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A NEW ATTEMPT ON GOLDBACH CONJECTURE

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Dedicated to Professor Su Bu-chinY on the Occasion of his 80th Birthday and his 50th Year of Educational Work

Let N be a large even integer and $D(N)$ denote the number of the ways of representing N as a sum of two primes, that is

$$D(N)=\sum_{N=p_1+p_2}1. \tag{1}$$

By cycle method, we can derive that

$$D(N)=\mathfrak{S}(N)\frac{N}{\log^2 N}+R, \tag{2}$$

where

$$\mathfrak{S}(N)=2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|N,p>2}\left(1+\frac{1}{p-2}\right)$$

$$R=\left(\sum_{q>Q}\frac{\mu^2(q)}{\phi^2(q)}C_q(-N)\right)\frac{N}{\log^2 N}+\int_E S^2(\alpha,\ N)e^{-2\pi i\alpha N}d\alpha \tag{3}$$

$$S(\alpha,\ N)=\sum_{p\leqslant N}e^{2\pi i\alpha p},\quad C_q(-N)=\sum_{h=1}^{q}{}' e^{\frac{-2\pi i N h}{q}},$$

$Q=\log^{16}N$ and the E denotes the supplement interval as usual. This suggests us to conjecture that the main term of $D(N)$ is $\mathfrak{S}(N)\frac{N}{\log^2 N}$, that is

$$D(N)\sim\mathfrak{S}(N)\frac{N}{\log^2 N}. \tag{4}$$

It is well known that the difficulty in proving this conjecture is to deal with the integral in the remainder term R. So far as we know, up to now, the cycle method might be the unique approach* which suggests us to conjecture (4) is true. In this paper, we shall give another method which also suggests us to conjecture (4) is true . It seems to be more direct and elementary than the cycle method.

For convenience, we consider

$$\hat{D}(N)=\sum_{N=d+d'}\Lambda(d)\Lambda(d')=\sum_{d<N}\Lambda(d)\Lambda(N-d)$$

in place of $D(N)$. It is easy to see that

$$D(N)=\frac{\hat{D}(N)}{\log^2 N}\left[1+O\left(\frac{\log\log N}{\log N}\right)\right]+O\left(\frac{N}{\log^3 N}\right).$$

Now We shall prove the following theorems:

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* Recently Prof. Hua, L. K. has proposed a different new method on this line, however not yet published.

Theorem I. *Let N be a large even integer. Then for*

$$Q=\sqrt{N}\log^{-20}N,$$

we have

$$\hat{D}(N)=\mathfrak{S}(N)N+\hat{R}, \tag{5}$$

where $\mathfrak{S}(N)$ is defined by (3)

$$\hat{R}=R_1+R_2+R_3+O(N\log^{-1}N), \tag{6}$$

and

$$R_1=\sum_{n<N}\Big(\sum_{\substack{d_1\mid n\\ d_1\le Q}}a(d_1)\Big)\Big(\sum_{\substack{d_2\mid N-n\\ (d_2,N)=1\\ d_2>Q}}a(d_2)\Big),$$

$$R_2=\sum_{n<N}\Big(\sum_{\substack{d_1\mid n\\ d_1>Q}}a(d_1)\Big)\Big(\sum_{\substack{d_2\mid N-n\\ (d_2,N)=1\\ d_2\le Q}}a(d_2)\Big)$$

$$R_3=\sum_{n<N}\Big(\sum_{\substack{d_1\mid n\\ d_1>Q}}a(d_1)\Big)\Big(\sum_{\substack{d_2\mid N-n\\ (d_2,N)=1\\ d_2>Q}}a(d_2)\Big),$$

$$a(m)=\mu(m)\log m.$$

Theorem II. *By means of Bombieri Theorem, we have*

$$R_1=R_2=O(N\log^{-1}N). \tag{7}$$

First of all, we prove some lemmas as follows:

Lemma 1. *Let m be a positive integer, and $m\leqslant N^{c_1}$. Then for*

$$\sigma\geqslant 1-\frac{c_2}{\sqrt{\log N}}\geqslant 1/2,$$

we have

$$\prod_{p\mid m}\Big(1-\frac{1}{p^s}\Big)^{-1}\ll\log^{c_3}N. \tag{8}$$

Proof Put $T=e^{\sqrt{\log N}}$, we have

$$\Big|\prod_{p\mid m}\Big(1-\frac{1}{p^s}\Big)^{-1}\Big|\leqslant\prod_{p\mid m}\Big(1-\frac{1}{p^\sigma}\Big)^{-1}=\prod_{p\mid m}\Big(1+\frac{1}{p^\sigma-1}\Big)$$

and

$$\log\prod_{p\mid m}\Big(1+\frac{1}{p^\sigma-1}\Big)\leqslant\sum_{p\mid m}\frac{1}{p^\sigma-1}\ll\sum_{p\mid m}\frac{1}{p^\sigma}$$

$$=\sum_{\substack{p\mid m\\ p\le T}}\frac{1}{p^\sigma}+\sum_{\substack{p\mid m\\ p>T}}\frac{1}{p^\sigma}=\Sigma_1+\Sigma_2.$$

Furthermore, we have

$$\Sigma_1\ll\log\log N$$

and

$$\Sigma_2\ll T^{-1/2}\log N\ll 1.$$

Since $\sigma\geqslant 1/2$, summing the above up, the Lemma is proved.

Lemma 2. *Let m be a positive integer $m\leqslant N^{c_1}$. Then we have*

$$\sum_{\substack{d\le N\\ (d,m)=1}}\frac{\mu(d)}{d}\ll e^{-c_3\sqrt{\log N}} \tag{9}$$

and

$$\sum_{\substack{d\le N\\ (d,m)=1}}\frac{\mu(d)}{d}\log d=-\frac{m}{\phi(m)}+O(e^{-c_4\sqrt{\log N}}). \tag{10}$$

Proof Put $X=N+1/2$ and

$$F(s)=\prod_{p|m}\left(1-\frac{1}{p^s}\right)\zeta(s).$$

Then
$$\sum_{\substack{d\le N\\(d,m)=1}}\frac{\mu(d)}{d}=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\frac{1}{F(1+w)}\frac{X^w}{w}\,dw+O\left(\frac{\log N}{T}\right),$$
where
$$b=\frac{1}{\log X},\quad T=e^{\sqrt{\log X}}.$$

Moving the line of the integral to $[c-iT,\ c+iT]$, $c=-\frac{c_5}{\sqrt{\log X}}$, and using Lemma 1, we have

$$\sum_{\substack{d\le N\\(d,m)=1}}\frac{\mu(d)}{d}\ll e^{-c_6\sqrt{\log N}}.$$

It is easy to prove by the method of Abel Summation that

$$\sum_{\substack{d=1\\(d,m)=1}}^{\infty}\frac{\mu(d)}{d}\log d=\sum_{\substack{d\le X\\(d,m)=1}}\frac{\mu(d)}{d}\log d+O(e^{-c_6\sqrt{\log X}}) \tag{11}$$

and

$$\sum_{\substack{d=1\\(d,m)=1}}^{\infty}\frac{\mu(d)}{d}\log d=\lim_{\sigma\to1+}\sum_{\substack{d=1\\(d,m)=1}}^{\infty}\frac{\mu(d)}{d^\sigma}\log d=-\left(\frac{1}{F(s)}\right)'_{s=1}. \tag{12}$$

From (11), (12) and

$$\left(\frac{1}{F(s)}\right)'_{s=1}=\prod_{p|m}\left(1-\frac{1}{p}\right)^{-1}=\frac{m}{\phi(m)}, \tag{13}$$

(10) is derived at once.

Lemma 3. *We have*

$$\sum_{\substack{n\le N\\(n,m)=1}}\frac{\mu(n)\log n}{\phi(n)}=-\mathfrak{S}(m)+O(e^{-c_6\sqrt{\log N}}).$$

Proof We have

$$\sum_{\substack{d\le N\\(d,m)=1}}\frac{\mu(d)\log d}{\phi(d)}=\sum_{\substack{d\le N\\(d,m)=1}}\frac{\mu(d)\log d}{d}\sum_{t|d}\frac{\mu^2(t)}{\phi(t)}=\sum_{\substack{t\le N\\(t,m)=1}}\frac{\mu^2(t)}{\phi(t)}\sum_{\substack{d\le N\\(d,m)=1\\t|d}}\frac{\mu(d)\log d}{d}$$
$$=\sum_{\substack{t\le N\\(t,m)=1}}\frac{\mu^2(t)}{\phi(t)}\sum_{\substack{v\le N/t\\(v,m)=1}}\frac{\mu(vt)\log vt}{vt}$$
$$=\sum_{\substack{t\le N\\(t,m)=1}}\frac{\mu^2(t)}{\phi(t)}\frac{\mu(t)}{t}\sum_{\substack{v\le N/t\\(v,mt)=1}}\frac{\mu(v)}{v}(\log v+\log t)$$
$$=\sum_{\substack{t\le N\\(t,m)=1}}\frac{\mu(t)}{t\phi(t)}\sum_{\substack{v\le N/t\\(v,mt)=1}}\frac{\mu(v)}{v}\log v+\sum_{\substack{t\le N\\(t,m)=1}}\frac{\mu(t)\log t}{t\phi(t)}\sum_{\substack{v\le N/t\\(v,mt)=1}}\frac{\mu(v)}{v}$$
$$=\Sigma_1+\Sigma_2.$$

By (10)

$$\Sigma_1=\sum_{\substack{t\le\sqrt{N}\\(t,m)=1}}\frac{\mu(t)}{t\phi(t)}\sum_{\substack{v\le N/t\\(v,mt)=1}}\frac{\mu(v)}{v}\log v+\sum_{\sqrt{N}<t\le N}=-\sum_{\substack{t\le\sqrt{N}\\(t,m)=1}}\frac{\mu(t)}{t\phi(t)}\frac{rt}{\phi(rt)}+O(e^{-c_7\sqrt{\log N}})$$
$$=-\frac{m}{\phi(m)}\sum_{\substack{t=1\\(t,m)=1}}^{\infty}\frac{\mu(t)}{\phi^2(t)}+O(e^{-c_8\sqrt{\log N}})=-\mathfrak{S}(m)+O(e^{-c_8\sqrt{\log N}}).$$

Similarly, by (9)

$$\Sigma_2=\sum_{\substack{t\le\sqrt{N}\\(t,m)=1}}\frac{\mu(t)\log t}{t\phi(t)}\sum_{\substack{v\le N/t\\(v,mt)=1}}\frac{\mu(v)}{v}+\sum_{\substack{\sqrt{N}<t\le N\\(t,m)=1}}\sum_{\substack{v\le N/t\\(v,mt)=1}}\frac{\mu(v)}{v}=O(e^{-c_5\sqrt{\log N}}).$$

Summing the above up, the Lemma is proved.

The proof of Theorem I. we have

$$\hat{D}(N)=-\sum_{n\le N}\Lambda(n)\sum_{d\mid N-n}a(d)$$

$$=-\sum_{n\le N}\Lambda(n)\sum_{\substack{d\mid N-n\\(d,N)=1}}a(d)-\sum_{n\le N}\Lambda(n)\sum_{\substack{d\mid N-n\\(d,N)>1}}a(d)=I_1+I_2. \tag{14}$$

It is evident that

$$I_2=O(N^{\frac{2}{3}}) \tag{15}$$

and

$$I_1=\sum_{n\le N}\sum_{d_1\mid n}a(d_1)\sum_{\substack{d_2\mid N-n\\(d_2,N)=1}}a(d_2)$$

$$=\sum_{n\le N}\Bigg(\sum_{\substack{d_1\mid n\\d_1\le Q}}a(d_1)+\sum_{\substack{d_1\mid n\\d_1>Q}}a(d_1)\Bigg)\Bigg(\sum_{\substack{d_2\mid N-n\\(d_2,N)=1\\d_2\le Q}}a(d_2)+\sum_{\substack{d_2\mid N-n\\(d_2,N)=1\\d_2>Q}}a(d_2)\Bigg) \tag{16}$$

$$=\Sigma_1+R_1+R_2+R_3,$$

where

$$\Sigma_1=\sum_{n\le N}\sum_{\substack{d_1\mid n\\d_1\le Q}}a(d_1)\sum_{\substack{d_2\mid N-n\\d_2\le Q\\(d_2,N)=1}}a(d_2).$$

It is easily seen that

$$\Sigma_1=\sum_{n\le N}\sum_{\substack{d_1\mid n\\d_1\le Q}}a(d_1)\sum_{\substack{d_2\mid N-n\\d_2\le Q\\(d_2,N)=1}}a(d_2)=\sum_{\substack{d_2\le Q\\(d_2,N)=1}}a(d_2)\sum_{\substack{d_1\le Q\\(d_1,d_2)=1}}a(d_1)\sum_{\substack{d_1n\le N\\d_1n\equiv N(d_2)}}1$$

$$=N\sum_{\substack{d_2\le Q\\(d_2,N)=1}}\frac{a(d_2)}{d_2}\sum_{\substack{d_1\le Q\\(d_1,d_2)=1}}\frac{a(d_1)}{d_1}+O(Q^2\log^2 N). \tag{17}$$

From (17) and Lemma 2 and Lemma 3, we have

$$\Sigma_1=N\sum_{\substack{d_2\le Q\\(d_2,N)=1}}\frac{\mu(d_2)\log d_2}{d_2}\sum_{\substack{d_1\le Q\\(d_1,d_2)=1}}\frac{\mu(d_1)\log d_1}{d_1}+O(N\log^{-1}N)$$

$$=-N\sum_{\substack{d_2\le Q\\(d_2,N)=1}}\frac{\mu(d_2)\log d_2}{\phi(d_2)}+O(N\log^{-1}N)$$

$$=\mathfrak{S}(N)N+O(N\log^{-1}N).$$

The Theorem is completed.

Now we are going to prove Theorem II.

The proof of Theorem II.

$$R_1=\sum_{n\le N}\sum_{\substack{d_1\mid n\\d_1\le Q}}\mu(d_1)\log d_1\sum_{\substack{d_2\mid N-n\\(d_2,N)=1\\d_2>Q}}\mu(d_2)\log d_2$$

$$=\sum_{d_1\le Q}\mu(d_1)\log d_1\Bigg(\sum_{\substack{n\le N\\n\equiv o(d_1)}}\sum_{\substack{d_2\mid N-n\\(d_2,N)=1\\d_2>Q}}\mu(d_2)\log d_2\Bigg)$$

$$=\sum_{d_1\le Q}\mu(d_1)\log d_1\Bigg(\sum_{\substack{n\le N\\n_1\equiv o(d)}}\Lambda(N-n)-\sum_{\substack{n\le N\\n\equiv o(d_1)}}\sum_{\substack{d_2\mid N-n\\(d_2,N)=1\\d_2\le Q}}\mu(d_2)\log d_2\Bigg)+O(N\log^{-1}N)$$

$$=\sum_{d_1<Q}\mu(d_1)\log d_1\left(\sum_{\substack{n\le N\\ n\equiv N(d_1)}}\Lambda(n)-\frac{N}{d_1}\sum_{\substack{d_2<Q\\ (d_2,d_1)=1}}\frac{\mu(d_2)\log d_2}{d_2}\right)+O(N\log^{-1}N)$$

$$=\sum_{d_1<Q}\mu(d_1)\log d_1\left(\sum_{\substack{u\le N\\ n\equiv N(d_1)}}\Lambda(n)-\frac{N}{\phi(d_1)}\right)+O(N\log^{-1}N)$$

$$=\sum_{\substack{d_1<Q\\ (d_1,N)=1}}\mu(d_1)\log d_1\left(\sum_{\substack{n\le N\\ n\equiv N(d_1)}}\Lambda(n)-\frac{N}{\phi(d_1)}\right)+O(N\log^{-1}N)$$

$$=O(N\log^{-1}N).$$

Similarly, We have

$$R_2=O(N\log^{-1}N).$$

The Theorem is completed.

Goldbach 猜想的一种新尝试

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摘 要

设 N 为大偶数，以 $D(N)$ 表示将 N 表成两个素数之和的表法个数，即

$$D(N)=\sum_{N=p_1+p_2}1.$$

Hardy 和 Littlewood 利用"圆法"证明了下面的结果

$$D(N)=\mathfrak{S}(N)\frac{N}{\log^2 N}+R, \tag{1}$$

这里

$$\mathfrak{S}(N)=2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p\mid N\\ p>2}}\left(1+\frac{1}{p-2}\right), \tag{2}$$

$$R=\left(\sum_{q>Q}\frac{\mu^2(q)}{\phi^2(q)}C_q(-N)\right)\frac{N}{\log^2 N}+\int_E S^2(\alpha,N)e^{-2\pi i\alpha N}d\alpha \tag{3}$$

$$S(\alpha,N)=\sum_{p\le N}e^{2\pi i\alpha p},\ C_q(-N)=\sum_{h=1}^{q}e^{-2\pi i\frac{Nh}{q}},$$

$Q=\log^{16}N$, E 表示在通常意义下的余区间，这就提出了下面的猜想

$$D(N)\sim\mathfrak{S}(N)\frac{N}{\log^2 N}. \tag{4}$$

熟知 Goldbach 猜想的困难在于误差项 R 的处理，至今"圆法"是提出猜想(4)的唯一的方法，本文提出了另一种途径来研究猜想(4). 而且方法是初等的，看起来是更为直接的方法。令

$$\hat{D}(N)=\sum_{d<N}\Lambda(d)\Lambda(N-d).$$

显然 $$D(N)=\frac{\hat{D}(N)}{\log^2 N}\left[1+O\left(\frac{\log\log N}{\log N}\right)\right]+O\left(\frac{N}{\log^3 N}\right).$$

本文证明了下面两个定理:

定理 1 设 N 为大偶数, $Q=\sqrt{N}\log^{-20}N$, 则

$$\hat{D}(N)=\mathfrak{S}(N)N+\hat{R}, \tag{5}$$

这里

$$\hat{R}=R_1+R_2+R_3+O(N\log^{-1}N), \tag{6}$$

$$R_1=\sum_{n<N}\Bigg(\sum_{\substack{d_1|n\\ d_1<Q}}a(d_1)\Bigg)\Bigg(\sum_{\substack{d_2|N-n\\ (d_2,N)=1\\ d_2<Q}}a(d_2)\Bigg),$$

$$R_2=\sum_{n<N}\Bigg(\sum_{\substack{d_1|n\\ d_1<Q}}a(d_1)\Bigg)\Bigg(\sum_{\substack{d_2|N-n\\ (d_2,N)=1\\ d_2<Q}}a(d_2)\Bigg),$$

$$R_3=\sum_{n<N}\Bigg(\sum_{\substack{d_1|n\\ d_1>Q}}a(d_1)\Bigg)\Bigg(\sum_{\substack{d_2|N-n\\ (d_2,N)=1\\ d_2>Q}}a(d_2)\Bigg),$$

$$a(m)=\mu(m)\log m.$$

证明定理 1 的方法是初等的,这就建议我们提出猜想(4).

定理 2 用 Bombieri 定理可以证明

$$R_1=R_2=O(N\log^{-1}N).$$

从上面两个定理看出,研究 Goldbach 猜想的困难,在于处理余项 R_3.