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ON AN EXTENTION OF HADAMARD INEQUALITIES FOR CONVEX FUNCTIONS

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In 1893, Hadamard[1] pointed out that for any continuous convex function $f(x)$ on a closed interval $[a, b]$ $(a\neq b)$ the following inequalities

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\,dx\leqslant\frac{f(a)+f(b)}{2} \tag{1}$$

hold. Recently, Vasić and Lacković, and Lupas generalized (1) in [4, 5] and [3] respectively. In fact, they used the mean value of an integral of $f(x)$ over an interval, whose centre is $\xi=(pa+qb)/(p+q)$, to separate $f(\xi)$ and $(pf(a)+qf(b))/(p+q)$, $p, q>0$. However, Vasić and Lackovic[4] not only required, the convex function $f(x)$ to be twice differentiable, but also erroneously treated the integral remainder of the linear interpolation in their proof.

We shall prove a more general result which indicates the general Jenson inequality[2] can be interpolated by the mean value of a multiple integral as follows.

Theorem. *Let $f(x)$ be a continuous convex function on a closed interval $[a, b]$. Then for any $x_j\in[a, b]$, $p_j>0$, $Q_j=\sum_{\nu=j}^{n}p_\nu$ $(j=0, 1, \cdots, n)$ and α_i, β_i satisfying*

$$0\leqslant\alpha_i<\beta_i\leqslant 1,\quad (\alpha_i+\beta_i)/2=Q_i/Q_{i-1}\ (i=1, 2, \cdots, n), \tag{2}$$

the following inequalities

$$f\left(\sum_{j=0}^{n} P_j x_j/Q_0\right)\leqslant\frac{1}{|\Omega|}\int_\Omega f(x(t))\,dt\leqslant\sum_{j=0}^{n}p_j f(x_j)/Q_0 \tag{3}$$

hold, where

$$\Omega=[\alpha_1, \beta_1]\times[\alpha_2, \beta_2]\times\cdots\times[\alpha_n, \beta_n],$$
$$|\Omega|=(\beta_1-\alpha_1)(\beta_2-\alpha_2)\cdots(\beta_n-\alpha_n),$$
$$t=(t_1, t_2, \cdots, t_n),$$
$$x(t)=x_0(1-t_1)+\sum_{j=1}^{n-1}x_j(1-t_{j+1})\prod_{i=1}^{j}t_i+x_n\prod_{i=1}^{n}t_i. \tag{4}$$

Equality occurs in the left inequality of (3) if and only if the x_j are all equal or $f(x)$ is linear on the interval $[\min_{t\in\Omega}x(t), \max_{t\in\Omega}x(t)]$, while equality occurs in the right inequality of (3) if and only if the x_j are all equal or $f(x)$ is linear in an interval including all the x_j.

Proof Setting

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$$q_0=1-t_1,\ q_j=(1-t_{j+1})\prod_{i=1}^{j} t_i(j=1,\ \cdots,\ n-1),\ q_n=\prod_{i=1}^{n} t_i,$$

we have $\sum_{j=0}^{n} q_j=1$ and $q_j\geqslant 0(j=0,1,\cdots,n)$ for $t\in\Omega$. Thus applying Theorem 86 of [2]

$$f\Big(\sum_{j=0}^{n} q_jx_j\Big)\leqslant\sum_{j=0}^{n} q_j f(x_j)$$

to (4), we obtain

$$f(x(t))\leqslant\sum_{j=0}^{n-1}(1-t_{j+1})\prod_{i=1}^{j} t_if(x_j)+\prod_{i=1}^{n} t_if(x_n)\quad(t\in\Omega). \tag{5}$$

Integrating (5) with respect to t over Ω, a straightforward simplification with (2) yields

$$\begin{aligned}\int_\Omega f(x(t))dt&\leqslant\sum_{j=0}^{n-1}\prod_{i=1}^{j}\frac{\beta_i^2-\alpha_i^2}{2}\Big(\beta_{j+1}-\alpha_{j+1}-\frac{\beta_{j+1}^2-\alpha_{j+1}^2}{2}\Big)\cdot\sum_{i=j+2}^{n}(\beta_i-\alpha_i)f(x_j)\\&+\prod_{i=1}^{n}\frac{\beta_i^2-\alpha_i^2}{2}f(x_n)=\prod_{i=1}^{n}(\beta_i-\alpha_i)\Big[\sum_{j=0}^{n-1}\prod_{i=1}^{j}\frac{Q_i}{Q_{i-1}}\Big(1-\frac{Q_{j+1}}{Q_j}\Big)f(x_j)\\&+\prod_{i=1}^{n}\frac{Q_i}{Q_{i-1}}f(x_n)\Big]=|\Omega|\sum_{j=0}^{n}P_jf(x_j)/Q_0,\end{aligned} \tag{6}$$

which is equivalent to the right inequality of (3).

If all the x_j are not equal and $f(x)$ is not linear on the smallest closed interval including all the x_j, then by a use of Theorem 90 of [2], inequality (5) is always strict; and so is inequality (6).

Next we prove the left inequality of (3). According to Theorem 112 of [2], for

$$\xi=\sum_{j=0}^{n} p_jx_j/Q_0, \tag{7}$$

there exists a λ such that

$$f(x)\geqslant f(\xi)+\lambda(x-\xi)\quad(a\leqslant x\leqslant b). \tag{8}$$

Let $t\in\Omega$, (4) insures $x(t)\in[a,\ b]$. Now, setting $x=x(t)$ in (8) and integrating it, we obtain

$$\begin{aligned}\int_\Omega f(x(t))dt&\geqslant\int_\Omega[f(\xi)+\lambda(x(t)-\xi)]dt\\&=f(\xi)|\Omega|+\lambda\Big[\int_\Omega x(t)dt-\xi|\Omega|\Big].\end{aligned} \tag{9}$$

Set $f(x)=x$ in (6), a use of (7) yields

$$\int_\Omega x(t)dt=|\Omega|\sum_{j=0}^{n} p_jx_j/Q_0=\xi|\Omega|.$$

Combining the above equaliy with (9), we obtain

$$\frac{1}{|\Omega|}\int_\Omega f(x(t))dt\geqslant f(\xi), \tag{10}$$

which is the right inequality of (3).

If all the x_j are not equal and $f(x)$ is not linear on the smallest closed interval containing $\{x(t):t\in\Omega\}$, then there exists at least one endpoint in this interval at which inequality (8) is strict. Consequently, at a certain point in Ω

$$f(x(t)) > f(\xi) + \lambda(x(t) - \xi).$$

Hence, according to the continuity of $f(x(t))$ in $t \in \Omega$, inequality (10) is also strict in this case.

Set $n=1$, the above theorem implies

Corollary. *Let $f(x)$ be a continuous convex function on $[a, b]$. Then for any positive numbers p, q, u and v satisfying $(u+v)/2 = (pa+qb)/(p+q)$ and $a \leqslant u < v \leqslant b$. the following inequalities*

$$f\left(\frac{pa+qb}{p+q}\right) \leqslant \frac{1}{v-u}\int_u^v f(x)\,dx \leqslant \frac{pf(a)+qf(b)}{p+q}$$

hold. Moreover, the first and second equality signs hold only if $f(x)$ is linear on $[u, v]$ and $[a, b]$ respectively.

This corollary is the result of [3, 4, 5]. However, the equality case was not mentioned by the authors.

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关于凸函数的 Hadamard 不等式的拓广

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摘　　要

设 f 是区间 $[a, b]$ 上连续的凸函数. 我们证明了 Hadamard 的不等式

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\,dx\leqslant\frac{f(a)+f(b)}{2}$$

可以拓广成对 $[a, b]$ 中任意 $n+1$ 个点 $x_0, \cdots, x_n$ 和正数组 $p_0, \cdots, p_n$ 都成立的下列不等式

$$f\left(\frac{\sum\limits_{i=0}^{n} p_i x_i}{\sum\limits_{i=0}^{n} p_i}\right)\leqslant|\Omega|^{-1}\int_\Omega f(x(t))\,dt\leqslant\frac{\sum\limits_{i=0}^{n} p_i f(x_i)}{\sum\limits_{i=0}^{n} p_i},$$

式中 Ω 是一个包含于 n 维单位立方体的 n 维长方体,其重心的第 i 个坐标为 $\sum\limits_{j=i}^{n} p_j/\sum\limits_{j=i-1}^{n} p_i$, $|\Omega|$ 为 Ω 的体积,对 Ω 中的任意点 $t=(t_1, \cdots, t_n)$

$$\omega(t)=x_0(1-t_1)+\sum_{i=1}^{n-1} x_i(1-t_{i+1})\prod_{j=1}^{i} t_j+x_n\prod_{j=1}^{n} t_j.$$

不等式中两个等号分别成立的情形亦已被分离出来.

此不等式是著名的 Jensen 不等式的精密化.