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POTENTIALITY AND REVERSIBILITY FOR GENERAL SPEED FUNCTIONS (I)

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§ 1. Introduction

At first the Markov processes with infinite particle systems were proposed as models for the temporal evolution of such systems[8]. Then several concrete models were discussed[7,10,15]. In this paper a general model which includes most of the known models is proposed. It may be regarded simultaneously as an extension and generalization for the probability model of the multivariate linear Master equation for non-equilibrium systems[2].

In general, people agree with the reversibility of Markov processes which depicts the detailed balance (or Gibbs state) in statistical mechanics, where the reversible measure is just the defailed balance state. Therefore these are quite important problems: When does a reversible measure exist? When does only one exist? How does one get the construction of all reversible measures?

For discrete state spaces, there are a lot of investigations in [6]. Hou and Chen have established an abstract field, they discovered that potentiality describes the essential character of reversibility, and they solved potentiality and reversibility for Markov chains by using the field. There are also a lot of investigations for the reversibility of the Markov processes with infinite particle systems, but the known results are so limited even for the spin-fiip processes or the exclusion processes (See [10]). Recently, Ding and Chen[1] use the method of the field to investigate reversibility for the spin-flip processes with the nearest-neighbour speed functions, and give a necessary and sufficient condition for the existence and uniqueness of the reversible measures.

In § 2 we first extend the field to abstract state spaces, then we discuss a localization of a field with product state spaces. In § 3 we introduce both the concept of the speed functions with finite range and the cencept of quasi-reversibility. We

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show that quasi-reversibility is an extension of reversibility. In § 4 we investigate the relation between potentiality and quasireversibility for the speed functions with finite range, and prove that quasi-reversibility implies potentiality, and under (4.3) and (4.10) we prove that $\widetilde{\mathscr{G}}\subset\mathscr{G}(\mathscr{V})$, where $\widetilde{\mathscr{G}}$ is the closed convex hull in weak topology of all measures constructed by a specification $\mathscr{V}$, and $\mathscr{G}(\mathscr{V})$ is the set of all Gibbs states for $\mathscr{V}$. We also prove that each element of $\mathscr{G}(\mathscr{V})$ is a quasi-reversible measure. We give a construction of all quasi-reversible measures, and from this we obtain a necessary and sufficient condition for existence and uniqueness of the quasi-reversible measures.

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§ 2. Extension of field theory

Hou and Chen have established an abstract field theory for countable state spaces. In this section, we will extend the theory to any state spaces and give some of its local properties.

Let X be a non-empty set, T be an index set, and function $a(\cdot,\cdot,\cdot): T\times X\times X\to R\triangleq(-\infty,+\infty)$ satisfy the following two hypotheses:

(1) $\forall x, \tilde{x}\in X, x\neq\tilde{x}, \forall t\in T, a(t, x, \tilde{x})\geqslant 0$;

(2) $\forall x, \tilde{x}\in X, \forall t\in T, a(t, x, \tilde{x})=0\Leftrightarrow a(t, \tilde{x}, x)=0$.

Given $x, \tilde{x}\in X, x\neq\tilde{x}$, $\tilde{x}$ is called reachable directly from x at time t and denoted by $x\xrightarrow{t}\tilde{x}$, if $a(t, x, \tilde{x})>0$. $\tilde{x}$ is called reachable from x at time t, and we write $x\overset{t}{\rightsquigarrow}\tilde{x}$, if there is a number of $x^{(1)}, \cdots, x^{(n)}$ in X such that $x\xrightarrow{t}x^{(1)}\xrightarrow{t}x^{(2)}\to\cdots\xrightarrow{t}x^{(n)}\xrightarrow{t}\tilde{x}$. And $L(t)\triangleq(x, x^{(1)}, \cdots, x^{(n)}, \tilde{x})$ is called a path from x to $\tilde{x}$ at time t.

Let $\mathscr{A}(t)=\{a(t, x, \tilde{x}): x, \tilde{x}\in X\}$, $t\in T$. The set of all paths of $\mathscr{A}(t)$ is denoted by $\mathscr{L}(t)$. It is clear that $x\xrightarrow{t}\tilde{x}\Leftrightarrow\tilde{x}\xrightarrow{t}x$, $x\overset{t}{\rightsquigarrow}\tilde{x}\Leftrightarrow\tilde{x}\overset{t}{\rightsquigarrow}x$.

Define

(3) $\varphi(t, x, \tilde{x})=\log a(t, x, \tilde{x})-\log a(t, \tilde{x}, x)$, if $x\xrightarrow{t}\tilde{x}$;

(4) $\varphi(L(t))=\sum_{k=0}^{n}\varphi(t, x^{(k)}, x^{(k+1)})$, if $L(t)=(x\triangleq x^{(0)}, x^{(1)}, \cdots, x^{(n+1)}=\tilde{x})\in\mathscr{L}(t)$.

$\Phi(t)\triangleq\{\varphi(t,x,\tilde{x}): x,\tilde{x}\in X\}$ where $\varphi(t,x,\tilde{x})$ is undefined when $\tilde{x}$ is not reachable directly at time t. Then $(X, \mathscr{A}(t), \mathscr{L}(t), \Phi(t))$ (simply, $\mathscr{A}(t)$) is called a field and $\varphi(L(t))$ is called the work done by $\mathscr{A}(t)$ along $L(t)$. $\mathscr{A}(t)$ is called a potential field (or $\mathscr{A}(t)$ has a potential), if there is a real-valued function $V(\cdot,\cdot): T\times X\to R$ such that

(5) $\forall x, \tilde{x}\in X, x\xrightarrow{t}\tilde{x}, \varphi(t, x, \tilde{x})=V(t, \tilde{x})-V(t, x)$.

Then $V(t)=\{V(t, x); x\in X\}$, $t\in T$ is called a potential function of the potential field $\mathscr{A}(t)$. We say that $\mathscr{A}(t)$ is independent of the path if for any closed path $L(t)$ (i. e., $x^{(0)}=x^{(n+1)}$), $\varphi(L(t))=0$.

For and fixed $t\in T$, we define a relation "$\overset{t}{\sim}$" as follows:

(6) $\forall x, \tilde{x}\in X,\ x \overset{t}{\sim} x \Leftrightarrow x \overset{t}{\rightsquigarrow} x$ or $x=\tilde{x}$.

It is an equivalence relation. Thus, we may divide X into equivalent classes $\{X_l(t): l\in D(t)\}$. For each $l\in D(t)$, we choose $\Delta_l=\Delta_l(t)\in X_l(t)$ at will; for each $x\in X_l(t)$, $x\neq\Delta_l$, we also choose arbitrarily a path $L(t, \Delta_l, x)\triangleq(\Delta_l, x^{(1)}, \cdots, x^{(k)}, x)$, and put

(7) $$\begin{cases}\hat{a}(t, \Delta_l, x)\triangleq a(t, \Delta_l, x^{(1)})a(t, x^{(1)}, x^{(2)})\cdots a(t, x^{(k)}, x),\\ \hat{a}(t, x, \Delta_l)=a(t, x, x^{(k)})a(t, x^{(k)}, x^{(k-1)})\cdots a(t, x^{(1)}, \Delta_l).\end{cases}$$

Finally, $\mathscr{A}(t)$ is called weakly symmetrizable, if there is a family of real-valued functions $U(t)\triangleq\{u(t, x): x\in X\}$, $t\in T$ such that

(8) $\forall t\in T,\ \forall x\in X,\ u(t, x)>0$;

(9) $\forall t\in T,\ \forall x, \tilde{x}\in X,\ u(t, x)a(t, x, \tilde{x})=u(t, \tilde{x})a(t, \tilde{x}, x)$.

Then $U(t)$ is called a symmetrizing function of $\mathscr{A}(t)$.

(10) **Theorem.** *The following four statements are equivalent:*

(I) $\mathscr{A}(t)$ *is a potential field;*

(II) $\mathscr{A}(t)$ *is path-independent;*

(III) $\mathscr{A}(t)$ *is weakly symmetrizable;*

(IV) $\forall t\in T,\ \forall l\in D(t),\ \forall x, \tilde{x}\in X_l(t)$.

(11) $\hat{a}(t, \Delta_l, x)a(t, x, \tilde{x})\hat{a}(t, \tilde{x}, \Delta_l)=\hat{a}(t, \Delta_l, \tilde{x})a(t, \tilde{x}, x)\hat{a}(t, x, \Delta_l)$. *When one of the statements holds,* $\lambda(\cdot, \cdot): T\times X\longrightarrow R$.

(12) $$\lambda(t, x)\triangleq\begin{cases}1, & x=\Delta_l,\\ \dfrac{\hat{a}(t, \Delta_l, x)}{\hat{a}(t, x, \Delta_l)}, & x\neq\Delta_l,\ x\in X_l(t),\end{cases}\quad t\in T,\ x\in X$$

is a weakly symmetrizing function, and $\log\lambda(\cdot, \cdot)$ *is a potential of* $\mathscr{A}(t)$. *Furthermore, if* $\lambda'(\cdot, \cdot)$ *is another weakly symmetrizing function, then there is an* $\alpha_l(t)>0$ *for each* $l\in D(t)$ *such that*

$$\lambda'(t, x)=\alpha_l(t)\lambda(t, x),\quad \forall x\in X_l(t).$$

Finally, $\mathscr{A}(t)$ *is a potential field if and only if* $\mathscr{A}(t)$ *restricted on* $X_l(t)$ *is a potential field for every* $l\in D(t)$.

Proof (I)$\Rightarrow$(II). Let $V(\cdot, \cdot)$ be a potential function of $\mathscr{A}(t)$, $L(t)=(x=x^{(0)}, x^{(1)}, \cdots, x^{(n)}, x^{(n+1)}=x)$ be an arbitrary closed path, then

$$\varphi(L(t))=\sum_{k=0}^{n}\varphi(t, x^{(k)}, x^{(k+1)})=\sum_{k=0}^{n}[V(t, x^{(k+1)})-V(t, x^{(k)})]=0.$$

(II)$\Rightarrow$(IV). $\forall t\in T,\ \forall l\in D(t),\ \forall x, \tilde{x}\in X$, (11) is trivial if $x \overset{t}{\not\rightarrow} \tilde{x}$. Suppose $x\overset{t}{\rightarrow}\tilde{x}$. For an arbitrary but fixed path $L(t, \Delta_l, x)$ from Δ_l to x and a path $L(t, \Delta_l, \tilde{x})$ from Δ_l to $\tilde{x}$, it follows, on account of (II), that

$$\varphi(L(t, \Delta_l, x))+\varphi(t, x, \tilde{x})=\varphi(L(t, \Delta_l, \tilde{x})).$$

So by (3), (4) and (7),

$$\log\left[\frac{\hat{a}(t, \Delta_l, x)}{\hat{a}(t, x, \Delta_l)}\cdot\frac{a(t, x, \tilde{x})}{a(t, \tilde{x}, x)}\cdot\frac{\hat{a}(t, \tilde{x}, \Delta_l)}{\hat{a}(t, \Delta_l, \tilde{x})}\right]=0,$$

hence (11) holds.

(IV)$\Rightarrow$(III). Since $a(t, x, \tilde{x})=a(t, \tilde{x}, x)=0$ when x and $\tilde{x}$ do not belong to the same $X_l(t)$, $\lambda(\cdot, \cdot)$ is a weakly symmetrizing function of $\mathscr{A}(t)$ from (11) and (12).

(III)$\Rightarrow$(I). Let $u(\cdot, \cdot)$ be a weakly symmetrizing function. We take $V(t, x)\triangleq\log u(t, x)$, $t\in T$, $x\in X$. Then (5) follows from (9).

This proves the main part of the theorem, and it is very easy to check the truth of the other assertions

In the rest of this section we want to discuss the localization of a field with product state spaces.

Let S be an arbitriry set, $\mathscr{S}$ be the set of all subsets of S, $\mathscr{S}_f$ be the set of all finite subsets of S. Let Y_u be a non-empty set for every $u\in S$. Write $X=\prod_{u\in S} Y_u$, and let $(\prod_{u\in S} Y_u, \mathscr{A}(t), \mathscr{L}(t), \Phi(t))$ be a field which is called a product field. Simply, we will call $\mathscr{A}(t)$ a product field. $\mathscr{A}(t)$ is called a field with local character, if

(13) $\forall t\in T,\ x\overset{t}{\rightarrow}\tilde{x}\Rightarrow\{u\in S: x(u)\neq\tilde{x}(u)\}\in\mathscr{S}_f.$

Let $\Lambda\in\mathscr{S}_f$, $z\in\prod_{u\in S\setminus\Lambda} Y_u$. $\mathscr{A}(t)$ restricted on $\prod_{u\in\Lambda} Y_u\times\{z\}$ is denoted by $\mathscr{A}^z_\Lambda(t)\triangleq(a(t, x, \tilde{x}): x, \tilde{x}\in\prod_{u\in\Lambda} Y_u\times\{z\})$. Similarly we have $\mathscr{L}^z_\Lambda(t)$ and $\Phi^z_\Lambda(t)$. We will call the field $(\prod_{u\in\Lambda} Y_u\times\{z\}, \mathscr{A}^z_\Lambda(t), \mathscr{L}^z_\Lambda(t), \Phi^z_\Lambda(t))$ (simply, $(\mathscr{A}^z_\Lambda(t))$ a local field of $\mathscr{A}(t)$, A product field is called a local potential field, if $\forall\Lambda\in\mathscr{S}_f$, $\forall z\in\prod_{u\in S\setminus\Lambda} Y_u$, and $\mathscr{A}^z_\Lambda(t)$ is a potential field.

(14) **Theorem.** *A product field $\mathscr{A}(t)$ with local character is a potential field if and only if it is a local potential field.*

Proof Let $V(\cdot, \cdot)$ be a potential of $\mathscr{A}(t)$, then $V(\cdot, \cdot)$ restricted on $T\times(\prod_{u\in\Lambda} Y_u\times\{z\})$ is a potential of $\mathscr{A}^z_\Lambda(t)$, and $\mathscr{A}(t)$ is a local potential field. Conversely, it is enough to show that $\mathscr{A}(t)$ is path-independent. Let $L(t)=(x=x^{(0)}, x^{(1)}, \cdots, x^{(n)}, x^{(n+1)}=x)\in\mathscr{L}(t)$ be a closed path, then $\Lambda\triangleq\bigcup_{k=0}^{n}\{u\in S;\ x^{(k)}(u)\neq x^{(k+1)}(u)\}\in\mathscr{S}_f$ from (13). We write $z\triangleq x_{S\setminus\Lambda}$ (i. e., the projection of x on $\prod_{u\in S\setminus\Lambda} Y_u$), then $L(t)$ is also a path of $\mathscr{A}^z_\Lambda(t)$. Since the work done by the field $\mathscr{A}(t)$ alone $L(t)$ is the same as the work done by $\mathscr{A}^z_\Lambda(t)$ alone $L(t)$, and $\mathscr{A}^z_\Lambda(t)$ is a potential field, so $\varphi(L(t))=0$. The assertion is proved.

From this theorem, the potential problem of such $\mathscr{A}(t)$ reduces to one about countable state spaces when every Y_u is countable, and we can use[4].

§ 3. Reversibility and Quasi-Reversibility

In this section, we will discuss the relation between the reversibility of the transition function semigroup and the reversibiliiy of its generator. Then we will propose a quite general stochastic model for certain physical systems. The model includes spin-flip processes, exclusion processes and others. We will introduce a concept of quasi-reversibility for a semigroup generator, and shown that quasi-reversibility which is easily described is an extension of reversibility, and that they are equivalent under some conditions.

Chen[3] has proved that the reversibility of a stationary Markov process with stationary transition function $P(t, x, A)$ $(t\geqslant 0,\ x\in E,\ A\in\mathscr{E},\ \mathscr{E}$ includes all singletons $\{x\})$ is equivalent to that

(1) $\forall f,\ g\in b\mathscr{E},\ \int_E fP_tg d\mu=\int_E gP_tfd\mu,$

where $b\mathscr{E}$ is the set of all bounded $\mathscr{E}$-measurable functions, μ is a stationary measure, and

(2) $P_tf(x)=\int_E P(t,x,dy)f(y).$

When (1) holds, we say that $P(t, x, A)$ is reversible with respct to μ, and μ is a reversible measure with respect to $P(t, x, A)$. Perhaps, what we can know first is neither Markov processes nor their transition functions, but their generators. Therefore we will first discuss the relation between the reversibility of $P(t, x, A)$ and that of its generator.

Let $(E, \mathscr{E})$ be the metric measurable space and $\mathscr{E}$ its Borel σ-field. $b\mathscr{E}$ is a Banach space with supremum norm$\|\cdot\|$."$\xrightarrow{s}$" and "s-lim" denote strong convergence and strong limit respectively. We define the generator Ω of $\{P_t: t\geqslant 0\}$ as follows:

(3) $$\begin{cases}\mathscr{D}(\Omega)\triangleq\left\{f\in b\mathscr{E}: \dfrac{P_tf-f}{t}\xrightarrow{s} g\in b\mathscr{E}\right\},\\ \Omega f\triangleq \text{s-}\lim\limits_{t\to 0}\dfrac{1}{t}(P_tf-f),\ f\in\mathscr{D}(\Omega).\end{cases}$$

We will say that Ω is reversible with respect to probability measure μ if

(4) $\forall f,\ g\in\mathscr{D}(\Omega)\ \int f\Omega g d\mu=\int g\Omega f d\mu.$

(5) **Proposition.**

(i) *If a (contraction) semigroup $\{P_t: t\geqslant 0\}$ on $b\mathscr{E}$ is reversible with respect to probability measure μ, then so is its generator.*

(ii) *Suppose that a linear operator Ω on $b\mathscr{E}$ satisfies the conditions of the Hille-Yosida theorem (for dissipative case), then it generates unique contraction semigroup*

$\{P_t: t\geqslant 0\}$. *If Ω is reversible with respect to probability measure μ, then so is $\{P_t: t\geqslant 0\}$.*

Proof (i), (3), strong convergence and dominated convergence theorem imply (i).

(ii). From Hille-Yosida theorem, we obtain

(6) $\forall f\in b\mathscr{E},\ P_t f = s\text{-}\lim_{n\to\infty}(I-(t/n)\Omega)^{-n}f,$

and for every $\lambda>0$, and every $f, g\in b\mathscr{E}$ there are $\tilde{f}, \tilde{g}\in\mathscr{D}(\Omega)$ such that $\tilde{f}=(I-\lambda\Omega)^{-1}f$ and $\tilde{g}=(I-\lambda\Omega)^{-1}g$. It follows from

(4) that

$$\int f(I-\lambda\Omega)^{-1}g d\mu=\int [(I-\lambda\Omega)\tilde{f}]\,\tilde{g}d\mu=\int[(I-\lambda\Omega)\tilde{g}]\,\tilde{f}d\mu$$
$$=\int g(I-\lambda\Omega)^{-1}f d\mu.$$

By induction, it follows that

$$\int f(I-\lambda\Omega)^{-n}g d\mu=\int g(I-\lambda\Omega)^{-n}f d\mu,\ \forall f,\ g\in b\mathscr{E},\ \forall\lambda>0,\ \forall n\geqslant 1.$$

Taking $\lambda=\frac{t}{n}$ and letting $n\to\infty$, We obtain (1) from (6).

(7) **Remark.** An analogue of proposition (5) is also true if $b\mathscr{E}$ is replaced by a proper subspace of $b\mathscr{E}$ (for example, the following $\mathscr{E}$).

(8) In our model Ω is described with some speed functions. Now we introduce several notations which will be used throughout the rest of this paper.

Let S be a countable set, so be Y_u for every $u\in S$. But $|Y_u|\geqslant 2$, $\forall u\in S$. We write $X(\Lambda)\triangleq\prod_{u\in\Lambda}Y_u$, in particular $X=X(S)$. The projection of x on $X(\Lambda)$ is denoted by x_Λ. We write x_u instead of $x_{(u)}$ for every $u\in S$. We topologize X by giving Y_u the discrete topology and X the resulting product topology. $\mathscr{F}_0(\Lambda)$ is the Borel σ-field on $X(\Lambda)$, it is also the product σ-field on $X(\Lambda)$. We write $\mathscr{F}(\Lambda)\triangleq\mathscr{F}_0(\Lambda)\times X(S\backslash\Lambda)$ which is a sub-σ-field of $\mathscr{F}\triangleq\mathscr{F}_0(S)$.

Let $\mathscr{B}(\Lambda)$ be the set of all bounded $\mathscr{F}(\Lambda)$-measurable functions, $\mathscr{B}\triangleq\mathscr{B}(S)$, and $\mathscr{S}$ and $\mathscr{S}_f$ be the same as in § 2. For each integer N, let $\mathscr{S}_N=\{\Lambda\in\mathscr{S}: |\Lambda|=N\}$, where $|\Lambda|$ is the number of members of Λ. We denote by $\mathscr{A}=\bigcup_{\Lambda\in\mathscr{S}_f}\mathscr{B}(\Lambda)$ the set of bounded continuous cylinder functions on X, and by $\mathscr{C}=\mathscr{C}(X)$ the set of all bounded continous functions on X. Clearly $\mathscr{A}\subset\mathscr{C}\subset\mathscr{B}$. We denote by $\mathscr{P}(X)$ the set of all probability measures on $(X, \mathscr{F})$. For every $\Lambda\in\mathscr{S}$, $\mu\in\mathscr{P}(X)$, μ_Λ denotes the projection of μ on $(X(\Lambda), \mathscr{F}_0(\Lambda))$, i. e.

$$\mu_\Lambda(F)\triangleq\mu(F\times X(S\backslash\Lambda)),\ \forall F\in\mathscr{F}_0(\Lambda).$$

Finally, we write ${}^y_\Lambda x=y\times x_{S\backslash\Lambda}$ for every $\Lambda\in\mathscr{S}_f$, $y\in X(\Lambda)$. From now on, N is fixed.

(9) **Definition.** *We call $c(\cdot,\cdot,\cdot): \mathscr{S}_N\times X(\Lambda)\times X\to R^+=[0,\infty)$ to be a speed function, if*

(10) $\forall y\in X(\Lambda)$, $c(\Lambda, y, \cdot)$ *is* $\mathscr{F}$*-measurable*;

(11) $\forall x\in X$, $c(\Lambda, y, x)=0$ *if there is a* $u\in\Lambda$ *such that* $x_u=y_u$. *If* $c(\cdot, \cdot, \cdot)$ *also satisfies*

(12) *Co-zero*: $c(\Lambda, y, x)=0\Leftrightarrow c(\Lambda, x_\Lambda, {}^y_\Lambda x)=0$, *then we set*

(13) $$q(x, \tilde{x})=\begin{cases} c(\Lambda, y, x), \ \exists\Lambda\in\mathscr{S}_N, \ \exists y\in X(\Lambda) \text{ such that } {}^y_\Lambda x=\tilde{x}; \\ 0, \text{ otherwise,}\end{cases}$$

and as in § 2 *define a field* $Q=\{q(x, \tilde{x}): x, \tilde{x}\in X\}$ *which is called a speed function field. Clearly the field has local character propertyfand from Theorem* (2.14) *we obtain.*

(14) **Proposition.** *A speed function field has a potential if and only if is a local potential field.*

(15) **Definition.** *A speed function field Q is said to be with finite range, if its speed function satisfies*

(16) $\forall\Lambda\in\mathscr{S}_N$, $\exists\tilde{\Lambda}\supset\Lambda$, $\tilde{\Lambda}\in\mathscr{S}_f$ *such that* $\forall y\in X(\Lambda)$, $c(\Lambda, y, \cdot)$ *is* $\mathscr{F}(\tilde{\Lambda})$-*measurable.*

The minimal $\tilde{\Lambda}$ satisfying (16) is denoted by $r(\Lambda)$. Clearly every speed function with finite range is continuous. For every x, $\tilde{x}\in X$ from (16), we have $c(\Lambda, y, x)=c(\Lambda, y, \tilde{x})$ whenever $x_{\tilde{\Lambda}}=\tilde{x}_{\tilde{\Lambda}}$. So we often use $c(\Lambda, y, x_{\tilde{\Lambda}})$ instead of $c(\Lambda, y, x)$, and the cast of $q(x, \tilde{x})$ can be treated similarly.

(17) **Definition.** *Let Q be a speed function field with finite range. If there is* $\mu\in\mathscr{P}(X)$ *such that*

(18) $\forall\Lambda\in\mathscr{S}_f$, $\forall x\in X(\Lambda)$, $\mu_\Lambda(x)>0$;

(19) $\forall\Lambda\in\mathscr{S}_N$, $\forall\tilde{\Lambda}\supset r(\Lambda)$, $y\in X(\Lambda)$, $x\in X(\tilde{\Lambda})$, *and we have*

$$\mu_{\tilde{\Lambda}}(x)c(\Lambda, y, x)=\mu_{\tilde{\Lambda}}({}^y_\Lambda x)c(\Lambda, x_\Lambda, {}^y_\Lambda x),$$

i. e.

$$\mu_{\tilde{\Lambda}}(x)q(x, {}^y_\Lambda x)=\mu_{\tilde{\Lambda}}({}^y_\Lambda x)q({}^y_\Lambda x, x),$$

then we say that Q is quasi-reversible with respect to μ, *and* μ *is a quasi-reversible measure for Q.*

(20) **Definition.** *Let Q be a speed function field satisfying*

(21) $\forall\Lambda\in\mathscr{S}_N$, $\forall x\in X$, $\sum_{y\in X(\Lambda)} c(\Lambda, y, x)\leqslant c(\Lambda)$,

(22) $\forall u\in S$, $\sum_{u\in\Lambda\in\mathscr{S}_N} c(\Lambda)<\infty$.

We define a linear operator Ω: $\mathscr{D}(\Omega)\to\mathscr{B}$ as follows:

(23) $\Omega f(x)\triangleq\sum_{\Lambda\in\mathscr{S}_N}\sum_{y\in X(\Lambda)} c(\Lambda, y, x)\Delta^y_\Lambda f(x)$,

where $\mathscr{D}(\Omega)\supset\mathscr{A}$, and $\Delta^y_\Lambda f(x)\triangleq f({}^y_\Lambda x)-f(x)$.

The closure $\bar{\Omega}$ of Ω is called reversible if there is $\mu\in\mathscr{P}(X)$ such that (18) and

(24) $\forall f, g\in\mathscr{D}(\bar{\Omega})$, $\int f\bar{\Omega}g d\mu=\int g\bar{\Omega}f d\mu$

hold. Then μ is said to be a reversible measure for $\bar{\Omega}$ (or Ω, or $c(\cdot, \cdot, \cdot)$).

Remark. When $\mathscr{D}$ is a core for $\bar{\Omega}$ (i. e., $\bar{\Omega}$ restricted on $\mathscr{D}$ and Ω have the same closed extension), we can use $\mathscr{D}$ instead of $\mathscr{D}(\bar{\Omega})$ in (24). In particular, we can use $\mathscr{D}(\Omega)$ instead of $\mathscr{D}(\bar{\Omega})$ and Ω instead of $\bar{\Omega}$. For reversibility there is condition (18) in addition to (4). From now on, reversibility for speed function usually means Definition (20).

(25) **Theorem.** *Let Q be a speed function field with finite range. If Q is reversible with respect to μ, then Q is quasi-reversible with respect to μ.*

Proof Let $\Lambda\in\mathscr{S}_N$, $\tilde{\Lambda}\supset r(\Lambda)$; $z\in X(\tilde{\Lambda})$, $y\in X(\Lambda)$, $\forall u\in\Lambda$, $z_u\neq y_u$. Taking $f=I_{\{z\}\times X(S\setminus\tilde{\Lambda})}$, $g=I_{\{{}^y_\Lambda z\}\times X(S\setminus\tilde{\Lambda})}$, for every $x\in\{{}^y_\Lambda z\}\times X(S\setminus\tilde{\Lambda})$ we have

$$\Omega f(x)=\sum_{\Lambda'\in\mathscr{S}_N}\sum_{w\in X(\Lambda')}c(\Lambda',w,x)\left[I_{\{z\}\times X(S\setminus\tilde{\Lambda})}({}^w_{\Lambda'}x)-I_{\{z\}\times X(S\setminus\tilde{\Lambda})}(x)\right]$$
$$=c(\Lambda,z_\Lambda,x)I_{\{z\}\times X(S\setminus\tilde{\Lambda})}(z_\Lambda\times x_{S\setminus\Lambda}),$$

hence

$$\int g\Omega f d\mu=\int_{\{{}^y_\Lambda z\}\times X(S\setminus\Lambda)}c(\Lambda,z_\Lambda,x)\mu(dx)=c(\Lambda,z_\Lambda,{}^y_\Lambda z)\mu_{\tilde{\Lambda}}({}^y_\Lambda z).$$

Using ${}^y_\Lambda z$ instead of z in the preceding discussion, we obtain

$$\int f\Omega g d\mu=c(\Lambda,y,z)\mu_{\tilde{\Lambda}}(z).$$

So reversibility for Q implies quasi-reversibility for Q.

§ 4. Potentiality and Quasi-reversibility

In this section, we will discuss the relation between potentiality and quasi-reversibility for Q.

(1) **Theorem.** *Let Q be a speed function field with finite range. If Q is quasi-reversible, then it has a potential.*

Proof Let μ is a quasi-reversible measure for Q. We want to prove that Theorem (2.10) (iv) holds. Since field Q is independent of t, we can use $\rightarrow$, $\rightsquigarrow$, D and X_l instead of $\overset{t}{\rightarrow}$, $\overset{t}{\rightsquigarrow}$, $D(t)$ and $X_l(t)$ respectively.

From § 2, it is clear that $q(x,\tilde{x})=0$ whenever $x\not\rightarrow\tilde{x}$, and in this case (2.11) holds. It suffices to prove that (2.11) holds when $x\rightarrow\tilde{x}$. Therefore, by Definition (3.9), we may suppose $q(x,\tilde{x})>0$. Then there are $\Lambda\in\mathscr{S}_N$, $y\in X(\Lambda)$ such that $y_u\neq x_u$ for each $u\in\Lambda$ and $x={}^y_\Lambda x$. Hence $q(x,x)=c(\Lambda,y,x)$. From quasi-reversibility we have

(2) $\mu_{\tilde{\Lambda}}(x_{\tilde{\Lambda}})q(x,\tilde{x})=\mu_{\tilde{\Lambda}}(\tilde{x}_{\tilde{\Lambda}})q(\tilde{x},x)$, $\forall\tilde{\Lambda}\supset r(\Lambda)$.

In § 2 we have taken and fixed the pathes $L(\Delta_l,x)=(\Delta_l=x^{(0)},x^{(1)},\cdots,x^{(n)},x^{(n+1)}=x)$ and $L(\Delta_l,\tilde{x})=(\Delta_l=\tilde{x}^{(0)},\tilde{x}^{(1)},\cdots,\tilde{x}^{(m)},\tilde{x}^{(m+1)}=\tilde{x})$. Using the preceding discussion for each segment $(x^{(k)}\rightarrow x^{(k+1)}$ called a segment of the path $L(\Delta_l,x))$, we can always take a $\tilde{\Lambda}\in\mathscr{S}_f$ such that (2) and the following equations all hold:

$$\mu_{\tilde{\Lambda}}(x^{(i)}_{\tilde{\Lambda}})q(x^{(i)},x^{(i+1)})=\mu_{\tilde{\Lambda}}(x^{(i+1)}_{\tilde{\Lambda}})q(x^{(i+1)},x^{(i)}),\quad i=0,1,\cdots,n,$$

$$\mu_{\tilde{A}}(\tilde{x}_{\Lambda}^{(j)})q(\tilde{x}^{(j)}, \tilde{x}^{(j+1)}) = \mu_{\tilde{A}}(\tilde{x}_{\Lambda}^{(j+1)})q(\tilde{x}^{(j+1)}, \tilde{x}^{(j)}), \quad j=0, 1, \cdots, m.$$

Hence

$$\mu_{\tilde{A}}((\Delta_l)_{\tilde{A}})\hat{q}(\Delta_l, x) = \mu_{\tilde{A}}(x_{\tilde{A}})\hat{q}(x, \Delta_l),$$
$$\mu_{\tilde{A}}((\Delta_l)_{\tilde{A}})\hat{q}(\Delta_l, \tilde{x}) = \mu_{\tilde{A}}(\tilde{x}_{\tilde{A}})q(\tilde{x}, \Delta_l).$$

From this and (2), we obtain

$$\hat{q}(\Delta_l, x)q(x, \tilde{x})\hat{q}(\tilde{x}, \Delta_l) = \hat{q}(\Delta_l, x)q(x, \tilde{x})\hat{q}(\tilde{x}, \Delta_l).$$

This is Theorem (2.10) (iv), hence Q has a potential.

Until now, our hypotheses for the speed functions are (3.10), (3.11), (3.12), (3.16), (3.21) and (3.22). Conditions (3.10) and (3.11) are certainly necessary. Condition (3.12) is also necessary for potentiality. In order to make the closure of Ω be a semigroup generator, Conditions (3.21) and (3.22) are necessary, too (See the example in [10; II, § 1.1]). Therefore, the essential condition for the speed functions is (3.16).

In order to discuss the converse of (4.1), we have to use another hypothesis for the speed functions.

(3) **Hypothesis.** For every $\Lambda \in \mathscr{S}_f$, there is a $\tilde{\Lambda} \in \mathscr{S}_f$, $\tilde{\Lambda} \supset \Lambda$ such that for every $x, \tilde{x} \in X$, whenever $\{u; x_u \neq \tilde{x}_u\} \subset \Lambda$, there are $x^{(i)}$, $i=1, 2, \cdots, m$ such that $x \triangleq x^{(0)} \to x^{(1)} \to \cdots \to x^{(m)} \to x^{(m+1)} \triangleq x$ and $\{u; x_u^{(i)} \neq x_u^{(i+1)}\} \subset \tilde{\Lambda}$. Clearly, if $\tilde{\Lambda}$ satisfies the above condition, then every $\Lambda_1 \supset \tilde{\Lambda}$ satisfies (3) too. So there are minimal sets satisfying (3). But the minimal sets may be not unique, so we choose arbitrary one of them and write it as $\delta(\Lambda)$. It is clear that $\delta(\Lambda) \supset \Lambda$.

(4) **Remark.** If there is $\Lambda_0 \in \mathscr{S}_{N-1}$ such that $|Y_u| > 2$ for each $u \in \Lambda_0$, then (3) may hold. For example, (3) is true when the condition "$y \in X(\Lambda)$, $|\Lambda| = N$, $\forall u \in \Lambda$, $y_u \neq x_u \Rightarrow c(\Lambda, y, x) > 0$" holds.

However, (3) does not certainly hold when $|Y_u| \equiv 2 (u \in S)$. For example, we take $Y_u \equiv \{0, 1\}$, $N=2$; for every $\{u, v\} \subset S$, $y \in X(\{u, v\})$ we suppose $c(\{u, v\}, y, x) > 0$ whenever $y_u \neq x_u$, $y_v \neq x_v$, B_y taking $x = (0, 0, 0) \times z$, $\tilde{x} = (1, 1, 1) \times z$, $z \in X(S \setminus \{u_1, u_2, u_3\})$, we have $x \nsim \tilde{x}$, so (3) does not hold. But later we will discuss this case again in another way.

(5) **Remark.** Let Q be a speed function field, which satisfies (3), with finite range, then for every $\Lambda \in \mathscr{S}_f$, $x \in X$, $w \in X(\Lambda)$, $w \neq x_\Lambda$, there is certainly a path $L(x^{(0)}, x^{(1)}, \cdots, x^{(n+1)})$ from $x^{(0)} = x$ to $x^{(n+1)} = w \times x_{S \setminus \Lambda}$ such that $\{u \in S; x_u^{(i)} \neq x_u^{(i+1)}\} \subset \delta(\Lambda)$, $0 \leqslant i \leqslant n$. Put

$$(6) \begin{cases} \bar{q}(x^{(0)}, x^{(n+1)}) \triangleq \prod_{i=1}^{n} q(x^{(i)}, x^{(i+1)}), \\ \bar{q}(x^{(n+1)}, x^{(0)}) \triangleq \prod_{i=1}^{n} q(x^{(i+1)}, x^{(i)}) \end{cases}$$

and $\tilde{\Lambda} = \bigcup_{\substack{A \subset \delta(\Lambda) \\ |A| = N}} \mathrm{r}(A)$. Since Q has finite range, $\bar{q}(x, w \times x_{S \setminus \Lambda})$ and $\bar{q}(w \times x_{S \setminus \Lambda}, x)$ are $\mathscr{F}(\tilde{\Lambda})$-measurable, i. e.

(7) $\begin{cases}\forall x\in X,\ \forall w\in X(\Lambda),\ \bar q(x,\ w\times x_{S\setminus\Lambda})=\bar q(x_{\tilde\Lambda},\ w\times x_{\tilde\Lambda-\Lambda}),\\ \bar q(w\times x_{S\setminus\Lambda},\ x)=\bar q(w\times x_{\tilde\Lambda\setminus\Lambda},\ x_{\tilde\Lambda}).\end{cases}$

In addition, if Q has a potential, then it is clear by path-independence that

(8) $\forall w\in X(\Lambda),\ \lambda(x)\dfrac{\bar q(x,\ w\times x_{S\setminus\Lambda})}{\bar q(w\times x_{S\setminus\Lambda},\ x)}=\lambda(w\times x_{S\setminus\Lambda}).$

(9) **Corollary.** *Let Q be a speed function field which satisfies (3) and be quasi-reversible, then we have*

(10) $\forall \Lambda\in\mathscr{S}_f,\ \sum_{w\in X(\Lambda)}\lambda(w\times x_{S\setminus\Lambda})<\infty.$

Proof Unsing the notation in Remark (5), from (2) we obtain

$$\mu_{\tilde\Lambda}(x^{(i)}_{\tilde\Lambda})q(x^{(i)},\ x^{(i+1)})=\mu_{\tilde\Lambda}(x^{(i+1)}_{\tilde\Lambda})q(x^{(i+1)},\ x^{(i)}),\ 0\leqslant i\leqslant n.$$

So, for every $w\in X(\Lambda)$ we have

$$\mu_{\tilde\Lambda}(x_{\tilde\Lambda})\bar q(x,\ w\times x_{S\setminus\Lambda})=\mu_{\tilde\Lambda}(w\times x_{\tilde\Lambda\setminus\Lambda})\bar q(w\times x_{S\setminus\Lambda},\ x),$$

hence from Theorems (1), (2) and (8) we obtain

$$\sum_{w\in X(\Lambda)}\lambda(w\times x_{S\setminus\Lambda})=\lambda(x)\sum_{w\in X(\Lambda)}\frac{\bar q(x,\ w\times x_{S\setminus\Lambda})}{\bar q(w\times x_{S\setminus\Lambda},\ x)}=\lambda(x)\sum_{w\in X(\Lambda)}\frac{\mu_{\tilde\Lambda}(w\times x_{\tilde\Lambda\setminus\Lambda})}{\mu_{\tilde\Lambda}(x_{\tilde\Lambda})}$$
$$=\lambda(x)\frac{\mu_{\tilde\Lambda}(X(\Lambda)\times x_{\tilde\Lambda\setminus\Lambda})}{\mu_{\tilde\Lambda}(x_{\tilde\Lambda})}<\infty.$$

(11) **Lemma.** *Let Q be a speed function field, which has a potential and satisfies (3) and (10), with finite range. For every $\Lambda\in\mathscr{S}_f$ we put*

(12) $f^\Lambda(x)=\lambda(x)\setminus[\sum_{w\in X(\Lambda)}\lambda(w\times x_{S\setminus\Lambda})],\ x\in X.$

Then $\{f^\Lambda:\Lambda\in\mathscr{S}_f\}$ has the following properties:

(13) $\forall x\in X,\ f^\Lambda(x)>0;$

(14) $\forall\Lambda\in\mathscr{S}_f,\ \forall y\in X(S\setminus\Lambda),\ \sum_{z\in X(\Lambda)}f^\Lambda(z\times y)=1;$

(15) $\forall\Lambda\subset\tilde\Lambda\in\mathscr{S}_f,\ \forall x\in X,\ f^{\tilde\Lambda}(x)=f^\Lambda(x)\sum_{w\in X(\Lambda)}f^{\tilde\Lambda}(w\times x_{S\setminus\Lambda});$

(16) $\forall\Lambda\in\mathscr{S}_f,\ \exists\tilde\Lambda\in\mathscr{S}_f,\ \tilde\Lambda\supset r(\Lambda)$ *such that* f^Λ is $\mathscr{F}(\tilde\Lambda)$*-measurable.*

Proof First, we note that f^Λ is independent of the selection of Δ_l and $L(\Delta,\ _l x)$ (See § 2). In fact, if we choose another Δ'_l and $L'(\Delta'_l,\ x)$, and define $\lambda'(x)$ in the same way, then from potentiality for Q and Theorem (2.10) there is an $\alpha_l>0$ such that $\lambda(x)=\alpha_l\lambda'(x)$ for every $x\in X_l$. So from (3) we obtain

$$\forall\Lambda\in\mathscr{S}_f,\ \forall w\in X(\Lambda),\ x\rightsquigarrow w\times x_{S\setminus\Lambda},$$

hence $\lambda(x)$ and $\lambda'(x)$ define the same f^Λ.

Next, (13) and (14) are trivial. For every $\Lambda\subset\tilde\Lambda\in\mathscr{S}_f$ and every $x\in X$, we have

$$f^{\tilde\Lambda}(x)=\frac{\lambda(x)}{\sum_{\tilde w\in X(\tilde\Lambda)}\lambda(\tilde w\times x_{S\setminus\tilde\Lambda})}=\frac{\lambda(x)}{\sum_{w\in X(\Lambda)}\lambda(w\times x_{S\setminus\Lambda})}\cdot\sum_{w\in X(\Lambda)}\frac{\lambda(w\times x_{S\setminus\Lambda})}{\sum_{\tilde w\in X(\tilde\Lambda)}\lambda(\tilde w\times x_{S\setminus\tilde\Lambda})}$$
$$=f^\Lambda(x)\sum_{w\in X(V)}f^{\tilde\Lambda}(w\times x_{S\setminus\Lambda}),$$

so (15) holds.

Finally we will prove (16). We use the notation and result in Remark (5) to prove that f^{Λ} is $\mathscr{F}(\tilde{\Lambda})$-measurable for every $\Lambda \in \mathscr{S}_f$, i. e.

(17) $\forall z \in X(S \setminus \tilde{\Lambda}),\ f^{\Lambda}(x) = f^{\Lambda}(x_{\tilde{\Lambda}} \times z)$.

For fixed $x \in X$ there is an $l \in D$ such that $x \in X_l$. Since f^{Λ} is independent of the selection of Δ_l, we can suppose $x = \Delta_l$ without loss of generality, i.e. $\lambda(x) = \lambda(\Delta_l) = 1$. Let $x_{\tilde{\Lambda}} \times z \in X_{l'}$. From path-independence, (7) and (8), we obtain

$$\begin{aligned}\lambda(w \times x_{\tilde{\Lambda}\setminus\Lambda} \times z) &= \frac{\hat{q}(\Delta_{l'},\ w \times x_{\tilde{\Lambda}\setminus\Lambda} \times z)}{\hat{q}(w \times x_{\tilde{\Lambda}\setminus\Lambda} \times z,\ \Delta_{l'})} \\ &= \frac{\hat{q}(\Delta_{l'},\ x_{\tilde{\Lambda}} \times z)}{\hat{q}(x_{\tilde{\Lambda}} \times z,\ \Delta_{l'})} \cdot \frac{\bar{q}(x_{\tilde{\Lambda}} \times z,\ w \times x_{\tilde{\Lambda}\setminus\Lambda} \times z)}{\bar{q}(w \times x_{\tilde{\Lambda}\setminus\Lambda} \times z,\ x_{\tilde{\Lambda}} \times z)} \\ &\overset{(7)}{=} \lambda(x_{\tilde{\Lambda}} \times z) \frac{\bar{q}(x,\ w \times x_{S\setminus\Lambda})}{\bar{q}(w \times x_{S\setminus\Lambda},\ x)} \\ &\overset{(8)}{=} \lambda(x_{\tilde{\Lambda}} \times z)\, \lambda(w \times x_{S\setminus\Lambda}).\end{aligned}$$

hence

$$f^{\Lambda}(x_{\tilde{\Lambda}} \times z) = \frac{\lambda(x_{\tilde{\Lambda}} \times z)}{\sum\limits_{w \in X(\Lambda)} \lambda(w \times x_{\tilde{\Lambda}\setminus\Lambda} \times z)} = \frac{1}{\sum\limits_{w \in X(\Lambda)} \lambda(w \times x_{S\setminus\Lambda})} = f^{\Lambda}(x),$$

thus the proof is terminated.

(18) **Corollary.** $\forall \Lambda \in \mathscr{S}_f,\ f^{\Lambda} \in \mathscr{C}(X)$.

Now, we discuss how to co nstruct the quasi-reversible measures for Q.

(19) **Definition.** *The family* $\mathscr{V} = \{f^{\Lambda}: \Lambda \in \mathscr{S}_f\}$ *of the functions satisfying* (13)—(15) *is said to be a specificateon.* $\mu \in \mathscr{P}(X)$ *is called a Gibbs state with specification* $\mathscr{V}$, *if*

(20) $\forall \Lambda \in \mathscr{S}_f, \forall y \in X(\Lambda),\ \mu(\{y\} \times X(S \setminus \Lambda) \mid \mathscr{F}(S \setminus \Lambda)) = f^{\Lambda}(y \times (\cdot)_{S\setminus\Lambda}),\ \mu$—a. e.

The set of all Gibbs states with specification $\mathscr{V}$ *is dented by* $\mathscr{G}(\mathscr{V})$.

For every $z \in X(S \setminus \Lambda)$, $F \in \mathscr{F}$, we define

(21) $\mu_{\Lambda, z}(F) \triangleq \sum\limits_{y \in F(z)} f^{\Lambda}(y \times z)$,

where $F(z) \triangleq \{y \in X(\Lambda): y \times z \in F\}$, and let $\mathscr{G}_{\Lambda}(\Lambda \in \mathscr{S}_f)$ be the closed convex hull of all $\mu_{\Lambda, z}$, $z \in X(S \setminus \Lambda)$. Finelly, let

(22) $\mathscr{G} \triangleq \{\mu \in \mathscr{P}(X):\ \exists \Lambda_m \in \mathscr{S}_f,\ \Lambda_m \nearrow S,\ \exists \mu_m \in \mathscr{G}_{\Lambda_m}$ such that $\mu_m \overset{\omega}{\to} \mu\}$

where $\mu_m \overset{\omega}{\to} \mu$ means that μ_m is weakly convergent to μ.

(23) **Proposition.** *Let* Q *be a speed funetion field, which satisfies* (3) *and* (10), *with finite range, then* $\mathscr{V} = \{f^{\Lambda}: \Lambda \in \mathscr{S}_f\}$ *defined by* (12) *is a specification, every* f^{Λ} *is a continuous function on* X *and*

(24) $\mathscr{G} \subset \mathscr{G}(\mathscr{V})$.

Proof Lemma (11) and Corollary (18) imply the first assertion. It is clear that $\mu \in \mathscr{G}(\mathscr{V})$ is equivalent to the following

(25) $\forall \Lambda \in \mathscr{S}_f,\ \forall y \in X(\Lambda),\ \forall F \in \mathscr{F}_0(S \setminus \Lambda),\ \mu(\{y\} \times F) = \int_F f^{\Lambda}(y \times z)\, \mu_{S\setminus\Lambda}(dz)$.

hence it remains only to prove that (25) holds for

(26) $F=\{y_1\}\times X(S\backslash\tilde{\Lambda}),\ \tilde{\Lambda}\supset\Lambda,\ \tilde{\Lambda}\in\mathscr{S}_f,\ y_1\in X(\tilde{\Lambda}\backslash\Lambda)$.

Let $\mu\in\mathscr{G}$, then there are $\Lambda_m\in\mathscr{S}_f$, $\Lambda_m\uparrow S$ and $\mu_m\in\mathscr{G}_{\Lambda_m}$ such that $\mu_m\xrightarrow{\omega}\mu$. In order to prove that $\mu\in\mathscr{G}(\mathscr{V})$, we first prove that (25) holds for F in (26) and

(27) $\mu=\mu_{\Lambda_m,z_m}\Lambda_m\supset\tilde{\Lambda},\ z_m\in X(S\backslash\Lambda_m)$.

From (21) and (15) we obtain that the right hand side of (25)

$$=\int_{\{y_1\}\times X(S\backslash\tilde{\Lambda})} f^{\Lambda}(y\times z)(\mu_{\Lambda_m,z_m})_{S\backslash\Lambda}(dz)=\int_{\{y_1\}\times X(\Lambda_m\backslash\tilde{\Lambda})\times\{z_m\}} f^{\Lambda}(y\times z)(\mu_{\Lambda_m,z_m})_{S\backslash\Lambda}(dz)$$

$$=\sum_{y'\in X(\Lambda_m\backslash\tilde{\Lambda})} f^{\Lambda}(y\times y_1\times y'\times z_m)\sum_{w\in X(\Lambda)} f^{\Lambda_m}(w\times y_1\times y'\times z_m)$$

$$=\sum_{y'\in X(\Lambda_m\backslash\tilde{\Lambda})} f^{\Lambda_m}(y\times y_1\times y'\times z_m)=\mu_{\Lambda_m,z_m}(\{y\times y_1\}\times X(S\backslash\tilde{\Lambda})).$$

Furthermore (25) holds for F in (26) and the convex linear combination ν_m of $\mu_{\Lambda_m z_m}$ (Λ_m is fixed). Because $f^{\Lambda}(y\times\cdot)$ is a bounded continuous function on $X(S\backslash\Lambda)$, so (25) holds also for F in (26) and $\mu=\mu_m\in\mathscr{G}_{\Lambda_m}$ ane hence for $\mu\in\mathscr{G}$. Thus we have proved that $\mu\in\mathscr{G}(\mathscr{V})$.

(28) **Proposition.** *Let Q be a speed function field which satisfies* (3) *and* (10), *with finite range. If* $\mathscr{G}(\mathscr{V})\neq\phi$, *then Q is quasi-reversible and every* $\mu\in\mathscr{G}(\mathscr{V})$ *is a quasi-reversible measure for Q,*

Proof It suffices to prove that every $\mu\in\mathscr{G}(\mathscr{V})$ is a quasi-reversible measure for Q.

First, we want to prove (3.18). Take $\Lambda\in\mathscr{S}_f$, $y\in X(\Lambda)$, from (16), there is a $\tilde{\Lambda}\supset r(\Lambda)$, $\tilde{\Lambda}\in\mathscr{S}_f$ such that f^{Λ} is $\mathscr{F}(\tilde{\Lambda})$-measurable. We may assume $\tilde{\Lambda}\neq\Lambda$ without loss of generality. For every $\Lambda_1\supset\tilde{\Lambda}$, applying (25) to $F=\{w\}\times X(S\backslash\Lambda_1)$, $w\in X(\Lambda_1\backslash\Lambda)$, we obtain

(29) $\mu_{\Lambda_1}(y\times w)=f^{\Lambda}(y\times w\times\bar{z})\mu_{\Lambda_1\backslash\Lambda}(w)$, $w\in X(\Lambda_1\backslash\Lambda)$, where $\bar{z}$ is an arbitrary but fixed element in $X(S\backslash\Lambda_1)$. Since $\Lambda_1\backslash\Lambda\neq\emptyset$ and $\mu_{\Lambda_1\backslash\Lambda}$ is a probability measure, there is a $\bar{w}\in X(\Lambda_1\backslash\Lambda)$ such that $\mu_{\Lambda_1\backslash\Lambda}(\bar{w})>0$, hence $\mu_{\Lambda_1}(y\times\bar{w})>0$ from (13). Taking the summation for w in (29) over $X(\Lambda_1\backslash\Lambda)$, we obtain $\mu_{\Lambda}(y)>0$.

Next, we want to prove (3.19). $\tilde{\Lambda}$ remains as above. From (29) we have, for every $\Lambda_1\supset\tilde{\Lambda}$, $\Lambda_1\in\mathscr{S}_f$, $x\in X(\Lambda_1)$, $y\in X(\Lambda)$, that

$$\frac{\mu_{\Lambda_1}(x)}{\mu_{\Lambda_1}({}^{y}_{\Lambda}x)}=\frac{f^{\Lambda}(x\times z)}{f^{\Lambda}({}^{y}_{\Lambda}x\times z)}=\frac{\lambda(x\times z)}{\lambda({}^{y}_{\Lambda}x\times z)},\ z\in X(S\backslash\Lambda_1).$$

Hence, by the potentiality for Q and Theorem (2.10), Q is weakly symmetrizable and $\lambda(\cdot)$ is its symmetrizing function. Therefore, we have

$\forall\Lambda_1\in\mathscr{S}_f,\ \Lambda_1\supset\tilde{\Lambda},\ \forall x\in X(\Lambda_1)$,

$$\frac{\mu_{\Lambda_1}(x)}{\mu_{\Lambda_1}({}^{y}_{\Lambda}x)}=\frac{c(\Lambda,\ x_{\Lambda},\ {}^{y}_{\Lambda}x)}{c(\Lambda,\ y,\ x)},$$

whenever $c(\Lambda, y, x)>0$. For every $\Lambda_1\in\mathscr{S}_f$, $\tilde{\Lambda}\supset\Lambda_1\supset r(\Lambda)$, $x\in X(\Lambda_1)$, by the above equality (taking $\Lambda_1=\tilde{\Lambda}$ there), we obtain

$$\frac{\mu_{\Lambda_1}(x)}{\mu_{\Lambda_1}({}^y_\Lambda x)}=\frac{\sum_{w\in X(\tilde{\Lambda}\setminus\Lambda_1)}\mu_{\tilde{\Lambda}}(x\times w)}{\sum_{w\in X(\tilde{\Lambda}\setminus\Lambda_1)}\mu_{\tilde{\Lambda}}({}^y_\Lambda x\times w)}=\frac{\sum_{w\in X(\tilde{\Lambda}\setminus\Lambda_1)}c(\Lambda,\ x_\Lambda,\ {}^y_\Lambda x\times w)}{\sum_{w\in X(\tilde{\Lambda}\setminus\Lambda_1)}c(\Lambda,\ y,\ x\times w)}$$

$$=\frac{c(\Lambda,\ x_\Lambda,\ {}^y_\Lambda x)}{c(\Lambda,\ y,\ x)},\quad x\in X(\Lambda_1),$$

whenever $c(\Lambda, y, x)>0$. Hence (3.19) follows from the last two equations. The proof is terminated.

From Propositions (23) and (28) we know how to construct some quasi-reversible measures for Q from a specification $\mathscr{V}$. But we do not know until now whether these are all such measures. We will construct all quasi-reversible measures for Q in another way, and give the necessary and sufficient conditions for the existence and uniqueness of quasi-reversible measures.

(30) **Definition.** *Let Q be a speed function field, which satisfies* (3), *with finite range. For every $\Lambda\in\mathscr{S}_f$, we will call $\partial\Lambda\triangleq(\bigcup_{A\subset\partial(\Lambda),|A|=N}r(A))\setminus\Lambda$ the boundary of Λ.*

It is clear that $\tilde{\Lambda}=\Lambda\cup\partial\Lambda$ from Remark (5).

We define $\bar{q}(\cdot,\ \cdot)$ according to Remark (5). We choose an arbitrary but fixed $\theta\in X$, then from (7) we have

$$\forall y\in X(\Lambda),\ y\neq\theta_\Lambda,\ \forall z\in X(\partial\Lambda),\ \forall w\in X(S\setminus(\Lambda\cup\partial\Lambda)),$$
$$\bar{q}(\theta_\Lambda\times z\times w,\ y\times z\times w)=\bar{q}(\theta_\Lambda\times z,\ y\times z),$$
$$\bar{q}(y\times z\times w,\ \theta_\Lambda\times z\times w)=\bar{q}(y\times z,\ \theta_\Lambda\times z).$$

(31) **Proposition.** *Let Q be a speed function field, which has a potential and satisfies* (3), *with finite range, and let $\Lambda_m\in\mathscr{S}_f$, $\Lambda_m=\Lambda_{m-1}\cup\partial\Lambda_{m-1}$, $\Lambda_m\neq\Lambda_{m-1}$, $m\geqslant1$, $|\Lambda_0|\geqslant N$ and $\Lambda_m\uparrow S$. If the equations*

$$(32)\quad x_{m,z}=\sum_{w\in X(\partial\Lambda_m)}\frac{\bar{q}(\theta_{\Lambda_m}\times w,\ \theta_{\Lambda_{m-1}}\times z\times w)}{\bar{q}(\theta_{\Lambda_{m-1}}\times z\times w,\ \theta_{\Lambda_m}\times w)}x_{m+1,w},\qquad m\geqslant1,\ z\in X(\partial\Lambda_{m-1})$$

have a positive solution $\bar{x}_{m,z}(m\geqslant1,\ z\in X(\partial\Lambda_{m-1}))$ and

$$(33)\quad z_m\triangleq\sum_{\substack{y'\in X(\Lambda_{m-1})\\ z'\in X(\partial\Lambda_{m-1})}}\frac{\bar{q}(\theta_{\Lambda_{m-1}}\times z',\ y'\times z')}{\bar{q}(y'\times z',\ \theta_{\Lambda_{m-1}}\times z')}\bar{x}_{m,z'}<\infty,$$

where $\bar{q}(x,\ \tilde{x})\setminus\bar{q}(x,\ \tilde{x})=1$ when $x=\tilde{x}$ as a convention, and for every $y\in X(\Lambda_{m-1})$, $z\in X(\partial\Lambda_{m-1})$, we define

$$(34)\quad \mu_m(y\times z)\triangleq z_m^{-1}\frac{\bar{q}(\theta_{\Lambda_{m-1}}\times z,\ y\times z)}{q(y\times z,\ \theta_{\Lambda_{m-1}}\times z)}\bar{x}_{m,z},$$

then $\{\mu_m\}_{m\geqslant1}$ determines a unique quasi-reversible measure for Q, such that each μ_m is the projection of μ on $X(\Lambda_m)$.

Proof Clearly, μ_m defined in (34) is a positive probability measure on $X(\Lambda_m)$. We want to prove that

1° $\{\mu_m\}_{m\geqslant1}$ are consistent. From the potentiality for Q and (3) we have

$$\forall y\in X(\Lambda_{m-1}),\ \forall z\in X(\partial\Lambda_{m-1}),\ \forall w\in X(\partial\Lambda_m),$$
$$\bar{q}(\theta_{\Lambda_m}\times w,\ y\times z\times w)=\bar{q}(\theta_{\Lambda_m}\times w,\ \theta_{\Lambda_{m-1}}\times z\times w)\bar{q}(\theta_{\Lambda_{m-1}}\times z,\ y\times z),$$

$$\bar{q}(y\times z\times w,\ \theta_{\Lambda_m}\times w)=\bar{q}(\theta_{\Lambda_{m-1}}\times z\times w,\ \theta_{\Lambda_m}\times w)\bar{q}(y\times z,\ \theta_{\Lambda_{m-1}}\times z).$$

so form (3) and the fact that $\bar{x}_{m,z}$ satisfies (32), by simple calculation, it is not difficult to obtain

$$Z_{m+1}=Z_m,\quad m\geq 1.$$

From this equation, (34), (35), and the fact that $\bar{x}_{m,z}$ satisfies (32), we obtain

$$\sum_{w\in X(\partial\Lambda_m)}\mu_{m+1}(y\times z\times w)=Z_{m+1}^{-1}\sum_{w\in X(\partial\Lambda_m)}\frac{\bar{q}(\theta_{\Lambda_m}\times w,\ y\times z\times w)}{\bar{q}(y\times z\times w,\ \theta_{\Lambda_m}\times w)}\bar{x}_{m+1,w}$$
$$=Z_m^{-1}\frac{\bar{q}(\theta_{\Lambda_{m-1}}\times z,\ y\times z)}{\bar{q}(y\times z,\ \theta_{\Lambda_{m-1}}\times z)}\bar{x}_{m,z}=\mu_m(y\times z).$$

2° Let $\mathscr{A}_0\triangleq\bigcup_{m=1}^{\infty}\mathscr{F}(\Lambda_m)$, then $\mathscr{A}_0$ is a field and $\sigma(\mathscr{A}_0)=\mathscr{F}$, by Kolmogorov consistency theroem, there is a unique measure on $\mathscr{F}$, denoted by μ, such that μ_m is the projection of μ on $X(\Lambda_m)$.

3° μ is a quasi-reversible measure for Q.

For every $\Lambda\in\mathscr{S}_N$, $\tilde{\Lambda}\supset r(\Lambda)$, since $\Lambda_m\nearrow S$, there is an $m\geq 1$ such that $\Lambda_m\supset\Lambda$, $\Lambda_{m+1}\supset\tilde{\Lambda}$. Let $y\in X(\Lambda)$, $x\in X(\Lambda_{m+1})$. Clearly, (3.19) holds when $c(\Lambda,\ y,x)=c(\Lambda,\ y,\ x_{\tilde{\Lambda}})=0$. Thus we may assume that $\theta_{\Lambda_m}\times x_{\partial\Lambda_m}\leadsto x\to{}^y_\Lambda x$. From (34) and the path-independence, we obtain

$$\frac{\mu_{m+1}(x)}{\mu_{m+1}({}^y_\Lambda x)}=\frac{\bar{q}(\theta_{\Lambda_m}\times x_{\partial\Lambda_m},\ x)}{\bar{q}(x,\ \theta_{\Lambda_m}\times x_{\partial\Lambda_m})}\Big/\frac{\bar{q}(\theta_{\Lambda_m}\times x_{\partial\Lambda_m},\ y\times x_{\Lambda_{m+1}\setminus\Lambda})}{\bar{q}(y\times\theta_{\Lambda_{m+1}\setminus\Lambda},\ \theta_{\Lambda_m}\times x_{\partial\Lambda_m})}$$
$$=\frac{\bar{q}(y\times x_{\Lambda_{m+1}\setminus\Lambda},\ x)}{\bar{q}(x,\ y\times x_{\Lambda_{m+1}\setminus\Lambda})}=\frac{\bar{q}({}^y_\Lambda x_{\tilde{\Lambda}},\ x_{\tilde{\Lambda}})}{\bar{q}(x_{\tilde{\Lambda}},\ {}^y_\Lambda x_{\tilde{\Lambda}})},$$

hence

$$\mu_{m+1}(x)\bar{q}(x_{\tilde{\Lambda}},\ {}^y_\Lambda x_{\tilde{\Lambda}})=\mu_{m+1}({}^y_\Lambda x)\bar{q}({}^y_\Lambda x_{\tilde{\Lambda}},\ x_{\tilde{\Lambda}}).$$

Taking the summation for $x_{\Lambda_{m+1}\setminus\tilde{\Lambda}}$ over $X(\Lambda_{m+1}\setminus\tilde{\Lambda})$, we obtain (3.19). Therefore μ is a quasi-reversible measure for Q.

(36) **Theorem.** *Let Q be a speed function field with finite range, satisfing (3). Suppose that there are $\Lambda_m\in\mathscr{S}_f$ such that $\Lambda_m=\Lambda_{m-1}\cup\partial\Lambda_{m-1}$, $\Lambda_m\neq\Lambda_{m-1}$, $m\geq 1$, $|\Lambda_0|\geq N$ and $\Lambda_m\nearrow S$. Then*

(i) *Every quasi-reversible measure μ for Q can beobtained in the same manner as in Proposition* (31);

(ii) *If there are two positive solutions $\{x^{(i)}_{m,z}\colon m\geq 1,\ z\in X(\partial\Lambda_{m-1})\}$ of* (32) *satisfying* (33), $i=1,2$ *then the two quasi-reversible measures obtained in the preceding manner are the same if and only if there is an $\alpha>0$ such that*

(37) $x^{(1)}_{m,z}=\alpha x^{(2)}_{m,z}$, $m\geq 1$, $z\in X(\partial\Lambda_{m-1})$;

(iii) *A necessary and sufficient condition of the existence of a quasi-reversible measure for Q is that there exists a positive solution of* (32) *satisfying* (33); *a necessacy and sufficient condition of the existence of a unique quasi-reversible measure for Q is tha there is a unique linear-independent positive solution of* (32) *satisfying* (33).

Proof

1° Let μ be an arbitrary quasi-reversible measure for Q, $\tilde{\mu}_m(m\geqslant1)$ be the projection of μ on $X(\Lambda_m)$. Then from (3.19) and (7), we have

$$\tilde{\mu}_{m+1}(\theta_{\Lambda_{m-1}}\times z\times w)=\tilde{\mu}_{m+1}(\theta_{\Lambda_m}\times w)\bar{q}(\theta_{\Lambda_m}\times w,\ \theta_{\Lambda_{m-1}}\times z\times w)/\bar{q}(\theta_{\Lambda_m}\times z\times w,\ \theta_{\Lambda_m}\times w),$$

$$\forall m\geqslant1,\ \forall z\in X(\partial\Lambda_{m-1}),\ \forall w\in X(\partial\Lambda_m).$$

Taking the summation for w over $X(\partial\Lambda_m)$, we obtain

$$\tilde{\mu}_m(\theta_{\Lambda_{m-1}}\times z)=\sum_{w\in X(\partial\Lambda_m)}\frac{\bar{q}(\theta_{\Lambda_m}\times w,\ \theta_{\Lambda_{m-1}}\times z\times w)}{\bar{q}(\theta_{\Lambda_{m-1}}\times z\times w,\ \theta_{\Lambda_m}\times w)}\tilde{\mu}_{m+1}(\theta_{\Lambda_m}\times w),$$

hence $\tilde{x}_{m,z}\triangleq\tilde{\mu}_m(\theta_{\Lambda_{m-1}}\times z)$, $m\geqslant1$, $z\in X(\partial\Lambda_{m-1})$ is a positive solution of equations (32). It is easy to check (33).

Next, from (3), (3.19) and (7), we have

$$\tilde{\mu}_m(y\times z)=\tilde{\mu}_m(\theta_{\Lambda_{m-1}}\times z)\bar{q}(\theta_{\Lambda_{m-1}}\times z,\ y\times z)\backslash\bar{q}(y\times z,\ \theta_{\Lambda_{m-1}}\times z)$$

for every $y\in X(\Lambda_{m-1})$ and $z\in X(\partial\Lambda_{m-1})$. Taking $\tilde{x}_{m,z}$ instead of $\bar{x}_{m,z}$ in (33) and (34), we obtain

$$Z_m=1,\ \mu_m=\tilde{\mu}_m,$$

so μ is the same as the measure obtained from Proposition (31).

2° Let $\{x^{(i)}_{m,z}: m\geqslant1,\ z\in X(\partial\Lambda_{m-1})\}$, $i=1,\ 2$ be two positive solutions of (32) satisying (33). We define $Z^{(i)}_m$, $\mu^{(i)}_m i=1,\ 2,\ m\geqslant1$ as in Proposition (31).

If $x^{(i)}_{m,z}$ satisfy (37), then $Z^{(1)}_m=\alpha Z^{(2)}_m$ from (33), hence $\mu^{(1)}_m=\mu^{(2)}_m$ from (34), so they determine the same quasi-reversible measure for Q from Proposition (31). Conversely, if $x^{(i)}_{m,z}(i=1,\ 2)$ determine the same quasi-reversible measure μ for Q, then $\mu^{(1)}_m=\mu^{(2)}_m$, $m\geqslant1$ since $\mu^{(1)}_m$ and $\mu^{(2)}_m$ are the projection of μ on $X(\Lambda_m)$, In particular, from (34), we have

$$(Z^{(1)}_m)^{-1}x^{(1)}_{m,z}=\mu^{(1)}_m(\theta_{\Lambda_{m-1}}\times z)=\mu^{(2)}_m(\theta_{\Lambda_{m-1}}\times z)=(Z^{(2)}_m)^{-1}x^{(2)}_{m,z}$$

for every $z\in X(\partial\Lambda_{m-1})$. Moreover, we know from the proof of Proposition (31) that $Z^{(i)}_m$ are independent of m, so (37) holds.

Now we can conclude that there is an injective mapping between the set of all quasi-reversible measures for Q and the set of all equivalent classes of linear-independent positive solutions of equations (32) satisfying (33). This completes the proof.

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一般速度函数的有势性与可逆性 (I)

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摘　要

本文提出了一类一般的无穷质点系统的随机演化模型，它包括已有的大多数模型[7,10,15]为其特例，同时也可以认为是对非平衡系统的多元线性 Master 方程的概率模型[2]的推广与一般化.

§2 首先将场论[4]推广到一般状态空间(定理 (2.10))使之作为讨论问题的一个基本工具，然后讨论以无穷乘积空间为态空间的场的局部化(定理(2.14)). §3 引入有限程速度函数场(定义(3.15))和拟可逆测度(定义 (3.17))作为离散化的条件，并证明了拟可逆是可逆性的外延(定理(3.25)). §4 研究有限程速度函数的有势性与可逆性之间的关系，证明了拟可逆必有势(定理(4.1)). 反之，在速度函数有势且满足(4.3)与 (4.10) 的条件下，证明了关于规范$\mathscr{V}$的 Gibbs 态集 $\mathscr{G}(\mathscr{V})\supset\bar{\mathscr{G}}$ (命题 (4.23))且 $\mathscr{G}(\mathscr{V})$ 的每一元都是拟可逆测度(命题 (4.28))，其中$\bar{\mathscr{G}}$是由$\mathscr{V}$出发构造的测度的一切弱极限作成的集(定义 (4.19)). 给出了构造一切拟可逆测度的一种办法. 由此得出了拟可逆测度存在及唯一的充要条件(定理(4.36)).