficiently small positive number $\eta>0$ and for any finite complex number ζ, the series $\sum_n[\sigma_n(\zeta, it_1, \eta)]^\tau$ converges if $\tau>\rho+1$ and that for any positive number η and for any finite complex number ζ, with a possible exception, the series $\sum_n[\sigma_n(\zeta, it_1, \eta)]^\tau$ diverges if $\tau<\rho$.

In order to prove Theorem 3.2 we show first, as in the proof of Theorem 3.1, that there exists a point it_0 in any interval of length greater than $2\pi D$ on $\sigma=0$ such that for any $\eta>0$

$$\overline{\lim_{\sigma\to+0}}\ \frac{\overset{+}{\log}\overset{+}{\log} M_S(\sigma)}{\log(1/\sigma)}=\rho,$$

where $S=S(t_0, \eta)$. Then we can complete the proof as in the proof of Thearem 4.4 in [14] and in that of the previous Theorem.

We study now the case of infinite order (R). Suppose that the series (1.1) satisfies (1.2) and (1.4) and that

$$\overline{\lim_{\sigma\to+0}} \frac{\overset{+}{\log}\overset{+}{\log}\overset{+}{\log} M(\sigma)}{\log(1/\sigma)}=\rho_1(0<\rho_1<+\infty). \tag{3.3}$$

Construct[11, 14] a differentiable function $\rho_1(r)\,(r>0)$ such that

$$\lim_{r\to+\infty}\rho_1(r)=\rho_1,\ \overline{\lim_{r\to+\infty}}\rho_1'(r)r\log r=0 \tag{3.4}$$

and that

$$\overline{\lim_{\sigma\to+0}} \frac{\overset{+}{\log}\overset{+}{\log} M(\sigma)}{U_1(1/\sigma)}=1, \tag{3.5}$$

where $U_1(r)=r^{\rho_1(r)}$is a strictly increasing function. Let $k(r)$ be a continuons function defined for $r>0$ such that

$$\lim_{r\to+\infty}k(r)=k(0<k<+\infty). \tag{3.6}$$

We can show [11] thst

$$\lim_{r\to+\infty}\frac{U_1(k(r)r)}{U_1(r)}=k^{\rho_1}. \tag{3.7}$$

Put $U(r)=\exp[U_1(r)]$and $\rho(r)=U_1(r)/\log r$. Then $U(r)=r^{\rho(r)}$ and $\rho(r)$ is evidently an order (R–H) of $f(r)$ in $\sigma>0$.

Theorem 3. 3. *Suppose that the series* (1.1) *satisfies* (1.2), (2.1), (2.2) *and* (3.3) *and that* $f(s)$ *is of order* (R–H) $\rho(1/\sigma)$ *as defined above. Then in any interval of length* $2\pi D$ *on* $\sigma=0$, *there exists at least a Borel* (R–H) *point of order at least* $\rho\left(\frac{1}{6\sigma}\right)\left(1+\frac{\log 6}{\log\sigma}\right)$ *and at most* $\rho\left(\frac{6}{\sigma}\right)\left(1-\frac{\log 6}{\log\sigma}\right)$.

That is to say, in any interval of length $2\pi D$ on $\sigma=0$. There exists a point it_1 such that for any sufficiently small number $\eta>0$ and for any finite complex number ζ, the series $\sum_n[U(6/\sigma_n(\zeta,\ it_1,\ \eta))]^{-\tau}$ converges if $\tau>1$ and that for any positive number η and for any finite complex number ζ, with a possible exception, the series $\sum_n[U(1/6\sigma_n(\zeta,\ it_1,\ \eta)]^{-\tau}$ diverges if $\tau<1$.

Proof As in the proofs of the previous theorems we show that there exists a point it_0 in any interval of length greater than $2\pi D$ on $\sigma=0$ such that for any $\eta>0$

$$\overline{\lim_{\sigma\to+0}} \frac{\overset{+}{\log}\overset{+}{\log} M_S(\sigma)}{\log U(1/\sigma)}=1, \tag{3.8}$$

where $S=S(t_0, \eta)$.

Let ε be such that $0<\varepsilon<\pi$. The applications[9]

$$z=e^{-s+it_0},\quad Z=z^{\pi/2\varepsilon}\text{and } w=\frac{Z^2-1+2Z}{Z^2-1-2Z}$$

transform the domain $\sigma>0$, $|t-t_0|<\varepsilon$ into the domains $|z|<1$, $|\arg z|<\varepsilon$; $|Z|<1$,

$|\arg Z|<\frac{\pi}{2}$ and $|w|<1$; $s=i(t_0\pm\varepsilon)$ and $+\infty$ correspond respectively to $z=e^{\pm i\varepsilon}$ and 0; $Z=\pm i$ and 0; and $w=\pm i,\ -1$.

Under the above applications we obtain

$$f(s)=f_1(z)=f_2(Z)=f_3(w).$$

Put $M_S(\sigma)=\sup\limits_{|t-t_0|\leqslant\varepsilon}|f(\sigma+it)|$, $\overline{M}_S(\sigma)=\sup\limits_{|t-t_0|\leqslant\varepsilon}|f(x+it)|\ (x\geqslant\sigma,\ \sigma>0)$,

$$\overline{M}_1(r)=\sup_{|z|<r}|f_1(z)|\ (|\arg z|<\varepsilon,\ 0<r<1),$$

$$\overline{M}_2(R)=\sup_{|Z|<R}|f_2(Z)|\left(|\arg Z|<\frac{\pi}{2},\ 0<R<1\right),$$

$$M_2(R)=\sup_{|Z|=R}|f_2(Z)|\left(|\arg Z|<\delta\left(<\frac{\pi}{2}\right),\ R>C\right),$$

$$M_3(S)=\max_{|w|\leqslant S}|f_3(w)|\ (0<S<1),$$

where $S=S(t_0,\ \varepsilon)$. We have

$$M_S(\sigma)\leqslant\overline{M}_S(\sigma)\leqslant M(\sigma).$$

Hence[12, 14]

$$1=\overline{\lim_{\sigma\to+0}}\frac{\overset{+}{\log}\overset{+}{\log}\overline{M}_S(\sigma)}{\log U(1/\sigma)}=\overline{\lim_{\sigma\to+0}}\frac{\overset{+}{\log}\overset{+}{\log}\overline{M}_S(\sigma)}{\log U(1/(1-e^{-\sigma}))}$$

$$=\overline{\lim_{s\to+0}}\frac{\overset{+}{\log}\overset{+}{\log}\overline{M}_1(s)}{\log U(1/(1-s))}=\left(\frac{2\varepsilon}{\pi}\right)^{\rho_1}\overline{\lim_{R\to1-0}}\frac{\overset{+}{\log}\overset{+}{\log}\overline{M}_2(R)}{\log U(1/(1-R))}.$$

Since (3.8) holds for any $\eta>0$, we deduce as above

$$\overline{\lim_{R\to1-0}}\frac{\overset{+}{\log}\overset{+}{\log}M_2(R)}{\log U(1/(1-R))}=\left(\frac{\pi}{2\varepsilon}\right)^{\rho_1}.$$

By (3.11), when $|w|\leqslant S$, the corresponding Z satisfies $|Z|<(S+1)/2$. By (3.10), when Z satisfies $|Z|=3S-2$, $|\arg Z|<\delta$ and $C<3S-2<1$, the corresponding w satisfies $|w|<S$. Consequently we have

$$M_2(3S-2)\leqslant M_3(S)\leqslant\overline{M}_2((S+1)/2)$$

and

$$\left(\frac{\pi}{6\varepsilon}\right)^{\rho_1}=\overline{\lim_{S\to1-0}}\frac{\overset{+}{\log}\overset{+}{\log}M_2(3S-2)}{\log U(3/[1-(3S-2)])}\leqslant\overline{\lim_{S\to1-0}}\frac{\overset{+}{\log}\overset{+}{\log}M_3(S)}{\log U(1/(1-S))}$$

$$\leqslant\overline{\lim_{S\to1-0}}\frac{\overset{+}{\log}\overset{+}{\log}\overline{M}_2((S+1)/2)}{\log U(1/2[1-(S+1)/2])}=\left(\frac{\pi}{\varepsilon}\right)^{\rho_1}.$$

Hence by (3.7)

$$\overline{\lim_{S\to1-0}}\frac{\overset{+}{\log}\overset{+}{\log}M_3(S)}{\log U(k_1/(1-S))}=1, \tag{3.12}$$

where $k_1=k\pi/2\varepsilon$ and $1/3\leqslant k\leqslant 2$.

Let $T_3(S)$ be the Nevanlinna characteristic function of $f_3(w)$ ($|w|=S$). By (3.12) we have

$$\overline{\lim_{S\to1-0}}\frac{\overset{+}{\log}\overset{+}{\log}T_3(S)}{\log U(k_1/(1-S))}=1. \tag{3.13}$$

In fact, we have, by an inequality of Nevanlinna [6], for any $a>1$

$$\log M_3\left(\frac{1+(a-1)S}{a}\right)\geqslant T_3\left(\frac{1+(a-1)S}{a}\right)\geqslant\frac{1-S}{1+(2a-1)S}\log M_3(S),$$

whence

$$1=\overline{\lim_{S\to1-0}}\frac{\overset{+}{\log}\overset{+}{\log}M_3\left(\frac{1+(a-1)S}{a}\right)}{\log U(ak_1/(a-1)(1-S))}\geqslant\overline{\lim_{S\to1-0}}\frac{\overset{+}{\log}T_3\left(\frac{1+(a-1)S}{a}\right)}{\log U(ak_1/(a-1)(1-S))}$$

$$\geqslant\overline{\lim_{S\to1-0}}\frac{\overset{+}{\log}\overset{+}{\log}M_3(S)}{\log U(ak_1/(a-1)(1-S))}\geqslant\left(\frac{a-1}{a}\right)^{\rho_1}.$$

Since the above inequalities hold for any $a>0$, we get (3.13).

Hence[4] $f_3(w)$ has a Borel point $w_1(|w_1|=1)$ of order (H)

$$\rho(k_1/1-|w|)\left[1+\frac{\log k_1}{\log(1/1-|w|)}\right].^{[4]}$$

By the applications mentioned above we can complete the proof of Theorem 3.3.

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只在右半平面收敛的指数级数的增长性及值的分布

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摘 要

本文研究只在右半平面内收敛的指数级数，引进了(R-H)级概念；推广了 Anderson, J. M. 及 Binmore K. G. 的一个不等式，并且利用所得结果研究了上述级数在某些水平半带形上的增长性以及在虚轴上的 Picard 点及 Borel 点。