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POTENTIALITY AND REVERSIBILITY FOR GENERAL SPEED FUNCTIONS (II). COMPACT STATE SPACES

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In this part of the paper, we use the methods and results developed in part I[1] to the case of compact state spaces.

In § 5 we show that quasi-reversibility for a speed function satisfying (4.3) in [1] with finite range is equivalent to potentiality, $\widetilde{\mathscr{G}}=\mathscr{G}(\mathscr{V})$, and quasi-reversibility implies reversibility under certain conditions. In § 6 we discuss mainly the preceding problems for the exclusion processes, and obtain some ideal results. First we give the simplest criterions for potentiality, then we prove that for the exclusion processes there is always a reversible measure. If the speed functions have a potential, then the reversible measures can be described by the canonical Gibbs states constructed by the speed functions. Conversely, if there exists a positive reversible measure, then it has a potential and the set of all positive reversible measures coincides with the set of all positive canonical Gibbs states. The corresponding results for the spin-flip processes have been obtained in [3]. We also show that these results can be obtained by our methods.

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§ 5. For Compact State Spaces

In this section, we will suppose that Y_u is finite for every $u\in S$, hence $X=\prod_{u\in S}Y_u$ becomes a compact space. In this case the results of preceding sections can be improved.

(1) **Theorem.** *Let Q be a speed function field, which satisfies* (4.3), *with finite range, then Q is quasi-reversible if and only if Q has a potential.*

Proof The necessity follows from Theorem (4.1) in [1]. We want to prove the sufficiency.

Since Y_u is finite for every $u\in S$, (4.10) in [1] holds. hence $\widetilde{\mathscr{G}}\subset\mathscr{G}(\mathring{\mathscr{V}})$ from Proposition (4.23) in [1]. Taking $\Lambda_n\in\mathscr{S}_f$, $n\geqslant 1$, $\Lambda_n\uparrow S$ and $z_n\in X(S\setminus\Lambda_n)$, we have

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$\mu_{\Lambda_n, z_n} \in \mathscr{P}(X)$ from (4.14) and (4.21) in [1]. Because X is a compact space, from [6] there is a subsequence of $\{\mu_{\Lambda_n, z_n}\}_{n\geqslant 1}$ converging to μ in weak topology, so we have $\mu \in \widetilde{\mathscr{G}}$ from (4.22) in [1]. Thus $\mathscr{G}(\mathscr{V}) \neq \varnothing$ and from Proposition (4.28) in [1] this implies that Q is quasi-reversible.

The following theorem may be regarded as the construction definition for Gibbs states.

(2) **Theorem.** *Let Q be a speed function field, which has a potential and satisfies* (4.3) *with finite range, then we have*

(3) $$\widetilde{\mathscr{G}} = \mathscr{G}(\mathscr{V}).$$

Proof Proposition (4.23) in [1] and the fact that Y_u is finite imply (4.24) in [1]. Conversely, if $\mu \in \mathscr{G}(\mathscr{V})$, then from (4.20) in [1] for every $\Lambda \in \mathscr{S}_f$ we have

(4) $$\forall y \in X(\Lambda),\ \mu(\{y\} \times X(S\backslash\Lambda)) = \int_{X(S\backslash\Lambda)} f^\Lambda(y \times z)\mu_{S\backslash\Lambda}(dz).$$

For each $n \geqslant 1$, we put

(5) $$I_n = \{1, 2, \cdots, n\}^{X(\Lambda)},\quad A_{ni} \triangleq \prod_{w \in X(\Lambda)} A^w_{n, i(w)},$$

where $i = \{i(w):\ w \in X(\Lambda)\}$, $A^w_{n,i(w)} \triangleq \left\{z \in X(S\backslash\Lambda):\ \frac{i(w)-1}{n} \leqslant f^\Lambda(w \times z) < \frac{i(w)}{n}\right\}$ if $1 \leqslant i(w) \leqslant n-1$, and $A^w_{n,n} \triangleq \left\{z \in X(S\backslash\Lambda):\ \frac{n-1}{n} \leqslant f^\Lambda(w \times z) \leqslant 1\right\}$. We choose an arbitrary but fixed $z_{n_i} \in A_{n_i}$, $i \in I_n$. From (4), (5) and (4.21) in [1] we have

(6) $$\forall y \in X(\Lambda),\ \mu(\{y\} \times X(S\backslash\Lambda)) = \lim_{n\to\infty} \sum_{i \in I_n} \mu_{S\backslash\Lambda}(A_{n_i}).$$

$\mu_{\Lambda, z_i}(\{y\} \times X(S\backslash\Lambda))$. Put $\mu_{\Lambda,n} \triangleq \sum_{i \in I_n} \mu_{S\backslash\Lambda}(A_{n_i})\mu_{\Lambda, z_{n_i}} \in \widetilde{\mathscr{G}}_\Lambda$. Since X is a compact space, from [9] we know that there exist $\mu^\Lambda \in \widetilde{\mathscr{G}}$ and $\{n_k\}$ such that $\mu_{\Lambda, n_k} \xrightarrow{w} \mu^\Lambda$. So for every $y \in X(\Lambda)$ we have $\mu(\{y\} \times X(S\backslash\Lambda)) = \mu^\Lambda(\{y\} \times X(S\backslash\Lambda))$ from (6), hence

(7) $$\mu(A \times X(S\backslash\Lambda)) = \mu^\Lambda(A \times X(S\backslash\Lambda)),\ \forall A \subset X(\Lambda).$$

Therefore, for every $\Lambda_n \in \mathscr{S}_L$, $\Lambda_n \uparrow S$, we have $\mu^{\Lambda_n} \in \widetilde{\mathscr{G}}_{\Lambda_n}$, and for every $\Lambda \in \mathscr{S}_f$, $y \in X(\Lambda)$, we have

$$\lim_{n\to\infty} \mu^{\Lambda_n}(\{y\} \times X(S\backslash\Lambda)) = \mu(\{y\} \times X(S\backslash\Lambda))$$

from (7). This fact implies that $\mu^{\Lambda_n} \xrightarrow{w} \mu$, hence $\mu \in \widetilde{\mathscr{G}}$ The proof is completed.

What can we say more for $N=1$? First we have

(8) **Theorem.** *Let Q be a potential field which satisfies* (4.3) in [1] *and is defined by a speed function $c(u, y, x)$ $(u \in S,\ y \in Y_u,\ x \in X)$. and*

(9) $$\forall u \in S,\ \forall y \in Y_u,\ c(u, y, \cdot) \in \mathscr{C}(X).$$

Then $\mathscr{V} = \{f^\Lambda:\ \Lambda \in \mathscr{S}_f\}$ defined by (4.12) in [1] *is a specification, and both* (4.18) *and* (3) *hold.*

Proof The first assertion follows from the proof of Lemma (4.11). Now we want to prove (4.18): $\forall \Lambda \in \mathscr{S}_f$, $f^\Lambda \in \mathscr{C}(X)$.

Let $x^{(m)} \longrightarrow x (m \to \infty)$, then there is an m_0 such that $(x^{(m)})_u = x_u$ for every $m \geqslant m_0$ and $u \in \Lambda$. So $f^\Lambda(x^{(m)}) = f^\Lambda(x_\Lambda \times (x^{(m)})_{S\setminus\Lambda})$, and it suffices to prove that

(10) $$\forall \tilde{w} \in X(\Lambda),\ f^\Lambda(\tilde{w} \times \cdot) \in \mathscr{C}(X(S\setminus\Lambda)).$$

Fix $\theta \in X$, $w \in X(\Lambda)$. From (4.3) in [1], $N=1$, $\delta(\Lambda)=\Lambda$ and Remark (4.5) in [1] we can choose n, $w_i \in X(\Lambda)$, $i=1, 2, \cdots, n$ such that

$$w_0 \times z \triangleq \theta_\Lambda \times z \to w_1 \times z \to \cdots \to w_n \times z \to w_{n+1} \times z \triangleq w \times z$$

for every $z \in X(S\setminus\Lambda)$. From (9) we know that

$$\bar{q}(\theta_\Lambda \times z,\ w \times z) \triangleq \prod_{i=0}^{n} q(w_i \times z,\ w_{i+1} \times z) \text{ and } \bar{q}(w \times z,\ \theta_\Lambda \times z) \triangleq \prod_{i=0}^{n} q(w_{i+1} \times z,\ w_i \times z)$$

are continuous functions, so from (4.8) in [1] we obtain

$$f^\Lambda(\tilde{w} \times z) = \frac{\lambda(\tilde{w} \times z)}{\sum_{w \in X(\Lambda)} \lambda(w \times z)} = \frac{\bar{q}(\theta_\Lambda \times z,\ \tilde{w} \times z)}{\bar{q}(\tilde{w} \times z,\ \theta_\Lambda \times z)} \Big/ \sum_{w \in X(\Lambda)} \frac{\bar{q}(\theta_\Lambda \times z,\ w \times z)}{\bar{q}(w \times z,\ \theta_\Lambda \times z)},$$

hence $f^\Lambda(\tilde{w} \times z)$ is a continuous function in z, and (10) holds.

Finally, we can prove (3) in the same way as the proof of Theorem (2).

(11) **Remark.**

(i) In fact, Lemma (4.11) in [1], Corollary (4.18) in [1] and Proposition (4.23) in [1] except (4.16) in [1] remain valid for countable $Y_u (\forall u \in S)$ and $N=1$, if we use Condition (9) instead of the hypothesis of "finite range" in Lemma (4.11) in [1] and Proposition (4.23) in [1]. The proofs are just the same as that of Theorem (8).

(ii) In § 6 we will discuss the case of $N=1$ and $Y_u = \{0, 1\}$ $(u \in S)$, which concludes the spin-flip processes.

When will a quasi-reversible measure be a reversible measure? An answer is obtained for $N=1$.

(12) **Lemma.** *Let $N=1$, the speed function $c(u, y, \cdot) \in \mathscr{C}(X)$ is uniformly bounded for u and y. we define a linear operator Ω on $\mathscr{C}(X)$ as follows:*

(13) $$\begin{cases} \Omega f \triangleq \sum_{u \in S} \sum_{y \in Y_u} c(u, y, \cdot) \Delta_u^y f,\ f \in \mathscr{D}(\Omega), \\ \mathscr{D}(\Omega) \triangleq \{f \in \mathscr{C}(X):\ \sum_{u \in S} \sum_{y \in Y_u} \|\Delta_u^y f\| < \infty\}. \end{cases}$$

Suppose that the closure $\bar{\Omega}$ of Ω generates a unique Markov semigroup, then $\mathscr{A}$ is a core for $\bar{\Omega}$.

Proof It suffices to prove that for every $f \in \mathscr{D}(\Omega)$, $\varepsilon > 0$, there is a $g \in \mathscr{A}$ such that $\|g - f\| < \varepsilon$ and $\|\Omega g - \Omega f\| < \varepsilon$. We write $M \triangleq \sup\{c(u, y, x):\ u \in S,\ y \in Y_u,\ x \in X\}$, choose a $T \in \mathscr{S}_f$ such that

$$\sum_{u \notin T} \sum_{y \in Y_u} \|\Delta_u^y f\| < \left(\frac{\varepsilon}{4M} \wedge \frac{\varepsilon}{4}\right),$$

and choose a $T_1 \in \mathscr{S}_f$ such that $x_{T_1} = x'_{T_1} \Rightarrow |f(x) - f(x')| < \left(\frac{\varepsilon}{4MK} \wedge \frac{\varepsilon}{4}\right)$, where $k = \sum_{u \in T} |Y_u|$. Putting $g(x) \triangleq f(x_{T_1} \times \theta)$, where $\theta \in X(S\setminus T_1)$ arbitrarily, we have $\|g - f\| \leqslant \left(\frac{\varepsilon}{4MK} \wedge \frac{\varepsilon}{4}\right)$, and for every $x \in X$

$$|\Omega g(x)-\Omega f(x)|\leqslant\sum_{u\in T}\sum_{y\in Y_u}|c(u,\ y,\ x)(\Delta_u^y g(x)-\Delta_u^y f(x)|$$
$$+\sum_{u\notin T}\sum_{y\in Y_u}|c(u,\ y,\ x)(\Delta_u^y g(x)-\Delta_u^y f(x))|$$
$$\leqslant 2M\sum_{u\in T}\sum_{y\in Y_u}\|g-f\|+M\sum_{u\notin T}\sum_{y\in Y_u}[\|\Delta_u^y g\|+\|\Delta_u^y f\|]$$
$$\leqslant 2MK\|g-f\|+2M\sum_{u\notin T}\sum_{y\in Y_u}\|\Delta_u^y f\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

(14) **Theorem.** *Let $c(u,\ y,\ \cdot)\in\mathscr{C}(X)\ (u\in S,\ y\in Y_u)$ be a speed function with finite range which is uniformly bounded for u and y. Suppose that there are $\Lambda_m\in\mathscr{S}_f$, $\Lambda_m\uparrow S$ such that $\Lambda_m=\Lambda_{m-1}\cup\partial\Lambda_{m-1}$, $m\geqslant 1$, and the closure $\overline{\Omega}$ of Ω defined by (13) generates a Markov semigroup on $\mathscr{C}(X)$. Then $\overline{\Omega}$ is quasireversible if and only if it is reversible. In detail, μ is reversible for $\overline{\Omega}$ if and only if it is a quasi-reversible measure for $\overline{\Omega}$.*

Proof From Theorem (3.25) in [1] we know that reversibility implies quasereversibility. Conversely, let μ be a quasi-reversible measure for $\overline{\Omega}$, from Lemma (12), it suffices to show that

$$\forall f,\ g\in\mathscr{A},\ \int f\Omega g d\mu=\int g\Omega f d\mu,$$

and this is equivalent to

(15) $$\forall A,\ B\in\bigcup_{\Lambda\in\mathscr{S}_f}\mathscr{F}(\Lambda),\ \int I_A\Omega I_B d\mu=\int I_B\Omega I_A\, d\mu.$$

To prove this we first consider that

(16) $$A=\{z_1\}\times X(S\setminus\Lambda_{m-1}),\ B=\{z_2\}\times X(S\setminus\Lambda_{m-1}),\ z_1,\ z_2\in X(\Lambda_{m-1}).$$

The equality in (15) is trivial when $z_1=z_2$. Moreover

(17) $$\int I_A\Omega I_B d\mu=\sum_{u\in\Lambda_{m-1}}\int\sum_{y\in Y_u}c(u,\ y,\ x)[I_A(x)I_B({}_u^y x)-I_{AB}(x)]\mu(dx),$$

(18) $$\int I_B\Omega I_A d\mu=\sum_{u\in\Lambda_{m-1}}\int\sum_{y\in Y_u}c(u,\ y,\ x)[I_B(x)I_A({}_u^y x)-I_{AB}(x)]\mu(dx).$$

If there are $u_1,\ u_2\in\Lambda_{m-1}$, $u_1\neq u_2$ such that $(z_1)_{u_i}\neq(z_2)_{u_i}$, then for every $u\in\Lambda_{m-1}$, $x\in X$, we have

$$I_A(x)I_B({}_u^y x)-I_{AB}(x)=I_B(x)I_A({}_u^y x)-I_{AB}(x)=0.$$

Hence from (17) and (18) the equality in (15) holds for A and B in (16) in this case. If there is only a $u_0\in\Lambda_{m-1}$ such that $(z_1)_{u_0}\neq(z_2)_{u_0}$, and $z_1=(z_1)_{u_0}\times(z_2)_{\Lambda_{m-1}\setminus u_0}$, then from (17) and (18) (write $\mu_{\Lambda_m}=\mu_m$) we obtain

$$\int I_A\Omega I_B d\mu=\int_A c(u_0,\ (z_2)_{u_0},\ x)\mu(dx)=\sum_{w\in X(\partial\Lambda_{m-1})}c(\mu_0,\ (z_2)_{u_0},\ z_1\times w)\mu_m(z_1\times w),$$
$$\int I_B\Omega I_A\, d\mu=\sum_{w\in X(\partial\Lambda_{m-1})}c(u_0,\ (z_1)_{u_0},\ z_2\times w)\mu_m(z_2\times w).$$

Since μ is a quasi-reversible measure for $\overline{\Omega}$, for every $w\in X(\partial\Lambda_{m-1})$, we have

$$\mu_m(z_1\times w)c(u_0,\ (z_2)_{u_0},\ z_1\times w)=\mu_m(z_2\times w)c(u_0,\ (z_1)_{u_0},\ z_2\times w),$$

thus the equality in (15) holds too. Furthermore, the equality in (15) holds for any A and B in (16).

In the general case, it suffices to consider

$$A=\{z_1\}\times X(S\backslash T_1),\ B=\{z_2\}\times X(S\backslash T_2),\ T_i\in\mathscr{S}_f,\ z_i\in X(T_i).$$

Take $\Lambda_m\supset T_1\cup T_2$. Since both A and B are finite disjoint unions of the sets taking the form of $\{z\}\times X(S\backslash\Lambda_m)$, and the equality in (15) is closed under linear combination, the equality in (15) holds for these A and B. Therefore (15) is proved.

When does $\bar{\Omega}$ in Lemma (12) and Theorem (14) generate a Markov semigroup? We have a simple result as follows:

(19) **Theorem.** *Suppose that for every $\Lambda\in\mathscr{S}_N$, $y\in X(\Lambda)$, the speed function $c(\Lambda, y, \cdot)\in\mathscr{C}(X)$. If there is $c(\Lambda, y)$, $\Lambda\in\mathscr{S}_N$, $y\in X(\Lambda)$ such that*

(20) $$\forall\Lambda\in\mathscr{S}_N,\ \forall y\in X(\Lambda),\ c(\Lambda, y, \cdot)\leqslant c(\Lambda, y),$$

(21) $$\sup_u \sum_{u\in\Lambda\in\mathscr{S}_N}\sum_{y\in X(\Lambda)} c(\Lambda, y)=L<\infty;$$

(22) $\exists M>0$ *such that*

$$\sum_{\Lambda'\in\mathscr{S}_N}\sum_{y'\in X(\Lambda')} c(\Lambda', y')\|\Delta_{\Lambda'}^{y'}c(\Lambda, y, \cdot)\|\leqslant MC(\Lambda, y),$$

then by defining a linear operator Ω on $\mathscr{C}(X)$ as follows:

(23) $$\begin{cases}\Omega f=\sum_{\Lambda\in\mathscr{S}_N}\sum_{y\in X(\Lambda)} c(\Lambda, y, \cdot)\Delta_\Lambda^y f,\\ \mathscr{D}(\Omega)=\{f\in\mathscr{C}(X):\sum_{\Lambda\in\mathscr{S}_N}\sum_{y\in X(\Lambda)} c(\Lambda, y)\|\Delta_\Lambda^y f\|<\infty\},\end{cases}$$

the closure $\bar{\Omega}$ of Ω generates a unique Markov semigroup on $\mathscr{C}(X)$.

Proof An outline of the proof is: taking an arbitrary subset $\mathscr{S}_N^0$ of $\mathscr{S}_N$, and putting

$$\Omega_0 f=\sum_{\Lambda\in\mathscr{S}_N^0}\sum_{y\in X(\Lambda)} c(\Lambda, y, \cdot)\Delta_\Lambda^y f,$$

we can prove that $\|f\|\leqslant\|f-\lambda\Omega_0 f\|$ for every $\lambda>0$.

Then by using the notations in [7], and setting

$$U(\Lambda, y)f\triangleq\Delta_\Lambda^y f,$$
$$M(\Lambda, y)f\triangleq c(\Lambda, y, \cdot)f(\cdot),$$

it is very easy to prove that

$$\sum_{\Lambda'\in\mathscr{S}_N}\sum_{y'\in X(\Lambda')} c(\Lambda', y')r(U(\Lambda', y'), U(\Lambda, y))\leqslant 2NL,$$

$$\sum_{\Lambda'\in\mathscr{S}_N}\sum_{y'\in X(\Lambda')} c(\Lambda', y')\|[U(\Lambda', y'), M(\Lambda, y)]\|\leqslant Mc(\Lambda, y)$$

for every $\Lambda\in\mathscr{S}_N$ and $y\in X(\Lambda)$. Clearly, $\mathscr{D}(\Omega)\supset\mathscr{A}$, so Ω has a dense domain. Moreover, we have

(24) $$\forall f\in\mathscr{D}(\bar{\Omega}),\ \forall\lambda>0,\ f-\lambda\Omega f\geqslant 0\Rightarrow f\geqslant 0$$

and $\bar{\Omega}\mathbf{1}=0$, hence [7] is applicable.

(25) **Remark** condition (22) in Theorem (19) can be replaced by the following condition

(26) $\forall\Lambda\in\mathscr{S}_N, \forall y\in X(\Lambda)$,

$$\left[\frac{c}{2NL}\sum_{\substack{\Lambda'\in\mathscr{S}_N\\ \Lambda'\cap\Lambda=\phi}}\sum_{y'\in X(\Lambda')}\|\Delta_{\Lambda'}^{y'}c(\Lambda, y, \cdot)\|+\sup_{\Lambda'\in\mathscr{S}_N, y'\in X(\Lambda'), \Lambda\cap\Lambda\neq\phi}\|\Delta_{\Lambda'}^{y'}c(\Lambda, y, .)\|\right]\leqslant c(\Lambda, y),$$

where $c \triangleq \sup_{\Lambda \in \mathscr{S}_N, y \in X(\Lambda)} c(\Lambda, y)$.

§ 6. Potentiality and Reversibility for Exclusion Processes

The spin-flip processes and the exclusion processes[6] are two kinds of Markov processes with infinite particle systems. Many of the results in the preceding sections are appli-cable to these processes, but the exclusion processes don't satisfy condition (4.3) in [1], so there are some differences. Since there were discussions for spin-flip processes in [3]. we will emphatically discuss the exclusion processes in this section.

We take $Y_u \equiv \{0, 1\}$, $\forall u \in S$, and let $c(\cdot, \cdot, \cdot)$: $S \times S \times X \to R$ be non-negative and satisfy

(1) $c(u, v, x) > 0$ if and only if $x_u \neq x_v$ for $u, v \in S$, $u \neq v$;

(2) $c(u, v, \cdot) = c(v, u, \cdot) \in \mathscr{C}(X)$, $u, v \in S$.

If we put

$$c(\{u, v\}, y, x) \triangleq \begin{cases} c(u, v, x), & x_u \neq x_v, \ (y_u, y_v) = (x_v, x_u), \\ 0, & \text{otherwise}, \end{cases}$$

then $c(u, v, x)$ is a particular form of $c(\Lambda, y, x)$ for $|\Lambda| = 2$, but it does not satisfy (4.3) in [1]. Then, Ω defined in § 3 reduces to

(3) $\Omega f(x) = \frac{1}{2} \sum_{u, v \in S} c(u, v, x) \Delta_{(u,v)} f(x)$,

where

$$\Delta_{(u,v)} f(x) = f({}_{(u,v)}x) - f(x),$$

$$({}_{(u,v)}x) = \begin{cases} x_v, & w = u, \\ x_u, & w = v, \\ x_w, & w \neq u, v. \end{cases}$$

In order to generate a unique Markov semigroup for the closure of Ω defined as above, we need some conditions (for example, see Theorem (5.19)). If there are $c(u, v) = c(v, u)$ and M such that

(4) $\forall x \in X$, $\forall u, v \in S$, $c(u, v, x) \leq c(u, v)$;

(5) $\sup_u \sum_v c(u, v) < \infty$;

(6) $\forall u, v \in S$, $\sum_w \|\Delta_w c(u, v, \cdot)\| \leq M c(u, v)$;

then by writing

(7) $\mathscr{D}(\Omega) \triangleq \{f \in \mathscr{C}(X)$: $\sum_{u,v} c(u, v) \|\Delta_{(u,v)} f\| < \infty\}$,

we attain that the closure of Ω defined by (3) and (7) generates a unique Markov semigroup, the corresponding process being called an exclusion process.

Again we let $X = \{0, 1\}^S$, $c(\cdot, \cdot)$: $S \times X \to R$ be non-negative and satisfy

(8) $\forall u \in S$, $\forall x \in X$, $c(u, x) > 0$,

(9) $\forall u\in S,\ c(u,\ \cdot)\in\mathscr{C}(X)$,

then

$$c(\{u\},\ y,\ x)=\begin{cases}0, & y=x_u,\\ c(u,\ x), & y=1-x_u,\end{cases}\quad u\in S,\ y\in Y_u,\ x\in X$$

defines a speed function as in[1; § 3] for $N=1$ and $Y_u=\{0,\ 1\}$. So Ω defined in[1; § 3] determines a unique Markov semigroup under appropriate conditions(for example, see Theorem (5.19)), the corresponding process being called a spinflip process.

Clearly, both $c(u,\ v,\ x)$ and $c(u,\ x)$ satisfy the co-zero condition; so we can define their respective speed function fieldsas in[1; § 3], i. e.,

(10) $$q(x,\tilde{x})=\begin{cases}c(u,\ v,\ x), & \text{if } \tilde{x}={}_{(u,v)}x,\ x_u\neq x_v,\\ 0, & \text{other cases of } \tilde{x}\neq x;\end{cases}$$

(11) $$q(x,\ \tilde{x})\triangleq\begin{cases}c(u,\ x), & \tilde{x}={}_ux,\\ 0, & \text{other cases of } \tilde{x}\neq x,\end{cases}$$

where $({}_ux)_w=\begin{cases}x_w, & \text{if } w\neq u\\ 1-x_w, & \text{if } w=u\end{cases}$. In both cases,

we need not define $q(x,\ x)$ in the following discussions.

We will give the criteria for potentiality of the fields.

(12) **Theorem.** *Let $c(u,\ x)$ be a speed function satisfying* (8). *Then its field Q has a potential if and only if*

(13) $$c(u,\ x)c(v,\ {}_ux)c(u,\ {}_{uv}x)c(v,\ {}_vx)$$
$$=c(v,\ x)c(u,\ {}_vx)c(v,\ {}_{uv}x)c(u,\ {}_ux),\ \forall u,\ v\in S,\ \forall x\in X,$$

where ${}_{uv}x={}_{vu}x={}_u({}_vx)$.

Proof For the convenience of proving the theorem, we introduce several notations.

Define a transformation $\zeta(u)$: $X\to X$ as follows:

$$x\zeta(u)={}_ux,\ x\in X,\ u\in S,$$

then for every path $L=(x=x^{(0)},\ x^{(1)},\ \cdots,\ x^{(n)})$, we have

$$x^{(k)}=x\zeta(u_1)\zeta(u_2)\cdots\zeta(u_k)={}_{u_k\cdots u_2u_1}x,\ 1\leqslant k\leqslant n.$$

In the proof we also use $x\zeta(u_1)\cdots\zeta(u_n)$ to denote path L, where x denotes the start and $\zeta(u_i)$ denotes the segment $x^{(i-1)}\to x^{(i)}$. By definition(2.4) in[1] we have

(14) $\varphi(x\zeta(u_1)\cdots\zeta(u_n))=\varphi(x\zeta(u_1)\cdots\zeta(u_k))+\varphi(x^{(k)}\zeta(u_{k+1})\cdots\zeta(u_n)),\ 1\leqslant k\leqslant n-1.$

Using these notations, we can rewrite(13) as

(15) $\forall x\in X,\ \forall u,\ v\in S,\ u\neq v,\ \varphi(x\zeta(u)\zeta(v))=\varphi(x\zeta(v)\zeta(u))$. Clearly,

(16) $\forall x\in X,\ \forall u\in S,\ \varphi(u\zeta(u)\zeta(u))=0.$

Now we want to prove

(17) $\varphi(x\zeta(u_1)\cdots\zeta(u_n))=0$

for every closed path(i. e., $x^{(n)}=x$ for the above L).

Since every closed path consists of an even number of segments, $n=2m$, m being a positive integer. We use induction on m. When $m=1$, then $u_1=u_2$, hence (17) follows from (16). Suppose that (17) holds for $n=2(m-1)$. Then, when $n=2m$, there is a k such that $2\leqslant k\leqslant n$, $u_k=u_1$ and $u_l\neq u_1$ for $2\leqslant l\leqslant k-1$. Applying (14) and (15), we obtain

$$\begin{aligned}\varphi(x\zeta(u_1)\cdots\zeta(u_n)) &= \varphi(x\zeta(u_1)\zeta(u_2))+\varphi(({}_{u_1u_2}x)\zeta(u_3)\cdots\zeta(u_n))\\ &= \varphi(x\zeta(u_2)\zeta(u_1))+\varphi(({}_{u_2u_1}x)\zeta(u_3)\cdots\zeta(u_n))\\ &= \varphi(x\zeta(u_2))+\varphi(({}_{u_2}x)\zeta(u_1)\zeta(u_3)\cdots\zeta(u_n)).\end{aligned}$$

Similarly, applying repeatedly (14) and (15), and applying (16) and $u_k=u_1$, we obtain

$$\begin{aligned}\varphi(x\zeta(u_1)\cdots\zeta(u_n)) &= \varphi(x\zeta(u_2))+\varphi(({}_{u_2}x)\zeta(u_3))+\varphi(({}_{u_3u_2}x)\zeta(u_1)\zeta(u_4)\cdots\zeta(u_n))\\ &= \varphi(x\zeta(u_2)\zeta(u_3))+\varphi(({}_{u_3u_2}x)\zeta(u_1)\zeta(u_4)\cdots\zeta(u_n))=\cdots\\ &= \varphi(x\zeta(u_2)\cdots\zeta(u_{k-1}))+\varphi(({}_{u_{k-1}\cdots u_2}x)\zeta(u_1)\zeta(u_k)\cdots\zeta(u_n))\\ &= \varphi(x\zeta(u_2)\cdots\zeta(u_{k-1}))+\varphi(({}_{u_{k-1}\cdots u_2}x)\zeta(u_1)\zeta(u_k))\\ &\quad +\varphi(({}_{u_ku_1u_{k-1}\cdots u_2}x)\zeta(u_{k+1})\cdots\zeta(u_n))\\ &= \varphi(x\zeta(u_2)\cdots\zeta(u_{k-1})\zeta(u_{k+1})\cdots\zeta(u_n))=0.\end{aligned}$$

So (17) holds for every closed path, and the condition is sufficient.

Clearly, the condition is also necessary.

(18) **Theorem.** *Let $c(u, v, x)$ be a speed function satisfying* (1), *then its field Q has a potential if and only if*

(19) $$c(u, v, x)c(v, w, {}_{(u,v)}x)c(w, u, {}_{(w,u)}x) = c(u, w, x)c(w, v, {}_{(u,w)}x)c(v, u, {}_{(v,u)}x),\ x\in X,\ \forall u, v, w\in S.$$

Proof 1°. As the preceding proof, we define $x\ \zeta(u, v)={}_{(u,v)}x$ for $u\neq v$, $x_u\neq x_v$. But where $\zeta(u, v)$ is not a transformation on X because it is only defined on $\{x\in X:\ x_u\neq x_v\}$. We also use $x\zeta(u_1, v_1)\cdots\zeta(u_n, v_n)$ to denote the path $L=(x\triangleq x^{(0)}, x^{(1)}, \cdots, x^{(n)})$, which consists of $x=x^{(0)}$ and $x^{(k)}=x^{(k-1)}\ \zeta(u_k, v_k)$, $k=1, \cdots, n$, and we regard $\zeta(u_k, v_k)$ as the k'th segment of the path. By using these notations, it is very easy to prove that (19) is equivalent to

(20) $$\varphi(x\zeta(u, v)\zeta(v, w))=\varphi(x\zeta(u, w))\ \forall u, v, w\in S,$$
$$u, v, w \text{ are pairwise different},\ \forall x\in X,\ x_u\neq x_v=x_w.$$

Moreover, clearly we have $\zeta(u, v)=\zeta(v, u)$, $u\neq v$ and

(21) $$\forall u, v\in S, u\neq v, \forall x\in X, x_u\neq x_v, \varphi(x\zeta(u, v)\zeta(u, v))=0,$$

(22) $$\varphi(x\zeta(u_1, v_1)\cdots\zeta(u_n, v_n)) =\varphi(x\zeta(u_1, v_1)\cdots\zeta(u_k, v_k))+\varphi(x^{(k)}\zeta(u_{k+1}, v_{k+1})\cdots\zeta(u_n, v_n)).$$

2°. Under the conditions in (20), we have $x\zeta(u, v)\zeta(v, w)=x\zeta(u, w)$, so clearly (19) is necessary. We will prove its sufficiency. For this, we first prove

(23) $$\varphi(x\zeta(u_1, v_1)\zeta(u_2, v_2))=\varphi(x\zeta(u_2, v_2)\zeta(u_1, v_1)),$$

where u_1, v_1, u_2, v_2 are pairwise different and $x_{u_i}\neq x_{v_i}$, $i=1, 2$. We may and do assume that $x_{u_1}=x_{u_2}\neq x_{v_1}=x_{v_2}$. So $x\zeta(u_1, v_2)$ is well-defined. Note that

$$x\zeta(u_1, v_1)\zeta(u_2, v_2)\zeta(u_1, v_1)\zeta(u_2, v_2)=x,$$

$$x\zeta(u_1, v_1)\zeta(u_2, v_2)\zeta(u_1, v_1) = {}_{(u_2,v_2)}x,$$

and from (21), (22), (20) we have

$$\begin{aligned}
&\varphi(x\zeta(u_1, v_1)\zeta(u_2, v_2)\zeta(u_1, v_1)\zeta(u_2, v_2))\\
&\quad=\varphi(x\zeta(u_1, v_1)\zeta(u_2, v_2)\zeta(u_1, v_1))+\varphi(({}_{(u_2,v_2)}x)\zeta(u_2, v_2)\zeta(u_1, v_2))\\
&\qquad+\varphi(({}_{(u_1,v_2)}x)\zeta(u_1, v_2))\\
&\quad=\varphi(x\zeta(u_1, v_1)\zeta(u_2, v_2)\zeta(u_1, v_1))+\varphi(({}_{(u_2,v_2)}x)\zeta(u_2, u_1))+\varphi(({}_{(u_1,v_2)}x)\zeta(u_1, v_2))\\
&\quad=\varphi(x\zeta(u_1, v_1)\zeta(u_2, v_2)\zeta(u_1, v_1)\zeta(u_2, u_1))+\varphi(({}_{(u_1,v_2)}x)\zeta(u_1, v_2)).
\end{aligned}$$

In fact, the last three steps reduce $\zeta(u_2, v_2)\zeta(u_1, v_2)$ to $\zeta(u_2, u_1)$ in terms of (20). Similarly, the right-hand side above reduces to zero, hence (23) is proved.

3° We now prove that condition (20) is sufficient, i. e., for every closed path $L=(x^{(0)}, \cdots, x^{(n-1)}, x^{(0)})=x\zeta(u_1, v_1)\cdots\zeta(u_n, v_n)$, we have

$$\varphi(x\zeta(u_1, v_1)\cdots\zeta(u_n, v_n))=0. \tag{24}$$

We use induction for n. When $n=2$, (24) reduces to (21), and when $n=3$, from (22), (24) reduces to (21) too. Suppose that (24) holds for $n<m$, now we want to prove that (24) holds for $n=m$.

Since L is a closed path, each u_k and $v_k (1\leqslant k\leqslant m)$ appear an even number of times, and there is a k such that $2\leqslant k\leqslant m$ and

$$\{u_1, v_1\}\cap\{u_k, v_k\}\neq\varnothing,\ \{u_1, v_1\}\cap\{u_l, v_l\}=\varnothing,\ 2\leqslant l\leqslant k. \tag{25}$$

Since $\zeta(u, v)=\zeta(v, u)$, we can assume that

$$u_1=u_k \tag{26}$$

without loss of generality. At the moment we write $\tilde{x}\triangleq x\zeta(u_1, v_1)\zeta(u_2, v_2)$, then applying (22) and (23) again and again, we obtain

$$\begin{aligned}
&\varphi(x\zeta(u_1, v_1)\cdots\zeta(u_m, v_m))\\
&\quad=\varphi(x\zeta(u_2, v_2)\cdots\zeta(u_{k-1}, v_{k-1})\zeta(u_1, v_1)\zeta(u_k, v_k)\cdots\zeta(u_m, v_m)).
\end{aligned} \tag{27}$$

If $v_1=v_k$, then from (26) and (21), we obtain (at the moment we write $\hat{x}=x\zeta(u_2, v_2)\cdots\zeta(u_{k-1}, v_{k-1})$) that the right-hand side of (27)

$$=\varphi(x\zeta(u_2, v_2)\cdots\zeta(u_{k-1}, v_{k-1})\zeta(u_{k+1}, v_{k+1})\cdots\zeta(u_m, v_m)),$$

so (24) holds for $n=m$ from the hypothesis for $n=m-2$. If $v_1\neq v_k$, then from (26) and (20) we obtain that the right-hand side of (27)

$$=\varphi(x\zeta(u_2, v_2)\cdots\zeta(u_{k-1}, v_{k-1})\zeta(v_1, v_k)\zeta(u_{k+1}, v_{k+1})\cdots\zeta(u_m, v_m)),$$

so (24) holds also for $n=m$ from the hypothesis for $n=m-1$. Hence (24) holds for any n and the proof is completed.

We have an analogue of Theorem (5.8) for the spin-flip processes. Now we consider the similar case for the exclusion processes. By the equivalent relation (2.6) in [1], and from (1), the field determined by $c(u, v, x)$ divides X into the equivalent classes $\{X_l; l\in D\}$ as follows:

$$X_l=\{x\in X;\ \Lambda\triangleq\{u\in S;\ x_u\neq(\Delta_l)_u\}\in\mathscr{S}_f,\ \sum_{u\in\Lambda}x_u=\sum_{u\in\Lambda}(\Delta_l)_u\}, \tag{28}$$

where Λ_l is an arbitrary but fixed element of X_l. Letting

(29) $$X_k(\Lambda)=\{y\in X(\Lambda):\ |y|\triangleq\sum_{u\in\Lambda}y_u=k\},$$

and choosing an arbitrary but fixed $y^{(k)}\in X_k(\Lambda)$ (we add the suffix Λ, if necessary), we have

(30) $$\forall\Lambda\in\mathscr{S}_f,\ \forall y\in X_k(\Lambda),\ \exists y_1,\ y_2,\ \cdots,\ y_n \text{ such that}$$
$$y_0\times z\triangleq y^{(k)}\times z\to y_1\times z\to\cdots\to y_n\times z\to y^{(n+1)}\times z\triangleq y\times z \text{ for every } z\in X(S\backslash\Lambda).$$

Put

(31) $$\begin{aligned}\bar q(y^{(k)}\times z,\ y\times z)&=\prod_{i=0}^{n}q(y_i\times z,\ y_{i+1}\times z),\\ &\qquad\qquad\qquad\qquad \forall y\in X_k(\Lambda),\\ \bar q(y\times z,\ y^{(k)}\times z)&=\prod_{i=0}^{n}q(y_{i+1}\times z,\ y_i\times z),\end{aligned}$$

then we have

(32) **Lemma.** *Let Q have a potential. For every $\Lambda\in\mathscr{S}_f$, $k\in\{0,\ 1,\ \cdots,\ |\Lambda|\}$ we put*

(33) $$\begin{cases}\forall y\in X(\Lambda),\ \forall z\in X(S\backslash\Lambda),\\ f_k^\Lambda(y\times z)\triangleq I_{X_k(\Lambda)}(y)K^{-1}(\Lambda,\ z,\ k)\dfrac{\bar q(y^{(k)}\times z,\ y\times z)}{\bar q(y\times z,\ y^{(k)}\times z)},\\ K(\Lambda,\ z,\ k)\triangleq\displaystyle\sum_{w\in X_k(\Lambda)}\frac{\bar q(y^{(k)}\times z,\ w\times z)}{\bar q(w\times z,\ y^{(k)}\times z)},\end{cases}$$

then $\mathscr{V}_c\triangleq\{f_k^\Lambda:\ \Lambda\in\mathscr{S}_f,\ 0\leqslant k\leqslant|\Lambda|\}$ satisfies:

(34) $$0\leqslant f_k^\Lambda\in\mathscr{C}(X);\ f_k^\Lambda(x)>0 \textit{ if and only if } x_\Lambda\in X_k(\Lambda);$$

(35) $$\forall z\in X(S\backslash\Lambda),\ \sum_{y\in X_k(\Lambda)}f_k^\Lambda(y\times z)=1;$$

(36) $$\begin{gathered}\forall\Lambda\subset\tilde\Lambda\in\mathscr{S}_f,\ 0\leqslant\tilde k\leqslant|\tilde\Lambda|,\ \forall y_1\in X(\Lambda),\\ y_2\in X(\tilde\Lambda\backslash\Lambda),\ z\in X(S\backslash\tilde\Lambda),\\ f_{\tilde k}^{\tilde\Lambda}(y_1\times y_2\times z)=f_{|y_1|}^\Lambda(y_1\times y_2\times z)\sum_{y\in X_{|y_1|}(\Lambda)}f_{\tilde k}^{\tilde\Lambda}(y_1\times y_2\times z).\end{gathered}$$

Proof Since Q has a poten tial, $f_k^\Lambda(y\times z)$ is independent of the selection of $y^{(k)}$ and the path from $y^{(k)}\times z$ to $y\times z$, hence the definition of f_k^Λ is justified. It is clear that (34) and (35) hold except $f_k^\Lambda\in\mathscr{C}(X)$ which follows from the proof of Theorem (5.8). Now we want to prove (36).

(36) is trivial when $|y_1\times y_2|\neq\tilde k$. Suppose $|y_1\times y_2|=\tilde k$ and write $|y_1|=k$, then from the potentiality of Q and (33), we obtain

$$\begin{aligned}f_{\tilde k}^{\tilde\Lambda}(y_1\times y_2\times z)&=K^{-1}(\tilde\Lambda,\ z,\ \tilde k)\frac{\bar q(y_{\tilde\Lambda}^{(\tilde k)}\times z,y_1\times y_2\times z)}{\bar q(y_1\times y_2\times z,y_{\tilde\Lambda}^{(\tilde k)}\times z)}\\ &=K^{-1}(\tilde\Lambda,\ z,\ \tilde k)\frac{\bar q(y_{\tilde\Lambda}^{(\tilde k)}\times z,y^{(k)}\times y_2\times z)}{\bar q(y^{(k)}\times y_2\times z,y_{\tilde\Lambda}^{(\tilde k)}\times z)}\cdot\frac{\bar q(y^{(k)}\times y_2\times z,\ y_1\times y_2\times z)}{\bar q(y_1\times y_2\times z,\ y^{(k)}\times y_2\times z)}\\ &=f_k^\Lambda(y_1\times y_2\times z)\sum_{y\in X_k(\Lambda)}\frac{\bar q(y^{(k)}\times y_2\times z,\ y\times y_2\times z)}{\bar q(y\times y_2\times z,\ y^{(k)}\times y_2\times z)}\\ &\quad\cdot\frac{\bar q(y_{\tilde\Lambda}^{(\tilde k)}\times z,\ y^{(k)}\times y_2\times z)}{\bar q(y^{(k)}\times y_2\times z,\ y_{\tilde\Lambda}^{(\tilde k)}\times z)},\cdot K^{-1}(\tilde\Lambda,\ z,\ \tilde k)\\ &=f_k^\Lambda(y_1\times y_2\times z)\sum_{y\in X_k(\Lambda)}f_{\tilde k}^{\tilde\Lambda}(y\times y_2\times z).\end{aligned}$$

(37) **Definition.** *Let* $\mathscr{A}(S\setminus\Lambda)=\sigma\{X_k(\Lambda)\times X(S\setminus\Lambda),\ 0\leqslant k\leqslant|\Lambda|;\ \mathscr{F}(S\setminus\Lambda)\}$. $\mu\in\mathscr{P}(X)$ *is called a canonical Gibbs state corresponding to* $c(u, v, x)$ (or $\mathscr{V}_c$), *if for every* $\Lambda\in\mathscr{S}_f$, $y\in X(\Lambda)$, *we have*

(38) $\mu(\{y\}\times X(S\setminus\Lambda)\,|\,\mathscr{A}(S\setminus\Lambda))=f^{\Lambda}_{|(\cdot)_\Lambda|}(y\times(\cdot)_{S\setminus\Lambda})\ \mu$—a. e. *and the set of all canonical Gibbs states corresponding to* $c(u, v, x)$ (or $\mathscr{V}_c$) *is denoted by* $\mathscr{G}_c(\mathscr{V}_c)$.

For $\mathscr{V}_c$ *we define* $\overline{\mathscr{G}}_\Lambda(\Lambda\in\mathscr{S}_f)$ *to be the closed convex hull in weak topology of all the following* $\mu_{\Lambda,z,k}$, $z\in X(S\setminus\Lambda)$, $0\leqslant k\leqslant|\Lambda|$:

(39) $$\forall F\in\mathscr{F},\ \mu_{\Lambda,z,k}(F)\triangleq\sum_{y\in F(z)}f^{\Lambda}_k(y\times z),$$

and put

(40) $$\overline{\mathscr{G}}\triangleq\{\mu\in\mathscr{P}(X):\ \exists\Lambda_m\in\mathscr{S}_f,\ \Lambda_m\uparrow S \text{ and } \exists\mu_m\in\overline{\mathscr{G}}_{\Lambda_m} \text{ such that } \mu_m\xrightarrow{w}\mu\}.$$

(41) **Theorem.** *If Q has a potential, then*

$$\mathscr{G}_c(\mathscr{V}_c)=\overline{\mathscr{G}}.$$

Proof Just similar to the proofs of Proposition (4.23) in [1] and Theorem (5.2).

From now on, we will call a measure satisfying (3.18) in [1] a positive measure, and denote the set of all positive probability measures on X by $\mathscr{P}_+(X)$, Similarly we have the set $\mathscr{R}$ of all reversible measures and the set $\mathscr{R}_+$ of all positive reversible measures. Clearly $\mathscr{R}_+=\mathscr{R}\cap\mathscr{P}(X)$. We will establish the relation between $\mathscr{R}$ and $\mathscr{G}_c(\mathscr{V}_c)$ when Q has a potential.

(42) **Lemma.** $\mu\in\mathscr{R}$ *if and only if*

(43) $\forall f\in\mathscr{C}(X)$, $\forall u, v\in S$,

$$\int c(u, v, x)f(x)\mu(dx)=\int c(u, v, x)f({}_{(u,v)}x)\mu(dx).$$

Proof See [5 Lemma 2.15].

(44) **Lemma.** $\mu\in\mathscr{R}$ *if and only if*

$$\forall\Lambda\in\mathscr{S}_f,\ \forall y\in X(\Lambda),\ \forall\{u, v\}\subset\Lambda,$$

(45) $$c(u, v, y\times(\cdot)_{S\setminus\Lambda})\mu(\{y\}\times X(S\setminus\Lambda)\,|\,\mathscr{A}(S\setminus\Lambda))$$
$$=c(u, v, {}_{(u,v)}y\times(\cdot)_{S\setminus\Lambda})\mu(\{{}_{(u,v)}y\}\times X(S\setminus\Lambda)\,|\,\mathscr{A}(S\setminus\Lambda)).\quad \mu\text{—a. e.}$$

Proof It is easy to prove that if $\mu\in\mathscr{R}$, then (43) holds for $f=I_{(\{y\}\cap X_k(\Lambda))\times F}$, $\Lambda\in\mathscr{S}_f$, $y\in X(\Lambda)$, $0\leqslant k\leqslant|\Lambda|$, $F\subset\mathscr{F}_0(S\setminus\Lambda)$. So from (43) we have

$$\int_{X_k(\Lambda)\times F}c(u, v, y\times(\cdot)_{S\setminus\Lambda})\,I_{\{y\}\times X(S\setminus\Lambda)}\,d\mu$$
$$=\int_{X_k(\Lambda)\times F}c(u, v, {}_{(u,v)}y\times(\cdot)_{S\setminus\Lambda})I_{\{(u,v)y\}\times X(S\setminus\Lambda)}\,d\mu$$

for every $u, v\in\Lambda$, hence

$$\int_{X_k(\Lambda)\times F}c(u, v, y\times(\cdot)_{S\setminus\Lambda})\mu(\{y\}\times X(S\setminus\Lambda)\,|\,\mathscr{A}(S\setminus\Lambda))d\mu$$
$$=\int_{X_k(\Lambda)\times F}c(u, v, {}_{(u,v)}y\times(\cdot)_{S\setminus\Lambda})\mu(\{{}_{(u,v)}y\}\times X(S\setminus\Lambda)\,|\,\mathscr{A}(S\setminus\Lambda))d\mu,$$

and so (45) follows.

Conversely, if the condition is satisfied, then, taking $F\triangleq X(S\setminus\Lambda)$ in the preceding equations and deducing these equations inversely, we obtain that (43) holds for $f=I_{\{y\}\times X(S\setminus\Lambda)}$, $\Lambda\in\mathscr{S}_f$, $y\in X(\Lambda)$, u, $v\in\Lambda$, $u\neq v$. Now if $u\in\Lambda$ and $v\notin\Lambda$, then

$$\int c(u,\ v,\ \cdot)I_{\{y\}\times X(S\setminus\Lambda)}d\mu=\sum_{y_1=0,1}\int c(u,\ v,\ \cdot)I_{\{y\times y_1\}\times X(S\setminus(\Lambda\cup\{v\}))}d\mu$$

$$=\sum_{y_1=0,1}\int c(u,\ v,\ \cdot)I_{\{_{(u,v)}(y\times y_1)\}\times X(S\setminus(\Lambda\cup\{v\}))}d\mu$$

$$=\int c(u,\ v,\ x)I_{\{y\}\times X(S\setminus\Lambda)}(_{(u,v)}x)\mu(dx).$$

So (43) holds for $f\in\mathscr{A}$ and u, $v\in S$, hence (43) holds, i. e., $\mu\in\mathscr{R}$.

(46) **Lemma.** *If Q has a potential, then for every $\Lambda\in\mathscr{S}_f$, $y\in X(\Lambda)$, $z\in X(S\setminus\Lambda)$, $\{u,\ v\}\subset\Lambda$ and $k\in\{0,\ 1,\ \cdots,\ |\Lambda|\}$ we have*

(47) $$c(u,\ v,\ y\times z)f_k^\Lambda(y\times z)=c(u,\ v,\ _{(u,v)}y\times z)f_k^\Lambda(_{(u,v)}y\times z).$$

Proof f_k^Λ exists from Lemma (32). (47) is trivial when $y_u=y_v$ or $|y|\neq k$. Suppose $y^{(k)}\times z\rightsquigarrow y\times z$, then from the path-independence and (33), we have

$$\frac{f_k^\Lambda(y\times z)}{f_k^\Lambda(_{(u,v)}y\times z)}=\frac{\bar q(y^{(k)}\times z,y\times z)}{\bar q(y\times z,\ y^{(k)}\times z)}\Big/\frac{\bar q(y^{(k)}\times z,\ y\times z)}{\bar q(_{(u,v)}y\times z,\ y^{(k)}\times z)}$$

$$=\frac{\bar q(_{(u,v)}y\times z,\ y\times z)}{q(y\times z,\ _{(u,v)}y\times z)}=\frac{c(u,\ v,\ _{(u,v)}y\times z)}{c(u,\ v,\ y\times z)}.$$

(48) **Theorem.** *If Q has a potential, then $\mathscr{R}=\mathscr{G}_c(\mathscr{V}_c)$.*

Proof If $\mu\in\mathscr{G}_c(\mathscr{V}_c)$, then from (38) and (47) we know that for every $\Lambda\in\mathscr{S}_f$, $y\in X(\Lambda)$, $\{u,\ v\}\subset\Lambda$, the left-hand side of (45)

$$=c(u,\ v,\ y\times(\cdot)_{S\setminus\Lambda})f^\Lambda_{|(\cdot)_\Lambda|}(y\times(\cdot)_{S\setminus\Lambda})$$

$$=c(u,\ v,\ _{(u,v)}y\times(\cdot)_{S\setminus\Lambda})f^\Lambda_{|(\cdot)_\Lambda|}(_{(u,v)}y\times(\cdot)_{S\setminus\Lambda})$$

$$=\text{the right-hand sied of (45)}.\quad \mu\text{—a. e.}$$

So $\mu\in\mathscr{R}$ follows from Lemma (44).

Conversely, if $\mu\in\mathscr{R}$, then for every $\Lambda\in\mathscr{S}_f$, $k\in\{0,\ 1,\ \cdots,\ |\Lambda|\}$, $y\in X_k(\Lambda)$, $\{u,\ v\}\subset\Lambda$, from (45) and (47), we obtain

$$\frac{\mu(\{y\}\times X(S\setminus\Lambda)\,|\,\mathscr{A}(S\setminus\Lambda))}{f_k^\Lambda(y\times(\cdot)_{S\setminus\Lambda})}=\frac{\mu(\{_{(u,v)}y\}\times X(S\setminus\Lambda)\,|\,\mathscr{A}(S\setminus\Lambda))}{f_k^\Lambda(_{(u,v)}y\times(\cdot)_{S\setminus\Lambda})}.\quad \mu\text{—a. e.}$$

Since $y\times z\rightsquigarrow\tilde y\times z$ for every $\tilde y\in X_k(\Lambda)$, $z\in X(S\setminus\Lambda)$, from the above equation and (35) we obtain $\forall\tilde y\in X_k(\Lambda)$,

$$\frac{\mu(\{y\}\times X(S\setminus\Lambda)\,|\,\mathscr{A}(S\setminus\Lambda))}{f_k^\Lambda(y\times(\cdot)_{S\setminus\Lambda})}=\frac{\mu(\{\tilde y\}\times X(S\setminus\Lambda)\,|\,\mathscr{A}(S\setminus\Lambda))}{f_k^\Lambda(\tilde y\times(\cdot)_{S\setminus\Lambda})}$$

$$=\frac{\mu(X_k(\Lambda)\times X(S\setminus\Lambda)\,|\,\mathscr{A}(S\setminus\Lambda))}{\sum_{\tilde y\in X_k(\Lambda)}f_k^\Lambda(\tilde y\times(\cdot)_{S\setminus\Lambda})}=I_{X_k(\Lambda)}((\cdot)_\Lambda).\quad \mu\text{—a. e.}$$

Hence, for every $\Lambda\in\mathscr{S}_f$, $y\in X(\Lambda)$ we have

$$\mu(\{y\}\times X(S\setminus\Lambda)\,|\,\mathscr{A}(S\setminus\Lambda))=f^\Lambda_{|y|}(y\times(\cdot)_{S\setminus\Lambda})I_{X_{|y|}(\Lambda)}((\cdot))$$

$$=f^\Lambda_{|(\cdot)_\Lambda|}(y\times(\cdot)_{S\setminus\Lambda}).\quad \mu\text{—a. e.}$$

Therefore $\mu\in\mathscr{G}_c(\mathscr{V}_c)$.

(49) **Remark.** Theorem (48) is obtained in [4] for some special $c(u, v, x)$ (See [6; II, § 1.3]). The analogues corresponding to Lemma (42), (44), (46) and Theorem (48) (taking $c(u, x)$, ${}_ux$, $\mathscr{F}(S\backslash\Lambda)$, f^Λ and $\mathscr{G}(\mathscr{V})$ instead of $c(u, v, x)$, ${}_{(u,v)}x$, $\mathscr{A}(S\backslash\Lambda)$, f_k^Λ and $\mathscr{G}_c(\mathscr{V}_c)$ respectively) are obtained also for spin-flip processes[3]. But our proofs are simpler. In [3] Tang has proved that an infinitesimal generator of a spin-flip process is reversible if and only if the corresponding field has a potential. But the condition of potentiality is not necessary for an exclusion process. We will discuss these problems.

(50) **Theorem.** *Any speed function for an exclusion process is reversible (even condition (1) is not required). In detail, if we put*

$$\mathbf{0}_u\equiv 0,\ \mathbf{1}_u\equiv\mathbf{1},\ \forall u\in S$$

and

$$\nu_0(\mathbf{0})=\nu_1(\mathbf{1})=1,\quad \nu_0,\ \nu_1\in\mathscr{P}(Z)$$

then ν_0 and ν_1 are reversible measures for the speed function.

Proof It suffices to consider ν_0 The theorem follows from.

$$\int f\Omega g d\nu_0=f(\mathbf{0})\Omega g(\mathbf{0})=\frac{1}{2}f(\mathbf{0})\sum_{u,v}c(u,\ v,\ \mathbf{0})\ [g({}_{(u,v)}\mathbf{0})-g(\mathbf{0})]=0=\int g\Omega f d\mu.$$

Theorem (50) can be generalized. Suppose that the field Q restricted on X_l, is symmetrizable[2], i. e., $Q_l\triangleq\{q(x,\tilde{x}):\ x,\ \tilde{x}\in X_l\}$ has a potential and

$$\sum_{x\in X_l}\hat{q}(\Delta_l,\ x)/\hat{q}(x,\ \Delta_l)<+\infty.$$

Put

$$\mu_l(x)\triangleq\frac{\hat{q}(\Delta_l,\ x)}{\hat{q}(x,\ \Delta_l)}\Big[\sum_{\tilde{x}\in X_l}\frac{\hat{q}(\Delta_l,\ \tilde{x})}{\hat{q}(\tilde{x},\ \Delta_l)}\Big]^{-1},$$

then $\{\log\mu_l(x):\ x\in X_l\}$ is a potential for Q_l.

(51) **Proposition.** *If Q_l is symmetrizable, then*

(52) $$\mu\in\mathscr{P}(X),\ \mu(A)\triangleq\sum_{x\in A\cap X_l}\mu_l(x),\ A\in\mathscr{F}$$

is a reversible measure Conversely, if $\mu\in\mathscr{R}$ and $\mu(X_l)>0$, then Q is symmetrizable.

Proof Since $\mu_l(x)\,(x\in X_l)$ is a symmetrizing distribution for Q, from (52) we have

$$\begin{aligned}\int c(u,\ v,\ x)f(x)\mu(dx)&=\sum_{x\in X_l}c(u,\ v,\ x)f(x)\mu_l(x)\\&=\sum_{x\in X_l}c(u,\ v,\ {}_{(u,v)}(x))f(x)\mu_l({}_{(u,v)}x)=\sum_{x\in X_l}c(u,\ v,\ x)f({}_{(u,v)}x)\mu_l(x)\\&=\int c(u,\ v,\ x)f({}_{(u,v)}x)\mu(dx)\end{aligned}$$

for every $f\in\mathscr{C}(X)$, so $\mu\in\mathscr{R}$ follows from Lemma (42).

Conversely, since X_l is countable, $\mu(X_l)>0$ implies that there is an $x\in X_l$, assume that $x=\Delta_l$ without loss of generality, such that $\mu(\Delta_l)>0$. But from Lemma (42) and the monotone class theorem, we obtain

$$c(u,\ v,\ x)\mu(x)=c(u,\ v,\ {}_{(u,v)}x)\mu({}_{(u,v)}x)$$

for every $u, v\in S$, $x\in X_l$, hence $\Delta_l \rightsquigarrow x$ for every $x\in X_l$. Therefore $\mu(x)>0(x\in X_l)$ and Q has a potential. Furthermore Q_l is symmetrizable from the fact $\mu(X_l)\leqslant 1$.

The proposition can be improved also. we can construct a reversible measure, even if Q has a potential but it is not symmetrizable.

(53) **Lemma.** *If Q_l has a potential, then*

$$Q_l(\Lambda)\triangleq\{q(y\times(\Delta_l)_{S\setminus\Lambda},\ \tilde{y}\times(\Delta_l)_{S\setminus\Lambda}):y,\ \tilde{y}\in X_l(\Lambda)\}$$

is symmetrizable, where $X_l(\Lambda)=\{y\in X(\Lambda): \sum\limits_{u\in\Lambda}y_u=\sum\limits_{u\in\Lambda}(\Delta_l)_u\}$, $\Lambda\in\mathscr{S}_f$.

Proof By the hypothesis, there is $\{v(x): x\in X_l\}$ such that $v(x)>0$, $\forall x\in X_l$ and

$$v(x)q(x,\ \tilde{x})=v(\tilde{x})q(\tilde{x},\ x),\ x,\ \tilde{x}\in X_l.$$

In particular, taking $x=y\times(\Delta_l)_{S\setminus\Lambda}$ and $\tilde{x}=\tilde{y}\times(\Delta_l)_{S\setminus\Lambda}$ for $y,\ \tilde{y}\in X_l(\Lambda)$, we know that $Q_l(\Lambda)$ has a potential and

$$\pi(y\times(\Delta_l)_{S\setminus\Lambda})=v(y\times(\Delta_l)_{S\setminus\Lambda})\Big[\sum_{\tilde{y}\in X_l(\Lambda)}v(\tilde{y}\times(\Delta_l)_{S\setminus\Lambda})\Big]^{-1}$$

is a symmetrizable distribution for $Q_l(\Lambda)$.

Now we set

$$\pi_{l,\Lambda}(y)\triangleq\pi(y\times(\Delta_l)_{S\setminus\Lambda}),\ y\in X_l(\Lambda),$$
$$\mu_{l,\Lambda}(F)\triangleq\pi_{l,\Lambda}(F((\Delta_l)_{S\setminus\Lambda})\cap X_l(\Lambda)),\ F\in\mathscr{F}.$$

Clearly $\mu_{l,\Lambda}\in\mathscr{P}(X)$. Taking $\Lambda_m\uparrow S$ such that $\mu_{l,\Lambda_m}\xrightarrow{w}\mu_l(m\to\infty)$, we have

(54) **Proposition.** $\mu_l\in\mathscr{R}$ *and* $\mu_l(\overline{X}_l)=1$,

where $\overline{X}_l$ is the closure of X_l in X.

Proof From Lemma (53), it is clear that

$$\int c(u,\ v,\ x)[f({}_{(u,v)}x)-f(x)]\mu_{l,\Lambda_n}(dx)=0,\ f\in\mathscr{C}(X)$$

when $\Lambda_n\supset\{u, v\}$, Since $c(u, v, \cdot)[f({}_{(u,v)}(\cdot))-f(\cdot)]\in\mathscr{C}(X)$, $\mu_{l,\Lambda_n}\xrightarrow{w}\mu_l$, from Lemma (42) we obtain $\mu_l\in\mathscr{R}$, and

$$\mu_l(\overline{X}_l)\geqslant\limsup_{n\to\infty}\mu_{l,\Lambda_n}(X_l)\geqslant\limsup_{n\to\infty}\pi_{l,\Lambda_n}(X_l(\Lambda))=1.$$

(55) **Theorem.** *If $\mathscr{R}_+\neq\emptyset$, then Q has a potential and $\mathscr{R}_+=\mathscr{P}_+(X)\cap\mathscr{G}_c(\mathscr{V}_c)$.*

Proof From the hypothesis and Lemma (44), we know that for every $\Lambda\in\mathscr{S}_f$ there is an N_Λ such that $\mu(N_\Lambda)=0$ and

(45)′ $$c(u,\ v,\ y\times x_{S\setminus\Lambda})\mu(\{y\}\times X(S\setminus\Lambda)\,|\,\mathscr{A}(S\setminus\Lambda))(x)$$
$$=c(u,\ v,\ {}_{(u,v)}y\times x_{S\setminus\Lambda})\mu(\{{}_{(u,v)}y\}\times X(S\setminus\Lambda)\,|\,\mathscr{A}(S\setminus\Lambda))(x)$$

for every $x\notin N_\Lambda$, $y\in X(\Lambda)$, $\{u,\ v\}\subset\Lambda$. Moreover, for each $k\in\{0,\ 1,\ \cdots,\ |\Lambda|\}$ and $y_0\in X_k(\Lambda)$, there is an $x^{(\Lambda,k)}\notin N_\Lambda$ such that $\mu(\{y_0\}\times X(S\setminus\Lambda)\,|\,\mathscr{A}(S\setminus\Lambda))(x^{(\Lambda,k)})>0$, because $\mu\in\mathscr{P}_+(X)$. So from (45)′, we have

$$\mu(\{y\}\times X(S\setminus\Lambda)\,|\,\mathscr{A}(S\setminus\Lambda))(x^{(\Lambda,k)})>0,\ y\in X_k(\Lambda).$$

So from this and (45)′, we know that (we write $z^{(\Lambda,k)}=(x^{(\Lambda,k)})_{S\setminus\Lambda}$)

$$c(u,\ v,\ y\times z^{(\Lambda,k)})c(v,\ w,\ {}_{(u,v)}y\times z^{(\Lambda,k)})c(w,\ u,\ {}_{(w,u)}y\times z^{(\Lambda,k)})$$
$$=c(u,\ w,\ y\times z^{(\Lambda,k)})c(w,\ v,\ {}_{(u,w)}y\times z^{(\Lambda,k)})c(v,\ u,\ {}_{(v,u)}y\times z^{(\Lambda,k)})$$

for $y\in X_k(\Lambda)$ and $u, v, w\in\Lambda$. This equality implies (19), because $\bigcup_{\Lambda\in\mathscr{S}_f}\bigcup_{0\leqslant k\leqslant|\Lambda|} X_k(\Lambda)\times\{z^{(\Lambda,k)}\}$ is dense in X and $c(u, v, \cdot)\in\mathscr{C}(X)$. Then Q has a potential from Theorem (18), and $\mathscr{R}_+=\mathscr{P}_+(X)\cap\mathscr{R}=\mathscr{P}_+(X)\cap\mathscr{G}_c(\mathscr{V}_c)$ follows from Theorem (48).

(56) **Remarke.** Using these proofs, we can easily prove that $\mathscr{R}\neq\emptyset\Rightarrow Q$ has a potential[3] for the spin-flip processes. In fact, that Q has a potential implies that $\mathscr{R}=\mathscr{G}(\mathscr{V})=\widetilde{\mathscr{G}}\neq\emptyset$. Conversely, if $\mathscr{R}\neq\emptyset$, then we obtain

$$c(u, y\times(\cdot)_{S\setminus\Lambda})\mu(\{y\}\times X(S\setminus\Lambda)\,|\,\mathscr{F}(S\setminus\Lambda))$$
$$=c(u, {}_uy\times(\cdot)_{S\setminus\Lambda})\mu(\{y\}\times X(S\setminus\Lambda)\,|\,\mathscr{F}(S\setminus\Lambda)).\ \mu\text{—a. e.}$$

This implies that there is an x^Λ such that

$$\mu(\{y\}\times X(S\setminus\Lambda)\,|\,\mathscr{F}(S\setminus\Lambda))(x^\Lambda)>0,\ \forall y\in X(\Lambda),$$

so (13) holds, and a potential.

Finally, we discuss the relation between reversibility and quasi-reversibility.

(57) **Theorem.** *Suppose that $c(u, v, \cdot)$ has a finite range.* $r(\{u, v\})$ is the same as (3.15) in [1]. *For every $\Lambda\in\mathscr{S}_f$, we put $\partial\Lambda\triangleq(\setminus\Lambda\bigcup_{u,v\in\Lambda} r(\{u, v\}))$ (See definition* (4.30) in [1]). *Also suppose that there are $\{\Lambda_m\}\subset\mathscr{S}_f$ such that $\Lambda_m=\Lambda_{m-1}\cup\partial\Lambda_{m-1}$, $\Lambda_m\neq\Lambda_{m-1}$, $m\geqslant1$, $\Lambda_m\uparrow S$. Then $\mu\in\mathscr{R}_+$ if and only if μ is a* quasi-*reversible measure for $c(u, v, \cdot)$.*

Proof From Theorem (3.25) in [1], positive reversibility implies quasi-reversibility. (But it is more simple if we use (43) and the proof of Theorem (3.25) in [1] to check the condition for quasi-reversibility.)

Conversely, suppose that μ is a quasi-reversible measure for $c(u, v, \cdot)$, then we have

$$c(u, v, y)\mu_\Lambda(y)=c(u, v, {}_{(u,v)}y)\mu_\Lambda({}_{(u,v)}y),\ y\in X(\Lambda) \tag{58}$$

for every $u, v\in S$ and $\Lambda\supset r\{u, v\}$ (here we use $c(u, v, y)$ instead of $c(u, v, y\times z)$, $z\in X(S\setminus\Lambda)$ arbitrarily).

Take $m\geqslant1$, $u, v\in\Lambda_{m-1}$, $y\in X(\Lambda_{m-1})$ and $f=I_{\{y\}\times X(S\setminus\Lambda)}$, then $r(\{u, v\})\subset\Lambda_m$, and

$$\int c(u, v, x)f(x)\mu(dx)=\int_{\{y\}\times X(\Lambda_m\setminus\Lambda_{m-1})} c(u, v, y\times w)\mu_m(d(y\times w))$$
$$=\sum_{w\in X(\partial\Lambda_{m-1})} c(u, v, y\times w)\mu_m(y\times w),$$

where $\mu_m\triangleq\mu_{\Lambda_m}$. Similarly, we have

$$\int c(u, v, x)f({}_{(u,v)}x)\mu(dx)=\sum_{w\in X(\partial\Lambda_{m-1})} c(u, v, {}_{(u,v)}y\times w)\mu_m({}_{(u,v)}y\times w),$$

so the equation in (43) holds from (58). Note that for every $\Lambda\in\mathscr{S}_f$, there is an $m\geqslant1$ such that $\Lambda\subset\Lambda_{m-1}$. By the preceding proof, it is easy to show that the equation in (43) holds for $f=I_{\{y\}\times X(S\setminus\Lambda)}$, $y\in X(\Lambda)$, $u, v\in\Lambda$, $u\neq v$. Hence (43) holds (See the last part of the proof of Lemma (44)). Therefore $\mu\in\mathscr{R}$, and $\mu\in\mathscr{P}_+(X)$ implies $\mu\in\mathscr{R}_+$.

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一般速度函数的有势性与可逆性

II. 紧空间情形

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摘 要

在这一部分中，我们应用前一部分的方法和结果于紧空间情形.

§5 就紧空间的情形证明了满足(I)中(4.3)的有限程速度函数的拟可逆性等价于有势性(定理(5.1)). 证明了 $\tilde{\mathscr{G}}=\tilde{\mathscr{G}}(\mathscr{V})$, 并推广了与此有关的自旋变相过程的结果(定理(5.2), (5.3)). 还证明了在某些条件下, 拟可逆与可逆的等价性(定理(5.14)). §6 我们着重讨论了排它过程(不限于有限程)的有势性与可逆性, 得到了较完整的结果. 首先给出了有势性的简洁判准(定理(6.18)), 这个判准原则上已不能再改善, 对排它过程, 证明了: 可逆测度必存在(定理(6.50)); 若速度函数有势, 则可逆测度由速度函数构造的典型Gibbs态来刻划(定理(6.48)); 反之, 若有正可逆测度存在, 则它有势且正可逆测度集与正典型 Gibbs 态集相等(定理(6.55)). 对于自旋变相过程的情形, 文中在有关地方注明用本文的方法也可得出[3]的相应结果.