Chin. Ann. of Math.
3 (6) 1982

THE DISTORTION THEOREMS FOR BIEBERBACH CLASS AND GRUNSKY CLASS OF UNIVALENT FUNCTIONS

REN FUYAO
(*Fudan University*)

§ 1. Introduction and Notations

Let B_1 denote the class of all of the functions $f(\zeta)$, analytic and univalent in the unit disk $|\zeta|<1$, such that $f(0)=0$ and $f(\zeta_1)f(\zeta_2)\neq 1$ for any ζ_1 and ζ_2 in $|\zeta|<1$. These are the functions of the so-called Bieberbach or Bieberbach-Eilenberg class. Let G_1 denote the class of all of the functions $f(\zeta)$, analytic and univalent in the unit disk $|\zeta|<1$, such that $f(0)=0$ and $f(\zeta_1)\overline{f(\zeta_2)}\neq -1$ for any ζ_1 and ζ_2 in $|\zeta|<1$, these are the functions of the so-called Grunsky class.

In this paper, by applying the area principle of Tao-Shing Shah[1], we obtained another necessary and sufficient condition, and proved some distortion theorems for these classes, particularly exponentiated Golusin inequalities and Fitz Gerald inequalities.

For the convenience of assertion, we introduce some notations. Let $f(\zeta)$ be the analytic and univalent function in $|\zeta|<1$ and $f(0)=0$. We define two functions of ζ and z in the unit disk by

$$f(\zeta,\ z)=f'(0)(f(\zeta)-f(z))\zeta z/(\zeta-z)f(\zeta)f(z)$$

and by

$$F(\zeta,\ z)=\ln f(\zeta,\ z),$$

taking the single-valued branch of the function which vanishes at $\zeta=0$. Since $f(\zeta)$ is univalent in the disk, these functions are regular in the unit disk, and hence the following expansion is valid in the unit disk

$$F(\zeta,\ z)=\sum_{m,n=1}^{\infty} d_{mn}\zeta^m z^n.$$

Similarly, if $f\in B_1$, we define

$$\varphi(\zeta,\ z)=1-f(\zeta)f(z)$$

and

$$\Phi(\zeta,\ z)=\ln\varphi(\zeta,\ z)=\sum_{m,n=1}^{\infty} C_{mn}\zeta^m z^n,$$

Manuscipt recived Dec. 8, 1980.

taking the single-valued branch of the function which vanishes at $\zeta=0$. Let $f\in G_1$, we define

$$\psi(\zeta, z)=1+f(\zeta)\overline{f(\bar{z})}$$

and

$$\Psi(\zeta, z)=\ln\psi(\zeta, z)=\sum_{m,n=1}^{\infty} e_{mn}\zeta^m z^n$$

which vanishes at $\zeta=0$.

Moreover, we define also

$$D_n(\zeta)=\sum_{m=1}^{\infty} d_{mn}\zeta^m, \qquad C_n(\zeta)=\sum_{m=1}^{\infty} C_{mn}\zeta^m, \qquad E_n(\zeta)=\sum_{m=1}^{\infty} e_{mn}\zeta^m,$$

$$D_n^{(p)}(\zeta)=\frac{d^p}{d\zeta^p}D_n(\zeta),\ C_n^{(p)}(\zeta)=\frac{d^p}{d\zeta^p}C_n(\zeta),\ E_n^{(p)}(\zeta)=\frac{d^p}{d\zeta^p}E_n(\zeta),$$

and

$$K(\zeta, \bar{z})=-\ln(1-\zeta\bar{z}), \qquad K^{(p,q)}(\zeta, z)=\frac{\partial^{p+q}}{\partial\zeta^p\partial\bar{z}^q}K(\zeta, \bar{z}),$$

where p and q are arbitrary nonnegative integers.

§ 2. On The Bieberbach Function

Theorem 1. *Let $f\in B_1$, $\{x_n\}$, $\{x'_n\}$, $\{y_n\}$ and $\{y'_n\}$ be arbitrary sequences of complex numbers such that*

$$X=\sum_{n=1}^{\infty}|x_n|^2/n<\infty, \qquad X'=\sum_{n=1}^{\infty}|x'_n|^2/n<\infty,$$
$$Y=\sum_{n=1}^{\infty}|y_n|^2/n<\infty, \qquad Y'=\sum_{n=1}^{\infty}|y'_n|^2/n<\infty. \tag{2.1}$$

Then we have

$$\left|\sum_{m,n=1}^{\infty}(C_{mn}x_m+d_{mn}y_m)x'_n\right|^2+\left|\sum_{m,n=1}^{\infty}(d_{mn}x_m+C_{mn}y_m)x'_n\right|^2$$
$$+\left|\sum_{m,n=1}^{\infty}(C_{mn}x_m+d_{mn}y_m)y'_n\right|^2+\left|\sum_{m,n=1}^{\infty}(d_{mn}x_m+C_{mn}y_m)y'_n\right|^2$$
$$\leqslant(X+Y)(X'+Y'). \tag{2.2}$$

Here the equality holds if and if for any positive integer n,

$$\sum_{m=1}^{\infty}(C_{mn}x_m+d_{mn}y_m)=\lambda_{11}\bar{x}'_n/n,$$
$$\sum_{m=1}^{\infty}(d_{mn}x_m+C_{mn}y_m)=\lambda_{12}\bar{x}'_n/n,$$
$$\sum_{m=1}^{\infty}(C_{mn}x_m+d_{mn}y_m)=\lambda_{21}\bar{y}'_n/n,$$
$$\sum_{m=1}^{\infty}(d_{mn}x_m+C_{mn}y_m)=\lambda_{22}\bar{y}'_n/n, \tag{2.3}$$

Where $\lambda_{ij}(i, j=1, 2)$ are constants such that

$$(|\lambda_{11}|^2+|\lambda_{12}|^2)X^2+(|\lambda_{21}|^2+|\lambda_{22}|^2)Y^2=(Y+X)(Y'+X'). \tag{2.4}$$

Proof By[1], since $f\in B_1$, for any $\{x_n\}$ and $\{y_n\}$ with

$$X=\sum_{n=1}^{\infty}|x_n|^2/n<\infty,\quad Y=\sum_{n=1}^{\infty}|y_n|^2/n<\infty$$

we have

$$\sum_{n=1}^{\infty}n\left\{\left|\sum_{m=1}^{\infty}(C_{mn}x_m+d_{mn}y_m)\right|^2+\left|\sum_{m=1}^{\infty}(d_{mn}x_m+C_{mn}y_m)\right|^2\leqslant X+Y,\right. \tag{2.5}$$

Here the equality holds if and only if the area of the complement of the union of the image $f(|\zeta|<1)$ and the image $1/f(|\zeta|<1)$ vanishes.

By Cauchy inequality and using (2.5), we get

the left-hand side of (2.2)

$$\leqslant(X'+Y')\sum_{n=1}^{\infty}n\left(\left|\sum_{m=1}^{\infty}(C_{mn}x_m+d_{mn}y_m)\right|^2+\left|\sum_{m=1}^{\infty}(d_{mn}x_m+C_{mn}y_m)\right|^2\right)$$
$$\leqslant(X'+Y')(X+Y). \tag{2.6}$$

By the necessary and sufficient condition for which the equality holds of Cauchy inequality, it follows that the equality holds for the first inequality of (2.6) if and only if (2.3) is valid. In this case, the left-hand side of (2.2) equals to

$$(|\lambda_{11}|^2+|\lambda_{12}|^2)(X')^2+(|\lambda_{21}|^2+|\lambda_{22}|^2)(Y').$$

Since the necessary and sufficient condition for which the equality holds for second inequality of (2.6), is that the area complementary to the union of image $f(|\zeta|<1)$ and image 1/f $(|\zeta|<1)$ vanishes, it follows that (2.4) is valid. This completes the proof.

Corollary 1. *Let $f\in B_1$, sequences $\{x_n\}$ and $\{x'_n\}$ satisfy the condition (2.1). We have*

$$\left|\sum_{m,n=1}^{\infty}d_{mn}x_m x'_n\right|^2+\left|\sum_{m,n=1}^{\infty}C_{mn}x_m x'_n\right|^2\leqslant X\cdot X', \tag{2.7}$$

where $X=\sum_{n=1}^{\infty}|x_n|^2/n$, $X'=\sum_{n=1}^{\infty}|x'_n|^2/n$. *The equality holds if and only if for all* $n\geqslant1$

$$\begin{cases}C_{mn}x_m=\lambda_1\bar{x}'_n/n,\\ d_{mn}x_m=\lambda_2\bar{x}'_n/n,\end{cases} \tag{2.8}$$

where λ_1 and λ_2 are constants such that

$$|\lambda_1|^2+|\lambda_2|^2=X/X'. \tag{2.9}$$

Theorem 2. *Let $f\in B_1, p, q$ be any nonnegative integers, and let $\zeta_1,\cdots,\zeta_N$; $\zeta'_1,\cdots,\zeta'_{N'}$ be two systems of distinguished points in $|\zeta|<1$. Then for any nonzero complex numbers $\{\eta_\mu\}$ and $\{\eta'_\nu\}$ $(\mu=1,\cdots,N;\nu=1,\cdots,N')$, we have*

1)

$$\sum_{n=1}^{\infty}n\left\{\left|\sum_{\mu=1}^{N}\eta_\mu C_n^{(p)}(\zeta_\mu)+\sum_{\nu=1}^{N'}\eta'_\nu D_n^{(q)}(\zeta'_\nu)\right|^2+\left|\sum_{\mu=1}^{N}\eta_\mu D^{(p)}(\zeta_\mu)+\sum_{\nu=1}^{N'}\eta'_\nu C_n^{(q)}(\zeta'_\nu)\right|^2\right\}$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu K^{(p,p)}(\zeta_\mu,\bar{\zeta}_\nu)+\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar{\eta}'_\nu K^{(q,q)}(\zeta'_\mu,\bar{\zeta}'_\nu), \tag{2.10}$$

the equality holds if and only if the area of the complement of the union of image $f(|\zeta|<1)$ and image $1/f(|\zeta|<1)$ to be vanishes,

2)
$$\left|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu F^{(p,q)}(\zeta_\mu,\zeta'_\nu)\right|^2+\left|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu \Phi^{(p,q)}(\zeta_\mu,\zeta'_\nu)\right|^2$$
$$\leqslant \sum_{\mu,\nu=1}^{N}\eta_\mu\bar\eta_\nu K^{(p,p)}(\zeta_\mu,\bar\zeta_\nu)+\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar\eta'_\nu K^{(q,q)}(\zeta'_\mu,\bar\zeta'_\nu), \tag{2.11}$$

The equality holds only if
$$\begin{cases}\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu \Phi^{(p,q)}(\zeta_\mu,\zeta'_\nu)=\lambda_1\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar\eta'_\nu K^{(q,p)}(\zeta'_\mu,\bar\zeta'_\nu),\\ \sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu F^{(p,q)}(\zeta_\mu,\zeta'_\nu)=\lambda_2\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar\eta'_\nu K^{(q,p)}(\zeta'_\mu,\bar\zeta'_\nu),\end{cases} \tag{2.12}$$

where λ_1 and λ_2 are constants such that
$$|\lambda_1|^2+|\lambda_2|^2=\sum_{\mu,\nu=1}^{N}\eta_\mu\bar\eta_\nu K^{(p,p)}(\zeta_\mu,\bar\zeta_\nu)\Big/\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar\eta'_\nu K^{(q,q)}(\zeta'_\mu,\zeta'_\nu) \tag{2.13}$$

To prove these, we set
$$x_n=\sum_{\mu=1}^{N}\eta_\mu\frac{\partial^p}{\partial\zeta_\mu^p}\zeta_\mu^n=n(n-1)\cdots(n-p+1)\sum_{\mu=1}^{N}\eta_\mu\zeta_\mu^{n-p},\ n\geqslant p,$$
$$x'_n=y_n=\sum_{\nu=1}^{N'}\eta'_\nu\frac{\partial^q}{\partial\zeta_\nu^q}\zeta_\nu'^n=n(n=1)\cdots(n-q+1)\sum_{\nu=1}^{N'}\eta'_\nu\zeta_\nu'^{n-q},\ q\geqslant n.$$

It is obviously that
$$X=\sum_{n=1}^{\infty}|x_n|^2/n=\sum_{\mu,\nu=1}^{N}\eta_\mu\bar\eta_\nu K^{(p,p)}(\zeta_\mu,\bar\zeta_\nu)<\infty,$$
$$X'=Y=\sum_{n=1}^{\infty}|x'_n|^2/n=\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar\eta'_\nu K^{(q,q)}(\zeta'_\mu,\bar\zeta'_\nu)<\infty.$$

Therefore, (2.10) follows from inequality (2.5) and the equality holds if and only if the area of the complement of the union of image $f(|\zeta|<1)$ and image $1/f(|\zeta|<1)$ to be vanishes. From (2.7), (2.8) and (2.9), we get the second part of the theorem at once. This completes the proof.

This theorem has several corollaries.

Corollary 1. *Let $f\in B_1$, $\{\zeta_\mu\}$ and $\{\zeta'_\nu\}$ be the two systems of distinguished points in the unit disk $|\zeta|<1$, $\mu=1,\cdots,N$; $\nu=1,\cdots,N'$, then we have*
$$\left|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu\left(\frac{f'(\zeta_\mu)f'(\zeta'_\nu)}{(f(\zeta_\mu)-f(\zeta'_\nu))^2}-\frac{1}{(\zeta_\mu-\zeta'_\nu)^2}\right)\right|^2+\left|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu\frac{f'(\zeta_\mu)f'(\zeta'_\nu)}{(1-f(\zeta_\mu)f(\zeta'_\nu))^2}\right|^2$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\eta_\mu\bar\eta_\nu(1-\zeta_\mu\bar\zeta_\nu)^{-2}\cdot\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar\eta'_\nu(1-\zeta'_\mu\bar\zeta'_\nu)^{-2}. \tag{2.14}$$

Here the equality holds only if
$$\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu\left(\frac{f'(\zeta_\mu)f'(\zeta'_\nu)}{(f(\zeta_\mu)-f(\zeta'_\nu))^2}-\frac{1}{(\zeta_\mu-\zeta'_\nu)^2}\right)=\lambda_1\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar\eta'_\nu(1-\zeta'_\mu\bar\zeta'_\nu)^{-2},$$
$$\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu\left(\frac{f'(\zeta_\mu)f'(\zeta'_\nu)}{(1-f(\zeta_\mu)f(\zeta'_\nu))^2}\right)=\lambda_2\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar\eta'_\nu(1-\zeta'_\mu\bar\zeta_\nu)^{-2},$$

where λ_1, λ_2 are constants such that
$$|\lambda_1|^2+|\lambda_2|^2=\sum_{\mu,\nu=1}^{N}\eta_\mu\bar\zeta_\nu(1-\zeta_\mu\bar\zeta_\nu)^{-2}\Big/\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar\eta'_\nu(1-\zeta'_\mu\bar\zeta'_\nu)^{-2}.$$

Corollary 2. *Suppose $f\in B_1$, ζ is any point in the unit disk, then we have*

$$|\{f,\zeta\}|^2+36|f'(\zeta)/(1-f(\zeta)^2)|^4\leqslant 36(1-|\zeta|^2)^{-4}, \tag{2.15}$$

where $\{f,\zeta\}=(f''(\zeta)/f'(\zeta))'-\frac{1}{2}(f''(\zeta)/f'(\zeta))^2$ *denotes the Schwarz derivative of the function* f *at* ζ. *the equality holds only if* f *satisfies the following equation*

$$\{f,\zeta\}=6\lambda_1(f'(\zeta))^2/\lambda_2(1-f(\zeta)^2),$$

where λ_1, λ_2 *are constants such that*

$$|\lambda_1|^2+|\lambda_2|^2=1.$$

We know from [8] that

$$\{f,\zeta\}=6\frac{\partial^2}{\partial\zeta\partial z}\left(\ln\frac{f(\zeta)-f(z)}{\zeta-z}\right)\Big|_{\zeta=z}=\left(\frac{f''(\zeta)}{f'(\zeta)}\right)'-\frac{1}{2}\left(\frac{f''(\zeta)}{f'(\zeta)}\right)^2.$$

Set $N'=N=1$, $\eta_1'=\eta_1=\sqrt{6}$, $\zeta_1=\zeta$, $\zeta_1'=z$ in above corollary and let $z\to\zeta$, we obtain this corollary at once.

Theorem 3. *Suppose* f *is regular in the unit disk,* $f(0)=0$, $f'(0)\neq 0$. *Then the necessary and sufficient condition for* $f\in B_1$ *is that*

$$\frac{1}{\pi}\iint\limits_{|z|<1}\left|\frac{f'(\zeta)f'(z)}{(f(\zeta)-f(z))^2}-\frac{1}{(\zeta-z)^2}\right|^2d\sigma_z+\frac{1}{\pi}\iint\limits_{|z|<1}\left|\frac{f'(\zeta)f'(z)}{(1-f(\zeta)f(z))^2}\right|^2d\sigma_z \leqslant(1-|\zeta|^2)^{-2} \tag{2.16}$$

holds for any ζ *in the unit disk.*

Moreover, if $f\in B_1$, *the equality of* (2.16) *holds if and only if the area of the complent of the union of the image* $f(|\zeta|<1)$ *and the image* $1/f(|\zeta|<1)$ *vanishes.*

Now we prove the necessity. Assume $f\in B_1$. Since

$$\ln f(\zeta,z)=\sum_{m,n=1}^{\infty}d_{mn}\zeta^m z^n=\sum_{n=1}^{\infty}D_n(\zeta)z^n,$$

$$\ln\varphi(\zeta,z)=\sum_{m,n=1}^{\infty}C_{mn}\zeta^m z^n=\sum_{n=1}^{\infty}C_n(\zeta)z^n,$$

taking the differentiation on both sides of these last expressions for z, and ζ, then we get

$$\frac{f'(\zeta)f'(z)}{(f(\zeta)-f(z))^2}-\frac{1}{(\zeta-z)^2}=\sum_{n=1}^{\infty}nD_n'(\zeta)z^{n-1},$$

$$\frac{f'(\zeta)f'(z)}{(1-f(\zeta)f(z))^2}=\sum_{n=1}^{\infty}nC_n'(\zeta)z^n.$$

Therefore, by virtue of

$$\frac{1}{\pi}\iint\limits_{|z|<1}z^n\bar{z}^{n'}d\sigma_z=\begin{cases}0, & n'\neq n,\\ 1/(n+1), & n'=n,\end{cases} \tag{2.17}$$

for $n, n'=1, 2, \cdots$, we obtain that the left-hand side of (2.16) equals to

$$\sum_{n=1}^{\infty}n(|D_n'(\zeta)|^2+|C_n'(\zeta)|^2).$$

Applying Theorem 2, we can conclude that the inequality is true.

Now we show the sufficiency of the condition (2.16). Assuming condition (2.16) to be satisfied, we first show that $f'(\zeta)$ doesn't vanishinq $|\zeta|<1$. Otherwise, there exists at least a point ζ_0 in $|\zeta|<1$. such that $f'(\zeta_0)\neq 0$. It follows from (2.16) that

$$\frac{1}{\pi}\iint_{|z|<1}|\zeta_0-z|^{-4}d\sigma_z\leqslant(1-|\zeta_0|^2)^2.$$

But this is impossible, because for any sufficiently small positive number we have

$$\frac{1}{\pi}\iint_{|z|<1}|z-\zeta_0|^{-4}d\sigma_z\geqslant\frac{1}{\pi}\iint_{|z-\zeta_0|\leqslant\eta}|\zeta_0-z|^{-4}d\sigma_z>\frac{1}{\eta^2}.$$

Letting $\eta\to 0$ in the last inequality, we get

$$\frac{1}{\pi}\iint_{|z|<1}|\zeta_0-z|^{-4}d\sigma_z\geqslant\infty.$$

So $f'(\zeta)\neq 0$ for any ζ in $|\zeta|<1$.

Secondly we show the univalence of $f(\zeta)$ in $|\zeta|<1$. If it isn't true, then there exist at least ζ_0 and z_0 in the unit disk such that

$$f(\zeta_0)=f(z_0),\ f'(\zeta_0)\neq 0,\ f'(z_0)\neq 0.$$

Therefore, in some neighborhood at z_0, we have

$$f(z)=f(\zeta_0)+f'(z_0)(z-z_0)+O(|z-z_0|^2).$$

In view of this last expression, for any sufficiently small positive number η, we conclude that

$$\frac{1}{\pi}\iint_{|z|<1}\left|\frac{f'(\zeta_0)f'(z)}{(f(\zeta)-f(\zeta_0))^2}-\frac{1}{(\zeta-z)^2}\right|^2d\sigma_z$$

$$>\frac{1}{\pi}\iint_{|z-z_0|<\eta}\left|\frac{f'(\zeta_0)(f'(z_0)+O(|z-z_0|^2))}{(z-z_0)^2(f'(z_0)+O(|z-z_0|))^2}-\frac{1}{(\zeta_0-z)^2}\right|^2d\sigma_z>c/\eta^2,$$

where c is a nonzero constant. Letting $\eta\to+0$, from the last inequality and (2.16), we get $(1-|\zeta_0|^2)^{-2}\geqslant+\infty$. This is also impossible. Hence $f(\zeta)$ is univalent in $|\zeta|<1$.

Finnally, we show $f(\zeta_1)f(\zeta_2)\neq 1$ for any ζ_1 and ζ_2 in $|\zeta|<1$. To do this, we assume that there are ζ_0 and z_0 in the unit disk such that $f(\zeta_0).\ f(z_0)=1$. Then it follows from (2.16) that

$$\frac{1}{\pi}\iint_{|z|<1}\left|\frac{f'(\zeta_0)f'(z)}{(1-f(z)f(\zeta_0))^2}\right|^2d\sigma_z\leqslant(1-|\zeta_0|^2)^{-2}.$$

But this is impossible, as proved above. Thus we complete the proof of the sufficiency. Therefore, the assertion of the theorem is true.

Remark For Bieberbach-Eilenberg class B_1, we know the inequalities (2.5) are also the sufficient conditions, but that sufficient conditions is too difficult to check. The condition (2.16) is easier to check than (2.5). On the other hand, Theorem 2 really is the generlization of Bazilevic theorem too.

Theorem 4. *Let $f\in B_1$, both $\{\zeta_\mu\}$ $(=1, \cdots, N)$ and $\{\zeta'_\nu\}$ $(\nu=1, \cdots, N')$ be the distinguished points in $|\zeta|<1$, and l be any nonnegative integer. Then we have*

$$\left|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu(F(\zeta_\mu,\zeta'_\nu))^l\right|^2+\left|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu(\Phi(\zeta_\mu,\zeta'_\nu))^l\right|^2$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu(K(\zeta_\mu,\bar{\zeta}_\nu))^l\cdot\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar{\eta}'_\nu(K(\zeta'_\mu,\bar{\zeta}'_\nu))^l, \tag{2.18}$$

for any nonzero complex numbers $\{\eta_\mu\}$ *and* $\{\eta'_\nu\}$.

To prove this, we will proceed, as in[2, 3]. We now set

$$x_n(\mu)=\eta_\mu^{\frac{1}{l}}\zeta_\mu^n,\quad x'_n(\nu)=\eta_\nu'^{\frac{1}{l}}\zeta_\nu'^n$$

for all $n\geqslant1$, then the left-hand side of(2.18)equals to

$$I_1=\left|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\left(\sum_{m,n=1}^{\infty}d_{mn}x'_m(\nu)x_n(\mu)\right)^l\right|^2+\left|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\left(\sum_{m,n=1}^{\infty}C_{mn}x'_m(\nu)x_n(\mu)\right)^l\right|^2.$$

By Cauchy inequality once more, we get

$$I_1=\left|\sum_{n_1}\cdots\sum_{n_l}\left(\sum_{\mu=1}^{N}\prod_{j=1}^{l}x_{n_j}(\mu)\right)\times\sum_{m_1}\cdots\sum_{m_l}\left(\sum_{\nu=1}^{N'}\prod_{j=1}^{l}d_{m_jn_j}x'_{m_j}(\nu)\right)\right|^2$$
$$+\left|\sum_{n_1}\cdots\sum_{n_l}\left(\sum_{\mu=1}^{N}\prod_{j=1}^{l}x_{n_j}(\mu)\right)\times\sum_{m_1}\cdots\sum_{m_l}\left(\sum_{\nu=1}^{N'}\prod_{j=1}^{l}C_{m_jn_j}x'_{m_j}(\nu)\right)\right|^2$$
$$\leqslant\left(\sum_{n_1}\cdots\sum_{n_l}\prod_{j=1}^{l}\frac{1}{n_j}\left|\sum_{\mu=1}^{N}\prod_{j=1}^{l}x_{n_j}(\mu)\right|^2\right)\times I_2,$$

where

$$I_2=\sum_{n_1}\cdots\sum_{n_l}\prod_{j=1}^{l}n_j\left|\sum_{m_1}\cdots\sum_{m_l}\sum_{\nu=1}^{N'}\prod_{j=1}^{l}d_{m_jn_j}x'_{m_j}(\nu)\right|^2$$
$$+\sum_{n_1}\cdots\sum_{n_l}\prod_{j=1}^{l}n_j\left|\sum_{m_1}\cdots\sum_{m_l}\sum_{\nu=1}^{N'}\prod_{j=1}^{l}C_{m_jn_j}x_{m_j}(\nu)\right|^2.$$

Using the following Shah inequalities once more

$$\sum_{n=1}^{\infty}n\left(\left|\sum_{m=1}^{\infty}d_{mn}y_m\right|^2+\sum_{m=1}^{\infty}C_{mn}y_m\right|^2\Big)\leqslant\sum_{n=1}^{\infty}|y_n|^2/n,$$

it follows that

$$I_2=\prod_{j=1}^{l-1}\sum_{n_j}n_j\sum_{n_l}n_l\left(\left|\sum_{m_l}d_{m_ln_l}\left(\sum_{m_1}\cdots\sum_{m_{l-1}}\sum_{\nu=1}^{N'}\prod_{j=1}^{l-1}d_{m_jn_j}x'_{m_j}(\nu)\right)x'_{m_l}(\nu)\right|^2\right.$$
$$\left.+\left(\left|\sum_{m_l}C_{m_ln_l}\left(\sum_{m_1}\cdots\sum_{m_{l-1}}\sum_{\nu=1}^{N'}\prod_{j=1}^{l-1}C_{m_jn_j}x'_{m_j}(\nu)\right)x'_{m_l}(\nu)\right|^2\right)\right.$$
$$\leqslant\prod_{j=1}^{l-1}\sum_{n_j}n_j\sum_{n_l}\frac{1}{n_l}\left(\left|\sum_{m_1}\cdots\sum_{m_{l-1}}\sum_{\nu=1}^{N'}\prod_{j=1}^{l-1}d_{m_jn_j}x'_{m_j}(\nu)x'_{n_l}(\nu)\right|^2\right.$$
$$\left.+\left|\sum_{m_1}\cdots\sum_{m_{l-1}}\sum_{\nu=1}^{N'}\prod_{j=1}^{l-1}C_{m_jn_j}x'_{m_j}(\nu)x'_{n_l}(\nu)\right|^2\right)$$
$$\leqslant\prod_{j=2}^{l}\sum_{n_j}\frac{1}{n_j}\sum_{n_1}n_1\left(\left|\sum_{m_1}d_{m_1n_1}\left(\sum_{\nu=1}^{N'}\prod_{j=1}^{l}x'_{n_j}(\nu)\right)\right|^2+\left|\sum_{m_1}C_{m_1n_1}\left(\sum_{\nu=1}^{N'}\prod_{j=1}^{l}x'_{n_j}(\nu)\right)\right|^2\right)$$
$$\leqslant\prod_{j=1}^{l}\sum_{n_j}\frac{1}{n_j}\left|\sum_{\nu=1}^{N'}x'_{n_j}(\nu)\right|^2$$

Therefore, we obtain the left-hand side of (2.18)

$$I_1\leqslant\left(\sum_{n_1}\cdots\sum_{n_l}\prod_{j=1}^{l}\frac{1}{n_j}\left|\sum_{\mu=1}^{N}\prod_{j=1}^{l}x_{n_j}(\mu)\right|^2\right)\times\left(\sum_{n_1}\cdots\sum_{n_l}\prod_{j=1}^{l}\frac{1}{n_j}\left|\sum_{\nu=1}^{N'}\prod_{j=1}^{l}x'_{n_j}(\nu)\right|^2\right).$$

Since

$$\sum_{n_1}\cdots\sum_{n_l}\prod_{j=1}^{l}\frac{1}{n_j}\left|\sum_{\mu=1}^{N}\prod_{j=1}^{l}x_{m_j}(\mu)\right|^2=\sum_{\mu,\nu=1}^{N}\left(\sum_{n=1}^{\infty}\frac{1}{n}x_n(\mu)\overline{x_n(\nu)}\right)^l$$
$$=\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu(K(\zeta_\mu,\bar{\zeta}_\nu))^l,$$

and similarly

$$\sum_{n_1}\cdots\sum_{n_l}\prod_{j=1}^{l}\frac{1}{n_j}\left|\sum_{\nu=1}^{N'}\prod_{j=1}^{l}x'_{n_j}(\nu)\right|^2=\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar{\eta}'_\nu(K(\zeta'_\mu,\bar{\zeta}'_\nu))^l,$$

we get (2.18) at once, when substituting these two relations into the last inequality. This completes the proof.

From this theorem, we obtain immediately the following:

Corollary 1. *Let $f\in B_1$, l be any nonnegative real number, and $\{\zeta_\mu\}$ $(\mu=1,\cdots,N)$ be an arbitrary system of distinguished points in $|\zeta|<1$. We have*

$$\left|\sum_{\mu,\nu=1}^{N}\eta_\mu\eta_\nu(F(\zeta_\mu,\zeta_\nu))^l\right|^2+\left|\sum_{\mu,\nu=1}^{N}\eta_\mu\eta_\nu(\Phi(\zeta_\mu,\zeta_\nu)^l\right|^2$$
$$\leqslant\left(\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu(K(\zeta_\mu,\bar{\zeta}_\nu))^l\right)^2 \tag{2.19}$$

for any complex constants $\{\eta_\mu\}$.

In particular, when $l=1$, inequality (2.19) reduces to the inequality (3.5) of [1].

Corollary 2. *Let $f\in B_1$, p be any complex number, and $\{\zeta_\mu\}$ $(\mu=1,\cdots,N)$ be distinguished points. Then for any complex numbers $\{\eta_\mu\}$, we have*

$$\left|\sum_{\mu,\nu=1}^{N}\eta_\mu\eta_\nu(f(\zeta_\mu,\zeta_\nu))^p\right|\leqslant\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu(1-\zeta_\mu\bar{\zeta}_\nu)^{-|p|}, \tag{2.20}$$

$$\left|\sum_{\mu,\nu=1}^{N}\eta_\mu\eta_\nu(\varphi(\zeta_\mu,\zeta_\nu))^p\right|\leqslant\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu(1-\zeta_\mu\bar{\zeta}_\nu)^{-|p|}. \tag{2.21}$$

Proof By virtue of Tayler expansion of exponential function and using (2.19), it follows that

$$\left|\sum_{\mu,\nu=1}^{N}\eta_\mu\eta_\nu(f(\zeta_\mu,\zeta_\nu))^p\right|\leqslant\sum_{l=0}^{\infty}\frac{1}{l!}|p|^l\left|\sum_{\mu,\nu=1}^{N}\eta_\mu\eta_\nu(F(\zeta_\mu,\zeta_\nu))^l\right|$$
$$\leqslant\sum_{l=0}^{\infty}\frac{1}{l!}|p|^l\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu(K(\zeta_\mu,\bar{\zeta}_\nu))^l=\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu(1-\zeta_\mu\bar{\zeta}_\nu)^{-|p|}.$$

Similarly, we get (2.21).

Theorem 5. *Let $f\in B_1$, l be nonnegative integer, and both $\{\zeta_\mu\}$ $(\mu=1,2,\cdots,N)$ and $\{\zeta'_\nu\}$ $(\nu=1,2,\cdots,N')$ be arbitrary systems of distinguished points in $|\zeta|<1$. Then*

$$\left|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu(\ln f(\zeta_\mu,\zeta'_\nu)(\varphi(\zeta_\mu,\zeta'_\nu))^\varepsilon)^l\right|$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu(K(\zeta_\mu,\bar{\zeta}_\nu))^l\cdot\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar{\eta}'_\nu(K(\zeta'_\mu,\bar{\zeta}'_\nu))^l \tag{2.22}$$

holds for any nonzero constants $\{\eta_\mu\}$ and $\{\eta'_\nu\}$, $\varepsilon=1,-1$.

In particular, while $l=1$, this theorem reduce to the corresponding inequality of [1].

Proof We first see that the left-hand side of (2.22) equals to

$$I=\left|\sum_{\mu,\nu=1}^{N}\eta_\mu\eta'_\nu\left(\sum_{m,n=1}^{\infty}(d_{mn}+\varepsilon C_{mn})\zeta_\nu'^m\zeta_\mu^n\right)^l\right|.$$

By means of the same method as in above Theorem 4, using Cauchy inequality again and again and applying the following Shah inequality

$$\sum_{n=1}^{\infty}n\left|\sum_{m=1}^{\infty}(d_{mn}+\varepsilon C_{mn})x'_m\right|^2\leqslant\sum_{n=1}^{\infty}|x'_n|^2/n,$$

we can obtain

$$I\leqslant\sum_{\mu,\nu=1}^{N}\eta_\mu\eta_\nu(K(\zeta_\mu,\bar\zeta_\nu))^l\cdot\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar\eta'_\nu(K(\zeta'_\mu,\bar\zeta'_\nu))^l.$$

This completes the proof.

Corollary 1. *Suppose $f\in B_1$, $\varepsilon=1,\ -1$, and p be any complex number, then*

$$\left|\sum_{\mu,\nu=1}^{N}\eta_\mu\eta_\nu(f(\zeta_\mu,\ \zeta_\nu)(\varphi(\zeta_\mu,\ \zeta_\nu))^\varepsilon)^p\right|\leqslant\sum_{\mu,\nu=1}^{N}\eta_\mu\bar\eta_\nu(1-\zeta_\mu\bar\zeta_\nu)^{-|p|}\qquad(2.23)$$

for any nonzero complex constants $\{\eta_\mu\}$.

Theorem 6. *Let $f\in B_1$, l be any nonnegative integer and $\{\zeta_\mu\}$ $(\mu=1,\ \cdots,\ N)$ be a system of distinguished points in $|\zeta|<1$. Let $P_n(t)$ denote the n-th Faber polynomials generated by the function f, $\varepsilon=1,\ -1$, and*

$$g_n^{(\varepsilon)}(\zeta)=P_n(1/f(\zeta))-(\zeta^{-n}+\varepsilon\bar\zeta^n),$$

$$h_n(\zeta)=P_n(f(\zeta))+2n\gamma_n,\quad \ln(f(\zeta)/\zeta)=-2\sum_{n=1}^{\infty}\gamma_n\zeta^n.$$

Then for any nonzero complex constants $\{\eta_\mu\}$ we have

$$\frac{l}{2}\sum_{n=1}^{\infty}\left(\left|\sum_{\mu=1}^{N}\eta_\mu g_n^{(\varepsilon)}(\zeta_\mu)\right|^2+\left|\sum_{\mu=1}^{N}\eta_\mu h_n(\zeta_\mu)\right|^2\right)\Big/n$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\eta_\mu\bar\eta_\nu\ln(|f(\zeta_\mu,\ \zeta_\nu)|^{\varepsilon l}/|1-\zeta_\mu\bar\zeta_\nu|^l);\qquad(2.24)$$

$$\frac{l}{2}\sum_{n=1}^{\infty}\left|\sum_{\mu=1}^{N}(g_n^{(\varepsilon)}(\zeta_\mu)+\varepsilon' h_n(\zeta_\mu))\right|^2\Big/n$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\eta_\mu\bar\eta_\nu\ln(|f(\zeta_\mu,\ \zeta_\nu)(\varphi(\zeta_\mu,\ \zeta_\nu))^{\varepsilon'}|^{\varepsilon l}/|1-\zeta_\mu\bar\zeta_\nu|^l).\qquad(2.25)$$

Here the equalities hold if and only if the area of the complement of the union of image $f(|\zeta|<1)$ and $1/f(|\zeta|<1)$ vanishes. $\varepsilon'=1,\ -1$.

Proof Using the properties of the Faber polynomials, we get

$$-g_n^{(\varepsilon)}(\zeta)=n\sum_{m=1}^{\infty}d_{mn}\zeta^m+\varepsilon\bar\zeta^n,$$

$$-h_n(\zeta)=n\sum_{m=1}^{\infty}C_{mn}\zeta^m.$$

Therefore, applying (2.10) of Theorem 2 (here, we choose $p=q=0$), we obtain

$$\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu K(\zeta_\mu,\ \bar{\zeta}_\nu)+\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar{\eta}'_\nu K(\zeta'_\mu,\ \bar{\zeta}'_\nu)$$
$$\geqslant\sum_{n=1}^{\infty}n\Big(\Big|\sum_{\mu=1}^{N}\eta_\mu\Big(\sum_{m=1}^{\infty}d_{mn}\zeta_\mu^m\Big)+\sum_{\nu=1}^{N'}\eta'_\nu\Big(\sum_{m=1}^{\infty}C_{mn}\zeta_\nu'^m\Big)\Big|^2$$
$$+\Big|\sum_{\mu=1}^{N}\eta_\mu\Big(\sum_{m=1}^{\infty}C_{mn}\zeta_\mu^m\Big)+\sum_{\nu=1}^{N'}\eta'_\nu\Big(\sum_{m=1}^{\infty}d_{mn}\zeta_\nu'^m\Big)\Big|^2\Big)$$
$$=\sum_{n=1}^{\infty}\Big(\Big|\sum_{\mu=1}^{N}\eta_\mu g_n^{(\varepsilon)}(\zeta_\mu)+\sum_{\nu=1}^{N'}\eta'_\nu h_n(\zeta'_\nu)\Big|^2+\Big|\sum_{\mu=1}^{N}\eta_\mu h_n(\zeta_\mu)+\sum_{\nu=1}^{N'}\eta'_\nu g_n^{(\varepsilon)}(\zeta'_\nu)\Big|^2\Big)\Big/n$$
$$-2\varepsilon\,\mathrm{Re}\Big\{\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu F(\zeta_\mu,\ \zeta_\nu)+\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\bar{\eta}_\mu\eta'_\nu\Phi(\zeta_\mu,\ \zeta'_\nu)$$
$$+\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar{\eta}'_\nu F(\zeta'_\mu,\ \zeta'_\nu)+\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\bar{\eta}'_\nu\Phi(\zeta_\mu,\ \zeta'_\nu)\Big\}$$
$$-\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu K(\bar{\zeta}_\mu,\ \zeta_\nu)-\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar{\eta}'_\nu K(\bar{\zeta}'_\mu,\ \zeta'_\nu),$$

where $\{\zeta_\mu\}$ and $\{\zeta'_\nu\}$ both are the systems of the distinguished points in $|\zeta|<1$, $\{\eta_\mu\}$ and $\{\eta_\nu\}$ both are any complex constants.

We know that if $A(\zeta,\ z)=A(z,\ \zeta)$, then[2]

$$\mathrm{Re}\Big\{\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu A(\zeta_\mu,\ z_\nu)\Big\}=\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu\mathrm{Re}\{A(\zeta_\mu,\ z_\nu)\}$$

and $K(\zeta,\ \bar{z})=\overline{K(\bar{\zeta},\ z)}$. Therefore, it follows from the above inequality that

$$\frac{l}{2}\sum_{n=1}^{N}\Big\{\Big|\sum_{\mu=1}^{N}\eta_\mu g_n^{(\varepsilon)}(\zeta_\mu)+\sum_{\nu=1}^{N'}\eta'_\nu h_n(\zeta'_\nu)\Big|^2+\Big|\sum_{\mu=1}^{N}\eta_\mu h_n(\zeta_\mu)+\sum_{\nu=1}^{N'}\eta'_\nu g_n^{(\varepsilon)}(\zeta'_\nu)\Big|^2\Big\}\Big/n$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu\ln(|f(\zeta_\mu,\ \zeta_\nu)|^\varepsilon/|1-\zeta_\mu\bar{\zeta}_\nu|)^l$$
$$+\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar{\eta}'_\nu\ln(|f(\zeta'_\mu,\ \zeta'_\nu)|^\varepsilon/|1-\zeta'_\mu\bar{\zeta}'_\nu|)^l+\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\bar{\eta}'_\nu\ln|\varphi(\zeta_\mu,\ \zeta'_\nu)|^{\varepsilon l}$$
$$+\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\bar{\eta}_\mu\eta'_\nu|\varphi(\zeta_\mu,\ \zeta'_\nu)|^{\varepsilon l}.\tag{2.26}$$

Particularly, if we set $\eta'_\mu=0\ (\mu=1,\ \cdots,\ N')$, it follows (2.24) from (2.26); if we set $N'=N$, $\eta'_\mu=\varepsilon'\eta_\mu$, $\zeta'_\mu=\zeta_\mu$ for $\mu=1,\ \cdots,\ N$, it follows from (2.26) that (2.25) is valid. This completes the proof.

Theorem 7. *Let $f\in B_1$, l be nonnegative integer and $\{\zeta_\mu\}$ $(\mu=1,\ \cdots,\ N)$ be the distinguished points in $|\zeta|<1$. Let $\sum_{\mu,\nu=1}^{N}a_{\mu\nu}\eta_\mu\bar{\eta}_\nu\geqslant0$ for any complex numbers $\{\zeta_\mu\}$ $(\mu=1,\ \cdots,\ N)$ and $\varepsilon=1,\ -1$. Then we have*

$$\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu a_{\mu\nu}\Big|\frac{f(\zeta_\mu)f(\zeta_\nu)}{\zeta_\mu\zeta_\nu}\Big|^{\varepsilon l}\cdot\exp\Big\{\frac{l}{2}\sum_{n=1}^{\infty}\frac{1}{n}\big(g_n^{(\varepsilon)}(\zeta_\mu)\overline{g_n^{(\varepsilon)}(\zeta_\nu)}+h_n(\zeta_\mu)\overline{h_n(\zeta_\nu)}\big)\Big\}$$
$$\leqslant\sum_{\mu,\nu=1}^{N}a_{\mu\nu}\eta_\mu\bar{\eta}_\nu\Big|\frac{f'(0)(f(\zeta_\mu)-f(\zeta_\nu))}{\zeta_\mu-\zeta_\nu}\Big|^{\varepsilon l}\Big/|1-\zeta_\mu\bar{\zeta}_\nu|^l;\tag{2.27}$$
$$\sum_{\mu,\nu=1}^{N}a_{\mu\nu}\eta_\mu\bar{\eta}_\nu\Big|\frac{f(\zeta_\mu)f(\zeta_\nu)}{\zeta_\mu\zeta_\nu}\Big|^{\varepsilon l}\cdot\exp\Big\{\frac{l}{2}\sum_{n=1}^{\infty}\frac{1}{n}\big(g_n^{(\varepsilon)}(\zeta_\mu)+\varepsilon' h_n(\zeta_\mu)\big)\overline{\big(g_n^{(\varepsilon)}(\zeta_\nu)+\varepsilon' h_n(\zeta_\nu)\big)}\Big\}$$
$$\leqslant\Big|\sum_{\mu,\nu=1}^{N}a_{\mu\nu}\eta_\mu\bar{\eta}_\nu\Big|\frac{f'(0)(f(\zeta_\mu)-f(\zeta_\nu))}{\zeta_\mu-\zeta_\nu}\Big|^{\varepsilon l}\Big/|1-\zeta_\mu\bar{\zeta}_\nu|^l,\tag{2.28}$$

where $\{\eta_\mu\}$ $(\mu=1,\ \cdots,\ N)$ are any nonzero constants.

Proof Set

$$a^{(1)}_{\mu,\nu}=\ln(|f(\zeta_\mu,\ \zeta_\nu)|^{sl}/|1-\zeta_\mu\bar{\zeta}_\nu|^l,$$

$$a^{(2)}_{\mu,\nu}=\frac{l}{2}\sum_{n=1}^{\infty}(g_n^{(s)}(\zeta_\mu)\overline{g_n^{(s)}(\zeta_\nu)}+h_n(\zeta_\mu)\overline{h_n(\zeta_\nu)})/n$$

and $a^{(3)}_{\mu,\nu}=a^{(1)}_{\mu,\nu}-a^{(2)}_{\mu,\nu}$ for $\mu,\ \nu=1,\ \cdots,\ N$.

Evidently the matrix $(a^{(2)}_{\mu,\nu})$ is positive semi-definite. Applying Theorem 6, it follows that the matrixes $(a^{(1)}_{\mu,\nu})$ and $(a^{(3)}_{\mu,\nu})$ for $\mu,\ \nu=1,\ \cdots,\ N$ are both positive semi-definite. So from [4, p. 314, Lemma 1] we have

$$\sum_{\mu,\nu=1}^{N}a_{\mu\nu}\exp(a^{(2)}_{\mu,\nu})(\exp(a^{(3)}_{\mu,\nu})-1)\eta_\mu\bar{\eta}_\nu\geqslant 0$$

for any nonzero simultaneously complex numbers$\{\eta_\mu\}$. Namely

$$\sum_{\mu,\nu=1}^{N}a_{\mu\nu}\eta_\mu\bar{\eta}_\nu|f(\zeta_\mu,\ \zeta_\nu)|^{sl}/|1-\zeta_\mu\bar{\zeta}_\nu|^l\geqslant\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu a_{\mu\nu}\exp\{a^{(2)}_{\mu,\nu}\}.$$

Substituting $\eta_\mu|f(\zeta_\mu)/\zeta_\mu|^{sl}$ for η_μ in the last inequalities, we obtain at once the inequalities (2.27). Similarly, just as we have proved above, we can verify the inequalities (2.28). Thus we complete the proof.

§ 3. On The Grunsky Functions

Similarly, by virtue of the basic inequality for $f\in G_1$ from[1], we can prove the following results.

Theorem 1. *Suppose* $f\in G_1$, p, q *are any non-negative integers and* $\{\zeta_\mu\}$ $(\mu=1,\cdots,N)$, $\{\zeta'_\nu\}$ $(\nu=1,\cdots,N)$ *are both the systems of the distinguished points in* $|\zeta|<1$. *Then for any nozero simultaneously complex constants*$\{\eta_\mu\}$*and*$\{\eta'_\nu\}$, *we have*

1)

$$\sum_{n=1}^{\infty}n\left(\left|\sum_{\mu=1}^{N}\overline{\eta_\mu E_n^{(p)}(\zeta_\mu)}+\sum_{\nu=1}^{N'}\eta'_\nu D_n^{(q)}(\zeta'_\nu)\right|^2+\left|\sum_{\mu=1}^{N}\overline{\eta_\mu D_n^{(p)}(\zeta_\mu)}+\sum_{\nu=1}^{N'}\eta'_\nu E_n^{(q)}(\zeta'_\nu)\right|^2\right)$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu K^{(p,p)}(\zeta_\mu,\ \bar{\zeta}_\nu)+\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar{\eta}'_\nu K^{(q,q)}(\zeta'_\mu,\ \bar{\zeta}'_\nu),\tag{3.1}$$

Where the equality holds if and only if the area of the complent of the union of $f(|\zeta|<1)$and $1/f(|\zeta|<1)$vanishes;

2)

$$\left|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu F^{(p,q)}(\zeta_\mu,\ \zeta'_\nu)\right|+\left|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\bar{\eta}'_\nu\Psi^{(p,q)}(\zeta_\mu,\ \bar{\zeta}'_\nu)\right|$$
$$\leqslant\left(\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu K^{(p,p)}(\zeta_\mu,\ \bar{\zeta}_\nu)\cdot\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar{\eta}'_\nu K^{(q,q)}(\zeta'_\mu,\ \bar{\zeta}'_\nu)\right)^{\frac{1}{2}}\tag{3.2}$$

of which equality holds only if for $n=1,\ 2,\ \cdots$

$$\sum_{m=1}^{\infty}\left(d_{mn}\sum_{\mu=1}^{N'}\eta'_\mu\frac{\partial^q}{\partial\zeta'^q_\mu}\zeta'^m_\mu+\overline{e_{mn}\sum_{\mu=1}^{N'}\eta'_\mu\frac{\partial^q}{\partial\zeta'^q_\mu}\zeta'^m_\mu}\right)=\frac{\lambda}{n}\sum_{\mu=1}^{N}\eta_\mu\frac{\partial^p}{\partial\zeta^p_\mu}\zeta^n_\mu,\tag{3.3}$$

λ being constant such that

$$|\lambda|^2=\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar{\eta}'_\nu K^{(q,q)}(\zeta'_\mu,\bar{\zeta}'_\nu)\Big/\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu K^{(p,p)}(\zeta_\mu,\bar{\zeta}_\nu).$$

Corollary 1. *Let $f\in G_1$, $\{\zeta_\mu\}$ $(\mu=1,\cdots,N)$ and $\{\zeta'_\nu\}$ $(\nu=1,\cdots,N)$ be both the systems of the distinguished points in $|\zeta|<1$, then*

$$\Big|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu F(\zeta_\mu,\zeta'_\nu)\Big|+\Big|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\bar{\eta}'_\nu\Psi(\zeta_\mu,\bar{\zeta}'_\nu)\Big|$$
$$\leqslant\Big(\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu K(\zeta_\mu,\bar{\zeta}_\nu)\times\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar{\eta}'_\nu K(\zeta'_\mu,\bar{\zeta}'_\nu)\Big)^{1/2}. \tag{3.4}$$

In particular, while $N'=N$, $\zeta'_\mu=\zeta$, $\eta'_\mu=\eta_\mu$, it reduces to the inequalities (4.1) of [1].

Corollary 2. *Under the assumption of above Corollary 1, we have*

$$\Big|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu\Big(\frac{f'(\zeta_\mu)f'(\zeta'_\nu)}{(f(\zeta_\mu)-f(\zeta'_\nu))^2}-\frac{1}{(\zeta_\mu-\zeta'_\nu)^2}\Big)\Big|+\Big|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu\frac{f'(\zeta_\mu)\overline{f'(\zeta'_\nu)}}{(1+f(\zeta_\mu)\overline{f(\zeta'_\nu)})^2}\Big|$$
$$\leqslant\Big(\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu(1-\zeta_\mu\bar{\zeta}_\nu)^{-2}\cdot\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar{\eta}'_\nu(1-\zeta'_\mu\bar{\zeta}'_\nu)^{-2}\Big)^{1/2}. \tag{3.5}$$

This strenthens the corresponding results of [4].

Corollary 3. *Let $f\in G_1$, then for any point ζ in the unit disk we have*

$$|\{f,\zeta\}|+6|f'(\zeta)|^2/(1+|f(\zeta)|^2)^2\leqslant 6(1-|\zeta|^2)^{-2}. \tag{3.6}$$

This was proved by Beresniewiez-Rajca, olga[4].

Theorem 2. *Suppose $f(\zeta)$ is regular in the unit disk, $f(0)=0$ and $f'(0)\neq 0$, then the necessary and sufficient condition for $f\in G_1$ is that*

$$\frac{1}{\pi}\iint_{|z|<1}\Big(\Big|\frac{f'(\zeta)f'(z)}{(f(\zeta)-f(z))^2}-\frac{1}{(\zeta-z)^2}\Big|^2+\Big|\frac{f'(\zeta)\overline{f'(z)}}{(1+f(\zeta)\overline{f(z)})^2}\Big|^2\Big)d\sigma_z$$
$$\leqslant(1-|\zeta|^2)^{-2}, \tag{3.7}$$

for any point ζ in $|\zeta|<1$. Moreover, if $f\in G_1$, then the equality of (3.7) holds if and only if the area of the complement of the union of $f(|\zeta|<1)$ and $1/f(|\zeta|<1)$ vanishes.

Theorem 3. *Let $f\in G_1$, l be any non-negative integer, and $\{\zeta_\mu\}$ $(\mu=1,\cdots,N)$, $\{\zeta'_\nu\}$ $(\nu=1,\cdots,N')$ be any distinguished points in $|\zeta|<1$. Then we have*

$$\Big|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu(F(\zeta_\mu,\zeta'_\nu)^l\Big|^2+\Big|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu(\Psi(\zeta_\mu,\bar{\zeta}'_\nu))^l\Big|^2$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu(K(\zeta_\mu,\bar{\zeta}_\nu))^l\cdot\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar{\eta}'_\nu(K(\zeta'_\mu,\bar{\zeta}'_\nu))^l \tag{3.8}$$

for any complex numbers $\{\eta_\mu\}$ and $\{\eta'_\nu\}$.

Corollary. *Let $f\in G_1$, p be any complex number and let $\{\zeta_\mu\}$ $(\mu=1,\cdots,N)$ be any distinguished points in $|\zeta|<1$, then we have*

$$\Big|\sum_{\mu,\nu=1}^{N}\eta_\mu\eta_\nu(f(\zeta_\mu,\zeta_\nu))^p\Big|\leqslant\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu(1-\zeta_\mu\bar{\zeta}_\nu)^{-|p|}, \tag{3.9}$$
$$\Big|\sum_{\mu,\nu=1}^{N}\eta_\mu\eta_\nu(\psi(\zeta_\mu,\zeta_\nu))^p\Big|\leqslant\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu(1-\zeta_\mu\bar{\zeta}_\nu)^{-|p|} \tag{3.10}$$

for any nonzero complex numbers $\{\eta_\mu\}$.

Theorem 4. *Let $f\in G_1$, $\varepsilon=1, -1$, p be any complex number and l be any non-negative integer. Let $\{\zeta_\mu\}$ $(\mu=1, \cdots, N)$ and $\{\zeta'_\nu\}$ $(\nu=1, \cdots, N')$ be any distinguished points in $|\zeta|<1$. Then we have*

$$\left|\sum_{\mu=1}^{N}\sum_{\nu=1}^{N'}\eta_\mu\eta'_\nu[\ln(f(\zeta_\mu, \zeta'_\nu)(\psi(\zeta_\mu, \zeta'_\nu))^\varepsilon)]^l\right|^2 \leqslant \sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu(K(\zeta_\mu, \bar{\zeta}_\nu))^l\cdot\sum_{\mu,\nu=1}^{N'}\eta'_\mu\bar{\eta}'_\nu(K(\zeta'_\mu, \bar{\zeta}'_\nu))^l, \tag{3.11}$$

$$\left|\sum_{\mu,\nu=1}^{N}\eta_\mu\eta_\nu(f(\zeta_\mu, \zeta_\nu)(\psi(\zeta_\mu, \zeta_\nu))^\varepsilon)^p\right| \leqslant \sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu(1-\zeta_\mu\bar{\zeta}_\nu)^{-|p|} \tag{3.12}$$

for any complex numbers $\{\eta_\mu\}$ and $\{\eta'_\nu\}$.

Theorem 5. *Let $f\in G_1$, $\varepsilon=1, -1$, $l>0$ and $(a_{\mu,\nu})$ for $\mu, \nu=1, \cdots, N$ be positive semi-definite Hermite matrix. Let $P_n(t)$ be the n-th Faber polynomials generated by function f, and*

$$g_n^{(\varepsilon)}(\zeta)=P_n(1/f(\zeta))-(\zeta^{-n}+\varepsilon\bar{\zeta}^n),$$

$$h_n^*(\zeta)=P_n(-f(\zeta))+2\gamma_n, \quad \ln(f(\zeta)/\zeta)=2\sum_{n=0}^{\infty}\gamma_n\zeta^n$$

for $n\geqslant 1$. Then we have

$$\frac{l}{2}\sum_{n=1}^{\infty}\left(\left|\sum_{\mu=1}^{N}\eta_\mu g_n^{(\varepsilon)}(\zeta_\mu)\right|^2+\left|\sum_{\mu=1}^{N}\eta_\mu h_n^*(\zeta_\mu)\right|^2\right)\Big/n \leqslant \sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu\ln(|f(\zeta_\mu, \zeta_\nu)|^{\varepsilon l}/|1-\zeta_\mu\bar{\zeta}_\nu|^l); \tag{3.13}$$

$$\sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu a_{\mu\nu}\left|\frac{f(\zeta_\mu)f(\zeta_\nu)}{\zeta_\mu\zeta_\nu}\right|^{\varepsilon l}\cdot\exp\left\{\frac{l}{2}\sum_{n=1}^{\infty}\frac{1}{n}(g_n^{(\varepsilon)}(\zeta_\mu)\overline{g_n^{(\varepsilon)}(\zeta_\nu)}+h_n^*(\zeta_\mu)\overline{h_n^*(\zeta_\nu)})\right\} \leqslant \sum_{\mu,\nu=1}^{N}\eta_\mu\bar{\eta}_\nu a_{\mu\nu}|f'(0)(f(\zeta_\mu-f(\zeta_\nu))|^{\varepsilon l}/|\zeta_\mu-\zeta_\nu|^l|1-\zeta_\mu\bar{\zeta}_\nu|^l, \tag{3.14}$$

where $\{\zeta_\mu\}$ $(\mu=1, \cdots, N)$ are any distinguished points in $|\zeta|<1$, $\{\eta_\mu\}$ $(\mu=1, \cdots, N)$ are any complex numbers.

References

[1] Xia Dioxing, *Fudan journal*, **2**(1956), 133—145.
[2] Hu Ke, *Jiangxi normal college J.*, **1**(1979), 5—14.
[3] Ren Fuyao, *Fudan Journal*, **3**(1979), 69—75.
[4] Beresniewiez-Rajca, Olga, *Demonstratio Math.*, **9**: 3(1976), 307—319.
[5] Лебедев, Н. А., Принцип площадейй в теории однолистных функций, 1975.
[6] Bieberbach, L., *Math. Ann.*, **77**(1916), 153—172.
[7] Grunsky, H. *Math. Ann.*, **108**(1933), 190—196.
[8] Schober, G., Univalent Functions-Selected Topics, 1975.
[9] Базилевич, И. Е., *Матем сб.*, **74**(1967), 133—146.

Bieberbach 函数族和 Grunsky 函数族的另一种充要条件及偏差定理

任 福 尧
(复旦大学)

摘 要

本文主要利用夏道行[1]对于 Bieberbach 函数族和 Grunsky 函数族的面积定理证明这两种函数族的另一充要条件，以及指数化的 Golusin 不等式和 Fitz Gerald 不等式等深刻的偏差定理.