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SOME CHARACTERIZATIONS OF A FINITE SUPERSOLVABLE GROUP

CHEN ZHONGMU*
(*Xinan Normal College*)

Much of the characterizations of a finite supersolvable group has been derived by a number of former authors. In 1941, Iwasawa proved that a finite group is supersolvable if and only if every maximal chain of subgroups has the same length[1]. In 1954, Huppert proved that a finite group is supersolvable if and only if every maximal subgroup has a prime index ([2], or [5] Th. 10.5.8). In 1957, Mclain obtained that a finite group G is supersolvable if and only if there exists a subgroup of order d for every factor d of order h of every subgroup H of G[3]. The present author proved that a finite group is supersolvable if and only if the indices of every maximal chain of subgroups are all square free.

However, the proof of Huppert's theorem is complex and has used the representation theory (as [5]), or Gaschütz's theorem (as [6] Th. 9.3.8). The first part of this note gives a proof of Huppert's theorem without using knowledge of that kind and extends a little the sufficient condition of this theorem. The second part is an extension of Maclain's result.

§1. Another Proof of Huppert's Theorem and Extensions.

Lemma. *If N is a minimal normal subgroup of order p^α of a finite group G and G/N is supersolvable, then either* 1) *there is a maximal subgroup M of G such that $G=MN$, $M\cap N=E$, or* 2) *G has a normal subgroup of a prime order.*

Proof N is an elementary abelian group of order p^α. Since G/N is supersolvable, G has a normal series

$$G=G_0>G_1>\cdots>G_{k-1}>G_k>N=G_{k+1}, \tag{1}$$

whose indices $[G_i:G_{i+1}]=p_i$, p_i being a prime and $p_i\leqslant p_{i+1}$, $i=0, 1, \cdots, k$.

a) $[G_k:N]\neq p$, then the order of G_k is p_kp^α. Let P_k be a Sylow p_k-subgroup of G_k and the normalizer of P_k in G be $N(P_k)$. Since $G_k\trianglelefteq G$, by Frattini argument

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$G=N(P_k)G_k$ ([8], p. 129, IV. 2. f). Furthermore, $G_k=P_kN$ will imply $G=N(P_k)N$. If $N(P_k)=G$, then P_k is a normal subgroup of order P_k and 2) holds true. If $N(P_k)<G$, then there exists a maximal subgroup M of G, such that $N(P_k)\leqslant M<G$. Evidently $G=MN$. Now set $M\cap N=D$. If $D\neq E$, then $D<N$, since $M\not\geqslant N$. D is normal both in M and N, therefore $D\trianglelefteq\langle M, N\rangle=G$, contrary to the minimality of N.

b) $[G_k:N]=p$, then G_k is a p-group, p is the largest prime factor of $|G|$ and the Sylow p-subgroup $P\trianglelefteq G$, $P\geqslant G_k$. Let Z be the center of P. $N\cap Z\neq E$, since $N\trianglelefteq P$. $Z\geqslant N$ by the minimality of N. Since $G_k=\langle b, N\rangle$, G_k is abelian. If G_k is not elementary, then the order of b is p^2. From this, the characteristic subgroup $\mho_1(G_k)$ (subgroup consists of the p-th power of elements of G_k) is a normal subgroup of order p generated by b^p. Now suppose G_k is an elementary abelian group. Then every subgroup of G_k is normal in G_k. If these subgroups are also normal in G, then G has a normal subgroup of order p. If not, there exists G_i in normal series **(1)**, **such** that every subgroup of G_k is normal in G_{i+1} but not so in G_i.

i) $[G_i:G_{i+1}]=p$, then G_i is a p-group. Since every subgroup of G_k is normal in G_{i+1}, G_k is contained in the center of G_{i+1}. Let $G_i=\langle a, G_{i+1}\rangle$. $[G_i, G_k]$ is generated by the elements $[ga^r, nb^s]$, $g\in G_{i+1}$, $n\in N$. Since N is contained in the center Z of P and $G_k\leqslant Z(G_{i+1})$, we obtain

$[ga^r, nb^s]=[ga^r, b^s]=a^{-r}g^{-1}b^{-s}ga^rb^s=a^{-r}b^{-s}a^rb^s=[a^r, b^s]$. Evidently G_k/N is a normal subgroup of order p of G_i/N, hence G_k/N is contained in the center of G_i/N. Therefore $[G_k, G_i]\leqslant N\leqslant Z$ and $[a, b]\in N$. Since $[a, b]\neq 1$, the order of $[a, b]$ is p. $[a^r, b^s]=[a, b]^{rs}$, since $[a, b]\in Z$. Hence $[G_i, G_k]=\langle[a, b]\rangle$ is a normal subgroup of order p of G.

ii) $[G_i:G_{i+1}]=p_i\neq p$. Since $G_i=\langle a, G_{i+1}\rangle$, we can choose an element a whose order is a power of p_i. Transform G_k by a, the subgroups of order p of N transform to subgroups of N also, and the subgroups of G_k outside N is also a subgroup of G_k outside N. The number of subgroups of order p of G_k outside N is

$$\frac{p^{\alpha+1}-1}{p-1}-\frac{p^\alpha-1}{p-1}=p^\alpha.$$

The numbers of conjugates of these p^α subgroups of order p under transformations by a are powers of p_i. Since $p_i\neq p$, there exists a class which contains only one subgroup of order p. That is the existence of a normal subgroup of order p of G_i outside N. Let this subgroup to be $\langle b\rangle$. Now we shall prove that $\langle b\rangle$ is normal in G. If $\langle b\rangle$ does not, then the number of conjugates of $\langle b\rangle$ in G is greater than 1 and all of which are normal in G_k. They generate a normal subgroup B of G. Since $B\leqslant G_k$ and N is a minimal normal subgroup of G, $B=G_k$. Let Q_1 be a conjugate of $\langle b\rangle$. $\langle b\rangle Q_1$ is normal in G_i. Since Q_1 and $\langle b\rangle$ are outside N and $[G_k:N]=p$, $\langle b\rangle Q_1\cap N=Q_2$

is a normal subgroup of order p of G_i. Take c to be a generator of Q_1, such that bc is a generator of Q_2. Then $a^{-1}ba=b^r$, $a^{-1}ca=c^s$, $a^{-1}(bc)a=(bc)^t$. Comparing above three expansions, we have $r=s=t$. Thus, we have proved that all conjugates of b transformed by a are their powers and the exponents are the same. Since all the conjugates of b generate $B=G_k$, it is possible to choose a basis for G_k in the conjugates of b. Therefore every element of G_k transformed by a is equal to its power and the exponents are the same. Hence every subgroup of G_k is normal in G_i, contrary to the assumption, so that $\langle b\rangle$ is normal in G. And G has a normal subgroup of order p.

Proof of Huppert's Theorem We proceed by induction on the order of G. It is known that G is solvable. Let N be a minimal normal subgroup of G, $|N|=p^\alpha$. G/N is supersolvable by induction. If case 1) arises in the above lemma, we have $[G:M]=p^\alpha$, so $\alpha=1$ by the assumption. N is a normal subgroup of order p. Hence G. has normal subgroup P of order p in any case. By induction on G/P, G is supersolvable.

The following theorem is an extension of Huppert's theorem in the solvable case.

Theorem 1.1. *A finite solvable group G is supersolvable if the index of every maximal subgroup in G is square free.*

Proof The proof may be obtained as the previous one by the lemma. Now we prove this theorem by Huppert's theorem as follows. Let N be a minimal normal subgroup of G. Then N is an elementary abelian group of order p^α. If G has a maximal subgroup $M\not\geqslant N$, then $M\cap N=D<N$ and D is normal in M and N. Hence $D\trianglelefteq\langle M, N\rangle=G$. By the minimality of N, $D=E$ and so $[G:M]=p^\alpha$. Since the index is square free, $\alpha=1$ and G has a normal subgroup of order p. By induction on G/N, G is supersolvable.

Suppoes that every maximal subgroup M of G contains N. G/N is supersolvable by induction. Since M/N is a maximal subgroup of G/N, the index $[G:M]=[G/N: M/N]$ is a prime. Hence G is supersolvable by Huppert's theorem.

A little extension of [2] Th. 10 may be given here: If a finite group G has a normal subgroup $N\leqslant\Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G, then G is supersolvable if and only if G/N is supersolvable.

The assumption "G is solvable" is necessary for this theorem. For example, the simple group A_5 of order 60 has no subgroup of order 15. If not so, A_5 would have a permutation representation of degree 4. The representation is faithful since A_5 is simple. A_5 would be isomorphic to a subgroup of S_4. This is impossible. The Sylow 3-subgroups and 5-subgroups are not maximal subgroup of A_5 otherwise their normalizer should be themselves. Therefore $1+3k=20$ or $1+5k=12$ by Sylow theorem which is impossible too. Hence the index of every maximal subgroup is square free in A_5.

§ 2. Extensions of Mclain's Result

Theorem 2.1. *Let $h=|H|$, (where H is a subgroup of a finite group G,) P_h be the smallest prime factor of h and q_h be the largest If there exist subgroups of indices p_h and q_h in H for every subgroup H of a finite group G, then is supersolvable.*

Proof We use induction on the order of G. By induction every proper subgroup of G is supersolvable. If G is not supersolvable, then G is an "inner supersolvable group". G is Ω or Ω' ordered solvable group, where Ω is the set of all primes ordered by their natural order and the order of Ω' is in an opposite manner ([4] Th. 2.2.). Suppose q is the last prime factor of $|G|$. Then the index $[G: M]$ of every maximal subgroup M which contains the q-complement of G is divisible by q^2, contrary to the assumption. Hence G is supersolvable.

We shall prove a further theorem:

Theorem 2.2. *A finite group G is supersolvable if and only if there exist two chains of subgroups*

$$G=G_0>G_1>G_2>\cdots>G_s>E, \tag{2}$$

$$G=H_0>H_1>H_2>\cdots>H_s>E, \tag{3}$$

such that the indices $[G_0: G_1]$, $[G_1: G_2]$, $\cdots$, $[G_s: E]$ are primes from small ones to large ones and on the contrary, $[H_0: H_1]$, $[H_1: H_2]$, $\cdots$, $[H_s: E]$ are primes from large to small ones.

Proof The necessity is derived by refinememt theorem of supersolvable groups, ([5] Th. 10.5.5). We proceed by induction on the order g of G to prove the sufficiency. Since $[G_i: G_{i+1}]$ is the smallest prime factor of $|G_i|$, $G_{i+1}\lhd G_i$ ([7] p. 77, Ex. 5), $i=0, 1, \cdots, s$. Therefore G is solvable and (2) is a composition series of G. Hence G has a normal Sylow q-subgroup Q, where q is the greatest prime factor of g. G has a minimal normal subgroup N contained in the center of Q. N is an elementary abelian q-group. Now we shall prove that the order of N is q.

Consider the subgroup H_1 of G, $[G: H_1]=q$. H_1 had the series of subgroups $H_1>H_2>\cdots>H_s>E$, their indices are primes from large ones to small ones. Again, consider the series of subgroups of H_1

$$H_1=H_1\cap G_0\geqslant H_1\cap G_1\geqslant H_1\cap G_2\geqslant\cdots\geqslant H_1\cap G_s\geqslant E. \tag{4}$$

Since $G_i\lhd G_{i-1}$, $H_1\cap G_{i-1}/H_1\cap G_i\simeq(H_1\cap G_{i-1})\cup G_i/G_i$. Because $[G_{i-1}: G_i]$ is a prime, $[H_1\cap G_{i-1}: H_1\cap G_i]=[G_{i-1}: G_i]$ or 1. We may derive a series of subgroups of H_1 by deleting the multiple groups in (4), such that the indices are all primes from small to large ones. Hence H_1 is supersolvable by induction.

If $H_1 \cap N = D \neq E$, then D is normal in H_1. Since H_1 is supersolvable, H_1 has a minimal normal subgroup M of order q which is contained in D by the refinement theorem for chief series. Since N is contained in the center of Q, then so also do D and M. Hence $M \triangleleft Q$ and so $M \triangleleft \langle H_1, Q \rangle$. Since $[G: H] = q$ and Q is a Sylow q-subgroup, $G = \langle H_1, Q \rangle$. Therefore $N = M$ is a normal subgroup of order q of G by the minimality of N.

If $H_1 \cap N = E$, then $G = H_1 N$, $[G: H] = |N| = q$. Consider the factor group G/N. It is easy to show that

$$[HN/N: KN/N] = [H: K(H \cap N)]$$

where $H \geqslant K$ are any two subgroups of G. If $[H: K]$ is a prime, $[H: K(H \cap N)] = [H: K]$ or 1. Deleting the multiple groups in follow series of subgroups of G/N

$$G/N \geqslant G_1N/N \geqslant G_2N/N \geqslant \cdots \geqslant G_sN/N \geqslant E,$$

$$G/N \geqslant H_1N/N \geqslant H_2N/N \geqslant \cdots \geqslant H_sN/N \geqslant E,$$

we get two series of subgroups of G/N. The indices of one are primes from large to small ones, while another from small to large ones. Hence G is supersolvable by induction.

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有限超可解群的几个特征性质

陈 重 穆
(西南师范学院)

摘 要

这篇短文的第一部分给出 Huppert 定理:[2]“每极大子群有质数指数的有限群为超可解”的一个不用表示论及 Gaschütz 定理的证明. 该证明得自

定理 1 若有限群 G 有 p^{α} 阶极小正规子群 N 使 G/N 为超可解, 则或者 1) G 有极大子群 M 使 $G=MN$, $M\cap N=E$, 或者 2) G 有质数阶正规子群.

在可解时 Huppert 定理推广为:

定理 2 设 G 为有限可解群. 于是 G 为超可解当且仅当每极大子群在 G 内的指数不含平方因子.

单群 A_5 说明本定理的假设“G 可解”是必要的.

本文第二部分是 Mclain 定理[3] 的推广:

定理 3 设 $h=|H|$ 的最小质因子为 p_h, 最大质因子为 q_h, 若有限群 G 的每子群 H 对其阶 h 恒存在指数为 p_h 及 q_h 的子群, 则 G 为超可解.

更广泛的结论为:

定理 4 有限群 G 为超可解当且仅当存在 G 的两个子群链

$$G=G_0>G_1>G_2>\cdots>G_s>E,$$
$$G=H_0>H_1>H_2>\cdots>H_s>E,$$

使指数列 $[G_0:G_1]$, $[G_1:G_2]$, $\cdots$, $[G_s:E]$ 为从小到大的质数, 而 $[H_0:H_1]$, $[H_1:H_2]$, $\cdots$, $[H_s:E]$ 为从大到小的质数.