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广义黎曼——哈斯曼问题

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本文研究广义解析函数的广义黎曼——哈斯曼问题，这类问题的简单模型是 Векуа, И. Н.[1] 在研究曲面和壳体理论时提出的，它有着重要的理论和实用价值. 本文提出直接应用广义解析函数预解表达式来研究这类问题的最一般形式,并得到了完整的结果.

问题 ***R*-*H*** 寻求下述方程的分片正则解 $w(z)=\{w^+(z),\ w^-(z)\}$.

$$\frac{\partial w}{\partial \bar{z}}+B(z)\bar{w}=0,\ z\in E \tag{1}$$

假设 $B(z)\in C_\alpha^{n-1}(D^++L)$, $B(z)\in C_\alpha^{n-1}(D^-+L)$, $L\in C_\alpha^{n+1}$, $0<\alpha\leqslant 1$. 且当 $z\to\infty$ 时有 $|B(z)|\leqslant\frac{K}{|z|^{1+\varepsilon}}$, $K>0$, $\varepsilon>0$. 边值条件为

$$\sum_{k=1}^{n}\left\{a_k(t)\ \frac{\partial^k w^+}{\partial t^k}+b_k(t)\ \overline{\frac{\partial^k w^+}{\partial t^k}}\right\}_{t=\alpha(t)}-\sum_{k=1}^{n}\left\{c_k(t)\ \frac{\partial^k w^-}{\partial t^k}+d_k(t)\ \overline{\frac{\partial^k w^-}{\partial t^k}}\right\}=f(t),\ t\in L \tag{2}$$

此处 $a_k(t)$、$b_k(t)$、$c_k(t)$、$d_k(t)$、$f(t)$ 均为 H 类函数. $\alpha(t)$ 为映射 L 到自身的反向同胚映照，$\alpha'(t)\neq 0$ 属于 H 类函数，满足条件

$$\alpha[\alpha(t)]\equiv t,\ t\in L. \tag{3}$$

边值条件的系数满足正则性条件

$$c_n(t)\overline{b_n[\alpha(t)]}-\overline{d_n(t)}a_n[\alpha(t)]\neq 0,\ t\in L. \tag{4}$$

对于 $n=1$ 的情况，我们得到十分精确的结果.

§1. 问 题 *R*-*H*

利用广义解析函数的预解表达式[2,3]

$$w(z)=\Phi(z)+\iint_E\Gamma_1(z,\ t)\Phi(t)dT_t+\iint_E\Gamma_2(z,\ t)\overline{\Phi^-(t)}dT_t, \tag{5}$$

其中 $\Phi(z)$ 为分片全纯函数，和分片正则解之间有一一对应的关系，其预解核具有下述性质

$$\Gamma_1(z,\ t)=\frac{1}{\pi}\overline{B(t)}\Omega_2(z,\ t),\ \Gamma_2(z,\ t)=\frac{1}{\pi}B(t)\Omega_1(z,\ t), \tag{6}$$

$$\Omega_1(z,\ t)=\frac{1}{t-z}+O(\ln|t-z|),\ \Omega_2(z,\ t)=O(\ln|t-z|). \tag{7}$$

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如果记 $[\Phi^+(t)]^{(n)}-t^n[\Phi^-(t)]^{(n)}=\mu(t),\ t\in L$, 那末能写

$$\frac{1}{2\pi i}\int_L\frac{\mu(\tau)d\tau}{\tau-z}=\begin{cases}\Phi^{+(n)}(z), & z\in D^+,\\ z^n\Phi^{-(n)}(z), & z\in D^-.\end{cases}\tag{8}$$

于是最后得

$$\Phi^+(z)=\frac{(-1)^n}{(n-1)!}\frac{1}{2\pi i}\int_L\mu(\tau)(\tau-z)^{n-1}\ln\left(1-\frac{z}{\tau}\right)d\tau+\sum_{k=0}^{n-1}\frac{(C_{2k}+iC_{2k+1})}{k!}z^k,\ z\in D^+,\tag{9}$$

$$\Phi^-(z)=\frac{(-1)^n}{(n-1)!}\frac{1}{2\pi i}\int_L\frac{\mu(\tau)}{\tau^n}\left[(\tau-z)^{n-1}\ln\left(1-\frac{\tau}{z}\right)+\sum_{k=0}^{n-2}\beta_k\tau^{n-k-1}z^k\right]d\tau+C_{2n}+iC_{2n+1},\ z\in D^-.\tag{10}$$

此处 $\Phi^-(\infty)=C_{2n}+iC_{2n+1}$, $\beta_k=\frac{(-1)^{k+1}C_{n-1}^{k+1}}{1}+\frac{(-1)^{k+2}C_{n-1}^{k+2}}{2}+\cdots+\frac{(-1)^{n-1}C_{n-1}^{n-1}}{n-k-1}$, $\ln\left(1-\frac{z}{\tau}\right)$ 取在原点为零的分支, $\ln\left(1-\frac{\tau}{z}\right)$ 取在无穷处为零的分支. 将其代入(5)式, 交换积分次序得

$$w^+(z)=\frac{(-1)^n}{(n-1)!}\frac{1}{2\pi i}\int_L\mu(\tau)(\tau-z)^{n-1}\ln\left(1-\frac{z}{\tau}\right)d\tau+\frac{1}{2\pi i}\int_L H_1(z,\tau)\mu(\tau)d\tau+\frac{1}{2\pi i}\int_L H_2(z,\tau)\overline{\mu(\tau)d\tau}+\sum_{k=0}^{2n+1}c_kw_k^+(z),\ z\in D^+,\tag{11}$$

$$w^-(z)=\frac{(-1)^n}{(n-1)!}\frac{1}{2\pi i}\int_L\frac{\mu(\tau)}{\tau^n}\left[(\tau-z)^{n-1}\ln\left(1-\frac{\tau}{z}\right)+\sum_{k=0}^{n-2}\beta_k\tau^{n-k-1}z^k\right]d\tau+\frac{1}{2\pi i}\int_L H_1(z,\tau)\mu(\tau)d\tau+\frac{1}{2\pi i}\int_L H_2(z,\tau)\overline{u(z)d\tau}+\sum_{k=0}^{2n+1}c_kw_k^-(z),\ z\in D^-,\tag{12}$$

其中

$$H_1(z,\tau)=\frac{(-1)^n}{(n-1)!}\iint_{D^+}\Gamma_1(z,t)(\tau-t)^{n-1}\ln\left(1-\frac{t}{\tau}\right)dT_t+\frac{(-1)^n}{(n-1)!}\iint_{D^-}\Gamma_1(z,t)\left[(\tau-t)^{n-1}\ln\left(1-\frac{\tau}{t}\right)+\sum_{k=0}^{n-2}\beta_k\tau^{n-k-1}t^k\right]dT_k,$$

$$H_2(z,\tau)=\frac{(-1)^{n-1}}{(n-1)!}\iint_{D^+}\Gamma_2(z,t)\overline{(\tau-t)^{n-1}\ln\left(1-\frac{t}{\tau}\right)}dT_t+\frac{(-1)^{n-1}}{(n-1)!}\iint_{D^-}\Gamma_2(z,t)\cdot\overline{\left[(\tau-t)^{n-1}\ln\left(1-\frac{\tau}{t}\right)+\sum_{k=0}^{n-2}\beta_k\tau^{n-k-1}t^k\right]}dT_k,$$

$$k!\,w_{2k}(z)=\begin{cases}z^k+\iint_{D^+}\Gamma_1(z,t)t^kdT_t+\iint_{D^+}\Gamma_2(z,t)\bar t^kdT_t, & z\in D^+,\\ \iint_{D^+}\Gamma_1(z,t)t^kdT_t+\iint_{D^+}\Gamma_2(z,t)\bar t^kdT_t, & z\in D^-,\end{cases}$$

$$k!\,w_{2k+1}(z)=\begin{cases}iz^k+\iint_{D^+}\Gamma_1(z,t)it^kdT_t-\iint_{D^+}\Gamma_2(z,t)i\bar t^kdT_t, & z\in D^+,\\ \iint_{D^+}\Gamma_1(z,t)it^kdT_t-\iint_{D^+}\Gamma_2(z,t)i\bar t^kdT_t, & z\in D^-,\ k=0,1\cdots n-1,\end{cases}$$

$$w_{2n}(z)=\begin{cases}\iint\limits_{D^-}\Gamma_1(z,\ t)dT_t+\iint\limits_{D^-}\Gamma_2(z,\ t)\,dT_t,\ z\in D^+,\\ 1+\iint\limits_{D^-}\Gamma_1(z,\ t)dT_t+\iint\limits_{D^-}\Gamma_2(z,\ t)dT_t,\ z\in D^-.\end{cases}$$

对于(11)、(12)中 $w^+(z)$、$w^-(z)$ 其分别在 D^+、D^- 中正则，且 $w^-(\infty)$ 存在有限. $w^+(z)$、$w(z)$ 的表达式中第一项，其 1 到 $n-2$ 阶导数的核函数是连续的，$n-1$ 阶导数有对数型奇异性，n 阶导数出现柯西型奇异核. 下面证明 $H_2(z,\ \tau)$ 的 1 到 $n-1$ 阶导数所对应的核函数是连续的，n 阶导数具有对数型奇异性. 对 $H_1(z,\ \tau)$ 的 1 到 n 阶导数所对应的核函数均是连续的. 注意到(6)、(7)我们有

$$H_2(z,\ \tau)=\frac{(-1)^{n-1}}{(n-1)!}\frac{1}{\pi}\iint\limits_{D^+}\frac{B(t)}{t-z}\overline{(\tau-t)^{n-1}\ln\left(1-\frac{t}{\tau}\right)}dT_t+\frac{(-1)^{n-1}}{(n-1)!}\frac{1}{\pi}$$
$$\iint\limits_{D^-}\frac{B(t)}{t-z}\left[\overline{(\tau-t)^{n-1}\ln\left(1-\frac{\tau}{t}\right)+\sum_{k=0}^{n-2}\beta_k\tau^{n-k-1}t^k}\right]dT_t+\cdots. \tag{13}$$

讨论积分核奇异性的阶只要讨论前面部分. 为了简便起见记

$$B_1(t,\ \tau)=B(t)\,\overline{(\tau-t)^{n-1}\ln\left(1-\frac{t}{\tau}\right)},$$

$$B_2(t,\ \tau)=B(t)\left[\overline{(\tau-t)^{n-1}\ln\left(1-\frac{\tau}{t}\right)+\sum_{k=0}^{n-2}\beta_k\tau^{n-k-1}t^k}\right]$$

首先讨论(13)的第一项奇异性，为此讨论

$$K_1(z,\ t)=\frac{1}{\pi}\iint\limits_{D^+}\frac{B_1(t,\ \tau)}{t-z}\,dT_t$$

$$\begin{aligned}\frac{\partial K_1}{\partial z}&=\frac{1}{\pi}\iint\limits_{D^+}\frac{B_1(t,\ \tau)}{(t-z)^2}\,dT_t\\ &=-\frac{1}{\pi}\iint\limits_{D^+}\frac{\partial}{\partial t}\left(\frac{1}{t-z}\right)B_1(t,\ \tau)\,dT_t\\ &=-\frac{1}{\pi}\iint\limits_{D^+}\frac{\partial}{\partial t}\left[\frac{B_1(t,\ \tau)}{t-z}\right]dT_t\\ &\quad+\frac{1}{\pi}\iint\limits_{D^+}\frac{1}{t-z}\,\frac{\partial}{\partial t}\,B_1(t,\ \tau)\,dT_t.\end{aligned}$$

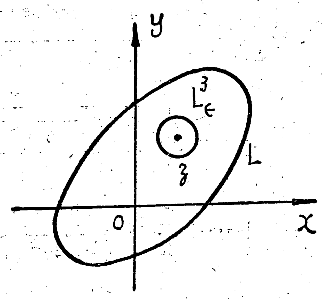

图 1

对于 $z\in D^+$ 的情况，注意到图 1 有

$$-\frac{1}{\pi}\iint\limits_{D^+}\frac{\partial}{\partial t}\left[\frac{B_1(t,\ \tau)}{t-z}\right]dT_t=\frac{1}{2\pi i}\int_L\frac{B_1(\sigma,\ \tau)}{\sigma-z}\,\overline{d\sigma}-\lim_{\varepsilon\to0}\frac{1}{2\pi i}\int_{L_\varepsilon^z}\frac{B_1(\sigma,\ \tau)}{\sigma-z}\,\overline{d\sigma}$$
$$=\frac{1}{2\pi i}\int_L\frac{B_1(\sigma,\ \tau)\,\overline{\sigma'(s)}^2\,d\sigma}{\sigma-z}$$

于是有

$$\frac{\partial K_1}{\partial z}=\frac{1}{2\pi i}\int_L\frac{B_1(\sigma,\ \tau)\overline{\sigma'(s)}^2d\sigma}{\sigma-z}+\frac{1}{\pi}\iint\limits_{D^+}\frac{1}{t-z}\,\frac{\partial}{\partial t}\,B_1(t,\ \tau)\,dT_t, \tag{14}$$

$$\frac{\partial^2K_1}{\partial z^2}=\frac{1}{2\pi i}\int_L\frac{B_1(\sigma,\ \tau)\overline{\sigma'(s)}^2d\sigma}{(\sigma-z)^2}+\frac{1}{\pi}\iint\limits_{D^+}\frac{1}{(t-z)^2}\,\frac{\partial}{\partial t}\,B_1(t,\ \tau)\,dT_t,$$

对上式第二个积分的性质是已知的，现讨论第一个积分，注意到围道 L 的弧长参数表示，$\sigma=\sigma(s)$，$0\leqslant s\leqslant l$.

$$\frac{1}{2\pi i}\int_L\frac{B_1(\sigma,\ \tau)\overline{\sigma'(s)^2}d\sigma}{(\sigma-z)^2}$$

$$=-\frac{1}{2\pi i}\int_L d\left(\frac{B_1(\sigma,\ \tau)\overline{\sigma'(s)^2}}{\sigma-z}\right)+\frac{1}{2\pi i}\int_L\frac{d[B_1(\sigma,\ \tau)\overline{\sigma'(s)^2}]}{\sigma-z}$$

$$=-\frac{1}{2\pi i}\int_0^l\frac{d}{ds}\left\{\frac{B_1[\sigma(s),\ \tau]\overline{\sigma'(s)^2}}{\sigma(s)-z}\right\}ds+\frac{1}{2\pi i}\int_0^l\frac{\frac{d}{ds}\{B_1[\sigma(s),\ \tau]\overline{\sigma'(s)^2}\}}{\sigma(s)-z}ds$$

$$=\frac{1}{2\pi i}\int_L\frac{\overline{\sigma'(s)}\frac{d}{ds}[B_1(\sigma,\ \tau)\overline{\sigma'(s)^2}]d\sigma}{\sigma-z}.$$

这里由于 $L\in C_\alpha^{n+1}$，故 $\left\{\frac{B_1[\sigma(s),\ \tau]\overline{\sigma'(s)^2}}{\sigma(s)-z}\right\}_0^l$ 为零，于是得

$$\frac{\partial^2K_1}{\partial z^2}=\frac{1}{2\pi i}\int_L\frac{\overline{\sigma'(s)}\frac{d}{ds}[B_1(\sigma,\ \tau)\overline{\sigma'(s)^2}]d\sigma}{\sigma-z}+\frac{1}{\pi}\iint_{D^+}\frac{1}{(t-z)^2}\frac{\partial^2}{\partial t^2}B_1(t,\ \tau)dT_t. \tag{15}$$

下面用数学归纳法证明一般公式，设 $n-1$ 阶导数有

$$\frac{\partial^{n-1}K_1}{\partial z^{n-1}}=\frac{1}{2\pi i}\int_L\frac{\overline{\sigma'(s)}\frac{d}{ds}\left\{\overline{\sigma'(s)}\frac{d}{ds}\left\{\cdots\frac{d}{ds}[B_1(\sigma,\ \tau)\overline{\sigma'(s)^2}]\right\}\cdots\right\}d\sigma}{\sigma-z}$$
$$+\frac{1}{2\pi i}\int_L\frac{\overline{\sigma'(s)}\frac{d}{ds}\left\{\overline{\sigma'(s)}\frac{d}{ds}\left\{\cdots\frac{d}{ds}\left\{\overline{\sigma'(s)}\left\{\frac{\partial}{\partial t}B_1(t,\ \tau)\right\}_{t=\sigma}\overline{\sigma'(s)^2}\right\}\cdots\right\}d\sigma}{\sigma-z}$$
$$+\cdots+\frac{1}{\pi}\iint_{D^+}\frac{1}{(t-z)^2}\frac{\partial^{n-1}}{\partial t^{n-1}}B_1(t,\ \tau)dT_t.$$

同样用上述方法将上式最后一项改写为

$$\frac{\partial^{n-1}K_1}{\partial z^{n-1}}=\frac{1}{2\pi i}\int_L\frac{\overline{\sigma'(s)}\frac{d}{ds}\left\{\overline{\sigma'(s)}\frac{d}{ds}\left\{\cdots\frac{d}{ds}[B_1(\sigma,\ \tau)\overline{\sigma'(s)^2}]\right\}\cdots\right\}d\sigma}{\sigma-z}$$
$$+\cdots+\frac{1}{2\pi i}\int_L\frac{\left\{\frac{\partial^{n-2}}{\partial t^{n-2}}B_1(t,\ \tau)\right\}_{t=\sigma}\overline{\sigma'(s)^2}d\sigma}{\sigma-z}$$
$$+\frac{1}{\pi}\iint_{D^+}\frac{1}{(t-z)}\frac{\partial^{n-1}}{\partial t^{n-1}}B_1(t,\ \tau)dT_t.$$

再对 z 求一次导数得

$$\frac{\partial^n K_1}{\partial z^n}=\frac{1}{2\pi i}\int_L \frac{\left\{\overline{\sigma'(s)}\frac{d}{ds}\left\{\overline{\sigma'(s)}\frac{d}{ds}\left\{\cdots\frac{d}{ds}\left[B_1(\sigma,\tau)\overline{\sigma'(s)^2}\right]\right\}\cdots\right\}d\sigma}{(\sigma-z)^2}$$

$$+\cdots+\frac{1}{2\pi i}\int_L \frac{\left\{\frac{\partial^{n-2}}{\partial t^{n-2}}B_1(t,\tau)\right\}_{t=\sigma}\overline{\sigma'(s)^2}\,d\sigma}{(\sigma-z)^2}$$

$$+\frac{1}{\pi}\iint_{D^+}\frac{1}{(t-z)^2}\frac{\partial^{n-1}}{\partial t^{n-1}}B_1(t,\tau)dT_t$$

$$=\frac{1}{2\pi i}\int_L \frac{\overline{\sigma'(s)}\frac{d}{ds}\left\{\overline{\sigma'(s)}\frac{d}{ds}\left\{\overline{\sigma'(s)}\frac{d}{ds}\left\{\cdots\frac{d}{ds}\left[B_1(\sigma,\tau)\overline{\sigma'(s)^2}\right]\right\}\cdots\right\}d\sigma}{\sigma-z}$$

$$+\frac{1}{2\pi i}\int_L \frac{\overline{\sigma'(s)}\frac{d}{ds}\left\{\overline{\sigma'(s)}\frac{d}{ds}\left\{\overline{\sigma'(s)}\frac{d}{ds}\left\{\cdots\frac{d}{ds}\left\{\overline{\sigma'(s)}\left\{\frac{\partial}{\partial t}B_1(t,\tau)\right\}_{t=\sigma}\overline{\sigma'(s)^2}\right\}\cdots\right\}d\sigma}{\sigma-z}$$

$$+\cdots+\frac{1}{2\pi i}\int_L \frac{\overline{\sigma'(s)}\frac{d}{ds}\left\{\left[\frac{\partial^{n-2}}{\partial t^{n-2}}B_1(t,\tau)\right]_{t=\sigma}\overline{\sigma'(s)^2}\right\}d\sigma}{\sigma-z}$$

$$+\frac{1}{\pi}\iint_{D^+}\frac{1}{(t-z)^2}\frac{\partial^{n-1}}{\partial t^{n-1}}B_1(t,\tau)dT_t. \tag{16}$$

对于 $z\in D^-$，上述计算同样是正确的. 由[1, 4]得知 $K_1(z,\tau)$ 直至 $n-1$ 阶导数均连续，其 n 阶导数仅有对数型奇异性. 再讨论(13)中的第二项，为此讨论

$$K_2(z,\tau)=\frac{1}{\pi}\iint_{D^-}\frac{B_2(t,\tau)}{t-z}dT_t,$$

$$\frac{\partial K_2}{\partial z}=\frac{1}{\pi}\iint_{D^-}\frac{B_2(t,\tau)}{(t-z)^2}dT_t$$

$$=-\frac{1}{\pi}\iint_{D^-}\frac{\partial}{\partial t}\left[\frac{B_2(t,\tau)}{t-z}\right]dT_t+\frac{1}{\pi}\iint_{D^-}\frac{1}{t-z}\frac{\partial}{\partial t}B_2(t,\tau)dT_t.$$

对于 $z\in D^-$ 的情况注意到图(2)有

$$-\frac{1}{\pi}\iint_{D^-}\frac{\partial}{\partial t}\left[\frac{B_2(t,\tau)}{t-z}\right]dT_t$$

$$=-\frac{1}{2\pi i}\int_L\frac{B_2(\sigma,\tau)\overline{d\sigma}}{\sigma-z}$$

$$+\lim_{R\to+\infty}\frac{1}{2\pi i}\int_{L_R}\frac{B_2(\sigma,\tau)\overline{d\sigma}}{\sigma-z}$$

$$-\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{L_\varepsilon^z}\frac{B_2(\sigma,\tau)\overline{d\sigma}}{\sigma-z}$$

$$=-\frac{1}{2\pi i}\int_L\frac{B_2(\sigma,\tau)\overline{\sigma'(s)^2}d\sigma}{\sigma-z}.$$

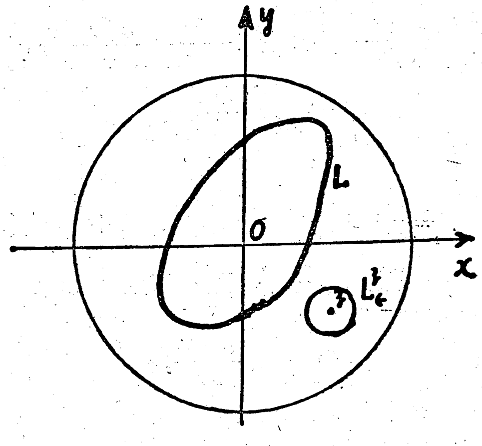

图 2

同理可得

$$\frac{\partial K_2}{\partial z}=-\frac{1}{2\pi i}\int_L \frac{B_2(\sigma,\tau)\overline{\sigma'(s)^2}d\sigma}{\sigma-z}+\frac{1}{\pi}\iint_{D^-}\frac{1}{t-z}\frac{\partial}{\partial t}B_2(t,\tau)dT_t,$$

$$\frac{\partial^2 K_2}{\partial z^2}=-\frac{1}{2\pi i}\int_L \frac{\overline{\sigma'(s)}\frac{d}{ds}[B_2(\sigma,\tau)\overline{\sigma'(s)^2}]d\sigma}{\sigma-z}+\frac{1}{\pi}\iint_{D^-}\frac{1}{(t-z)^2}\frac{\partial}{\partial t}B_2(t,\tau)dT_t.$$

和上面一样用数学归纳法可以证明

$$\begin{aligned}\frac{\partial^n K_2}{\partial z^n}=&-\frac{1}{2\pi i}\int_L \frac{\overline{\sigma'(s)}\frac{d}{ds}\left\{\overline{\sigma'(s)}\frac{d}{ds}\left\{\cdots\frac{d}{ds}[B_2(\sigma,\tau)\overline{\sigma'(s)^2}]\right\}\cdots\right\}d\sigma}{\sigma-z}\\&-\frac{1}{2\pi i}\int_L \frac{\overline{\sigma'(s)}\frac{d}{ds}\left\{\overline{\sigma'(s)}\frac{d}{ds}\left\{\cdots\frac{d}{ds}\left\{\overline{\sigma'(s)}\left\{\frac{\partial}{\partial t}B_2(t,\sigma)\right\}_{t=\sigma}\overline{\sigma'(s)^2}\right\}\cdots\right\}\right\}d\sigma}{\sigma-z}\\&-\cdots-\frac{1}{2\pi i}\int_L \frac{\overline{\sigma'(s)}\frac{d}{ds}\left\{\left[\frac{\partial^{n-2}}{\partial t^{n-2}}B_2(t,\tau)\right]_{t=\sigma}\overline{\sigma'(s)^2}\right\}\cdots\Big\}d\sigma}{\sigma-z}\\&+\frac{1}{\pi}\iint_{D^-}\frac{1}{(t-z)^2}\frac{\partial^{n-1}}{\partial t^{n-1}}B_2(t,\tau)dT_t.\end{aligned}$$

对于 $z\in D^+$ 上述计算同样是正确的. 由[1, 4]可以得出相应的结论. 对 $H_1(z,\tau)$ 经类似的讨论得知其1到 n 阶导数所对应的核函数均是连续的.

下面写出表达式

$$\left.\begin{aligned}&\frac{\partial^k w^+}{\partial z^k}=\int_L \overset{(k)}{K}_1(z,\tau)\mu(\tau)d\tau+\int_L \overset{(k)}{K}_2(z,\tau)\overline{\mu(\tau)d\tau}+\sum_{i=0}^{2n+1}c_i\frac{\partial^k w_i^+(z)}{\partial z^k},\ k=0,1\cdots n-1,\\&\frac{\partial^n w^+}{\partial z^n}=\frac{1}{2\pi i}\int_L\frac{\mu(\tau)d\tau}{\tau-z}+\int_L \overset{(n)}{K}_1(z,\tau)\mu(\tau)d\tau+\int_L \overset{(n)}{K}_2(z,\tau)\overline{\mu(\tau)d\tau}+\sum_{i=0}^{2n+1}c_i\frac{\partial^n w_i^+}{\partial z^n}\end{aligned}\right\}\tag{17}$$

其中 $\overset{(k)}{K}_1(z,\tau)$、$\overset{(k)}{K}_2(z,\tau)$, $k=0, 1\cdots n$ 是在 $\tau=z$ 附近只有弱奇性的正则核. 故有

$$\left.\begin{aligned}&\frac{\partial^k w^+}{\partial t^k}=\int_L \overset{(k)}{K}_1(t,\tau)\mu(\tau)d\tau+\int_L \overset{(k)}{K}_2(t,\tau)\overline{\mu(\tau)d\tau}+\sum_{i=0}^{2n+1}c_i\frac{\partial^k w_i^+}{\partial t^k},\ k=0,1\cdots n-1,\\&\frac{\partial^n w^+}{\partial t^n}=\frac{1}{2}\mu(t)+\frac{1}{2\pi i}\int_L\frac{\mu(\tau)d\tau}{\tau-t}+\int_L \overset{(n)}{K}_1(t,\tau)\mu(\tau)d\tau+\int_L \overset{(n)}{K}_2(t,\tau)\overline{\mu(\tau)d\tau}\\&\qquad+\sum_{i=0}^{2n+1}c_i\frac{\partial^n w_i^+}{\partial t^n},\end{aligned}\right\}\tag{18}$$

$$\left.\begin{aligned}&\frac{\partial^k w^-}{\partial z^k}=\int_L \overset{(k)}{K}_3(z,\tau)\mu(\tau)d\tau+\int_L \overset{(k)}{K}_4(z,\tau)\overline{\mu(\tau)d\tau}+\sum_{i=0}^{2n+1}c_i\frac{\partial^k w_i^-}{\partial z^k},\ k=0,1\cdots n-1,\\&\frac{\partial^n w^-}{\partial z^n}=\frac{1}{z^n 2\pi i}\int_L\frac{\mu(\tau)d\tau}{\tau-z}+\int_L \overset{(n)}{K}_3(z,\tau)\mu(\tau)d\tau+\int_L \overset{(n)}{K}_4(z,\tau)\overline{\mu(\tau)d\tau}\\&\qquad+\sum_{i=0}^{2n+1}c_i\frac{\partial^n w_i^-}{\partial z^n},\end{aligned}\right\}\tag{19}$$

其中 $\overset{(k)}{K}_3(z,\tau)$、$\overset{(k)}{K}_4(z,\tau)$, $k=0, 1\cdots n$ 是在 $\tau=z$ 附近最多只有弱奇性的正则核, 故有

$$\left.\begin{aligned}\frac{\partial^k w^-}{\partial t^k}&=\int_L \overset{(k)}{K}_3(t,\tau)\mu(\tau)d\tau+\int_L \overset{(k)}{K}_4(t,\tau)\overline{\mu(\tau)d\tau}+\sum_{i=0}^{2n+1}c_i\frac{\partial^k w_i^-}{\partial t^k},\ k=0,1\cdots n-1,\\ \frac{\partial^n w^-}{\partial t^n}&=-\frac{\mu(t)}{2t^n}+\frac{1}{t^n 2\pi i}\int_L\frac{\mu(\tau)d\tau}{\tau-t}+\int_L \overset{(n)}{K}_3(t,\tau)\mu(\tau)d\tau+\int_L \overset{(n)}{K}_4(t,\tau)\overline{\mu(\tau)d\tau}\\ &\quad+\sum_{i=0}^{2n+1}c_i\frac{\partial^n w_i^-}{\partial t^n}.\end{aligned}\right\}\tag{20}$$

将表达式(18)、(20)代入(2)经整理后得

$$\begin{aligned}K\mu\equiv&\frac{c_n(t)}{2}\frac{\mu(t)}{2}+\frac{d_n(t)}{\bar t^n}\frac{\overline{\mu(t)}}{2}+\frac{a_n[\alpha(t)]}{2}\mu[\alpha(t)]+\frac{b_n[\alpha(t)]}{2}\overline{\mu[\alpha(t)]}\\ &-\frac{c_n(t)}{t^n}\frac{1}{2\pi i}\int_L\frac{\mu(\tau)d\tau}{\tau-t}-\frac{d_n(z)}{\bar t^n}\overline{\frac{1}{2\pi i}\int_L\frac{\mu(\tau)d\tau}{\tau-t}}+\frac{a_n[\alpha(t)]}{2}\frac{1}{\pi i}\int_L\frac{\mu(\tau)d\tau}{\tau-\alpha(t)}\\ &+\frac{b_n[\alpha(t)]}{2}\overline{\frac{1}{\pi i}\int_L\frac{\mu(\tau)d\tau}{\tau-\alpha(t)}}+\int_L K_1(t,\tau)\mu(\tau)d\tau+\int_L K_2(t,\tau)\overline{\mu(\tau)d\tau}\\ &+\int_L K_3[\alpha(t),\tau]\mu(\tau)d\tau+\int_L K_4[\alpha(t),\tau]\overline{\mu(\tau)d\tau}=H(t),\ t\in L,\end{aligned}\tag{21}$$

其中 $K_i(t,\tau)$, $i=1、2、3、4$ 是 Fredholm 核. $H(t)=\sum_{k=0}^{2n+1}c_k g_k(t)+f(t),\ t\in L$,

$$\begin{aligned}g_k(t)=&-\sum_{i=0}^{n}\left\{a_i(t)\frac{\partial^i w_k^+}{\partial t^i}+b_i(t)\overline{\frac{\partial^i w_k^+}{\partial t^i}}\right\}_{t=\alpha(z)}+\sum_{i=0}^{n}\left\{c_i(t)\frac{\partial^i w_k^-}{\partial t^i}+d_i(t)\overline{\frac{\partial^i w_k^-}{\partial t^i}}\right\},\\ &k=0,1\cdots 2n+1.\end{aligned}\tag{22}$$

这样得到的问题(1)、(2)的解 $w(z)$ 由公式(5)、(8)所决定的密度函数 $\mu(t)$ 满足奇异积分方程(21). 反之方程(21)的解由公式(9), (10), (5)所确定的 $w(z)$ 为问题(1)、(2)的解. 对于这种含有共轭和位移的奇异积分方程已有作者[5]建立的 Noether 理论. 下面我们不具体解此方程而用其讨论原问题(1)、(2)的可解条件和解的个数.

为此考虑下述奇异积分算子所确定的齐次奇异积分方程

$$\begin{aligned}K_\nu\mu\equiv&\frac{c_n(t)}{t^n}\frac{\mu(t)}{2}+\frac{d_n(t)}{\bar t^n}\frac{\overline{\mu(t)}}{2}+\nu\frac{a_n[\alpha(t)]}{2}\mu[\alpha(t)]+\nu\frac{b_n[\alpha(t)]}{2}\overline{\mu[\alpha(t)]}\\ &-\frac{c_n(t)}{t^n}\frac{1}{2\pi i}\int_L\frac{\mu(\tau)d\tau}{\tau-t}-\frac{d_n(t)}{\bar t^n}\overline{\frac{1}{2\pi i}\int_L\frac{\mu(\tau)d\tau}{\tau-t}}+\nu\frac{a_n[\alpha(t)]}{2}\frac{1}{\pi i}\int_L\frac{\mu(\tau)d\tau}{\tau-\alpha(t)}\\ &+\frac{\nu b_n[\alpha(t)]}{2}\overline{\frac{1}{\pi i}\int_L\frac{\mu(\tau)d\tau}{\tau-\alpha(t)}}+\int_L K_1(t,\tau)\mu(\tau)d\tau+\int_L K_2(t,\tau)\overline{\mu(\tau)d\tau}\\ &+\nu\int_L K_3[\alpha(t),\tau]\mu(\tau)d\tau+\nu\int_L K_4[\alpha(t),\tau]\overline{\mu(\tau)d\tau}=0,\ t\in L,\ \nu=\pm1,\end{aligned}\tag{23}$$

k_+, k_- 分别为 $k_+\mu=0$, $k_-\mu=0$ 的线性无关解数. $k_+\mu=H$, 即为方程(21), 其对应的共轭齐次方程为

$$\begin{aligned}K'_\nu\varphi\equiv&\frac{c_n(t)}{t^n}\varphi(t)+\frac{\overline{d_n(t)t'^2(s)}}{t^n}\overline{\varphi(t)}-\nu a_n(t)\alpha'(t)\varphi[\alpha(t)]\\ &-\nu\overline{b_n(t)t'^2(s)\alpha'(t)\varphi[\alpha(t)]}+\frac{1}{\pi i}\int_L\frac{c_n(\tau)\varphi(\tau)d\tau}{\tau^n(\tau-t)}+\frac{1}{\pi i}\int_L\overline{\frac{d_n(\tau)\tau'^2(\sigma)\varphi(\tau)d\tau}{\tau^n(\tau-t)}}\end{aligned}$$

$$+\frac{\nu}{\pi i}\int_L \frac{a_n(\tau)\alpha'(\tau)\varphi[\alpha(\tau)]d\tau}{\tau-t}+\frac{\nu}{\pi i}\int_L \frac{\overline{b_n(\tau)\tau'^2(\sigma)\alpha'(\tau)\varphi[\alpha(\tau)]d\tau}}{\tau-t}$$
$$+\int_L K_1(\tau,t)\varphi(\tau)d\tau+\int_L \overline{K_2(\tau,t)\tau'^2(\sigma)\varphi(\tau)}d\tau$$
$$-\nu\int_L K_3(\tau,t)\alpha'(\tau)\varphi[\alpha(\tau)]d\tau-\nu\int_L \overline{K_4(\tau,t)\alpha'(\tau)\tau'^2(\sigma)\varphi[\alpha(\tau)]}d\tau$$
$$=0,\ t\in L,\ \nu=\pm1, \tag{24}$$

k'_+、k'_- 分别为其线性完全无关解数. 利用[5]中的结果，注意到这里 t 和 $\alpha(t)$ 反向，故有

$$A(t)-B(t)=\begin{bmatrix} \frac{c_n(t)}{t^n} & 0 & a_n[\alpha(t)] & 0 \\ \frac{\overline{d_n(t)}}{t^n} & 0 & \overline{b_n[\alpha(t)]} & 0 \\ 0 & b_n(t) & 0 & \frac{d_n[\alpha(t)]}{\alpha(t)^n} \\ 0 & \overline{a_n(t)} & 0 & \frac{\overline{c_n[\alpha(t)]}}{\alpha(t)^n} \end{bmatrix},$$

$$A(t)+B(t)=\begin{bmatrix} 0 & \frac{d_n(t)}{\bar{t}^n} & 0 & b_n[\alpha(t)] \\ 0 & \frac{\overline{c_n(t)}}{\bar{t}^n} & 0 & \overline{a_n[\alpha(t)]} \\ a_n(t) & 0 & \frac{c_n[\alpha(t)]}{\alpha(t)^n} & 0 \\ b_n(t) & 0 & \frac{\overline{d_n[\alpha(t)]}}{\alpha(t)^n} & 0 \end{bmatrix},$$

$$\det[A(t)-B(t)]$$
$$=\frac{-1}{t^n\alpha(t)^n}[c_n(t)\overline{b_n[\alpha(t)]}-\overline{d_n(t)}a_n[\alpha(t)]][b_n(t)\overline{c_n[\alpha(t)]}-\overline{a_n(t)}d_n[\alpha(t)]],$$

$$\det[A(t)+B(t)]$$
$$=\frac{-1}{\bar{t}^n\alpha(t)^n}[d_n(t)\overline{a_n[\alpha(t)]}-\overline{c_n(t)}b_n[\alpha(t)]][a_n(t)\overline{d_n[\alpha(t)]}-c_n[\alpha(t)]\overline{b_n(t)}].$$

问题的指数定义为

$$\varkappa=\operatorname{In}d_L\left\{\frac{\det[A(t)-B(t)]}{\det[A(t)+B(t)]}\right\}=\operatorname{In}d_L\left\{\left(\frac{\bar{t}^n\alpha(t)^n}{t^n\alpha(t)^n}\right)\left(\frac{c_n(t)\overline{b_n[\alpha(t)]}-\overline{d_n(t)}a_n[\alpha(t)]}{\overline{c_n(t)}b_n[\alpha(t)]-d_n(t)\overline{a_n[\alpha(t)]}}\right),\right.$$
$$\left.\left(\frac{b_n(t)\overline{c_n[\alpha(t)]}-\overline{a_n(t)}d_n[\alpha(t)]}{\overline{b_n(t)}c_n[\alpha(t)]-a_n(t)\overline{d_n[\alpha(t)]}}\right)\right\}=-4n+4\varkappa_1 \tag{25}$$

其中 $\varkappa_1=\operatorname{In}d_L\{c_n(t)\overline{b_n[\alpha(t)]}-\overline{d_n(t)}a_n[\alpha(t)]\}$.

对奇异积分方程(21)建立下述诸定理

定理 1 问题(1)、(2)当系数满足条件(4)时，奇异积分方程(21)是 Noether 型的.

定理 2 对齐次方程 $K_+\mu=0$ 其线性完全无关解数 $k_+=4(-n+\varkappa_1)+k'_++k'_--k_-$,

且当 $K_{-}\mu=0$ 的解数 k_{-} 小于 $4(-n+\varkappa_1)+k'_{+}+k'_{-}$ 时齐次方程 $K_{+}\mu=0$ 总有非零解.

定理 3　非齐次方程 $K_{+}\mu=H(t)$ 可解的充要条件是

$$\operatorname{Re}\int_L H(t)\varphi_j(t)dt=0,\quad j=1,\ 2\cdots k'_{+},\tag{26}$$

此处 $\varphi_j(t)$, $j=1$, $2\cdots k'_{+}$ 是共轭方程 $K'_{+}\varphi=0$ 线性完全解组.

对方程(21)来说(26)的形式为

$$\operatorname{Re}\int_L\left[f(t)+\sum_{k=0}^{2n+1}c_k g_k(t)\right]\varphi_j(t)dt=0,\quad j=1,\ 2\cdots k'_{+}.\tag{27}$$

记 $\operatorname{Re}\int_L g_k(t)\varphi_j(t)dt=g_{k_j}$, $-\operatorname{Re}\int_L f(t)\varphi_j(t)dt=f_j$, 则方程(27)可写为

$$\sum_{k=0}^{2n+1}c_k g_{k_j}=f_j,\quad j=1,\ 2\cdots k'_{+}.\tag{28}$$

记矩阵 $\|g_{k_j}\|$ 的秩为 r, 显然 $r\leqslant\min\{k'_{+1},\ 2n+2\}$.

利用上述关于奇异积分方程诸定理可得下述边值问题(1)、(2)的可解性条件和解的个数.

定理 4　非齐次问题(1)、(2)可解的充要条件是方程(28)系数矩阵当增加常数列 $\{f_j\}$ 形成增广矩阵时其秩不变.

此定理的另外形式是

$$\operatorname{Re}\int_L\lambda_m(t)f(t)dt=0,\quad m=1,\ 2\cdots k'_{+}-r,\tag{29}$$

其中 $\lambda_m(t)=\varphi_{r+m}+\sum_{i=1}^{r}a_{mi}\varphi_i(t)$, $\varphi_i(t)$, $i=1$, $2\cdots k'_{+}$ 为方程 $K'_{+}\varphi=0$ 的线性完全无关解, $\{a_{mi}\}$ 为确定的常数组.

定理 5　非齐次问题(1)、(2)对任意的右端均可解的充要条件是存在广义解析函数序列 $\{w_n^{+}(z)$、$w_n^{-}(z)\}$ 将其代入(2)的左端得 $\{f_n(t)\}$, 它在连续函数类中按实系数组合意义下是完备的.

定理 6　齐次问题(1)、$(2)^0$ 有 $2(n+1)+k_{+}-r$ 个线性完全无关解.

由于 $r\leqslant\min\{k'_{+},\ 2n+1\}$, 故 $2(n+1)+k_{+}-r\geqslant0$. 本定理具有确切的意义.

定理 7　齐次问题(1)、$(2)^0$ 恒有 $2\varkappa_1+2(1-n)+k'_{+}+k'_{-}-k_{-}-r$ 个线性无关解, 且当 $k_{-}<4\varkappa_1+2(1-n)+k'_{+}+k'_{-}-r$ 时恒有解.

§2. Векуа, И. Н. 模型的精确结果

Векуа, И. Н. 在[1]中提出研究 $n=1$ 的问题 R-H. 对此模型问题, 能得到十分精确的结果, 此处 R-H 边值条件的形式为

$$\left\{a_1(t)\frac{\partial w^{+}}{\partial t}+b_1(t)\overline{\frac{\partial w^{+}}{\partial t}}+a_0(t)w^{+}(t)+b_0(t)w^{-}(t)\right\}_{t=\alpha(z)}$$

$$=c_1(t)\frac{\partial w^{-}}{\partial t}+d_1(t)\overline{\frac{\partial w^{-}}{\partial t}}+c_0(t)w^{-}(t)+d_0(t)\overline{w^{-}(t)}+f(t),\quad t\in L,\tag{30}$$

其系数满足正则性条件

$$\overline{a_1[\alpha(t)]}c_1(t)-b_1[\alpha(t)]\overline{d_1(t)}\neq 0,\ t\in L. \tag{31}$$

由[6]知存在保角映射 $\zeta=\omega^+(z)$, $\zeta=\omega^-(z)$ 分别属于 $C^1_\alpha(\overline{D}^+)$、$C^1_\alpha(\overline{D}^-)$. 将域 $D^\pm$ 单叶映射为域 E^+、E^-, 由围道 Γ 分开, 其境界值满足边值条件: $\omega^+[\alpha(t)]=\omega^-(t)$, $t\in L$. $\omega^-(t)$ 由下述积分方程决定, 可用逐次逼近法求得

$$\omega^-(t)-\frac{1}{2\pi i}\int_L\left\{\frac{\alpha'(\tau)}{\alpha(\tau)-\alpha(t)}-\frac{1}{\tau-t}\right\}\omega^-(\tau)d\tau=t, \tag{32}$$

用上述结果求出 $\omega^-(t)$ 随之得 $\omega^+(t)=\omega^-[\beta(t)]$, 此处 $\beta(t)$ 是 $\alpha(t)$ 的反函数.

$$\omega^-(z)=z-\frac{1}{2\pi i}\int_L\frac{\omega^-(\tau)d\tau}{\tau-z},\ z\in D^+, \tag{33}$$

$$\omega^+(z)=\frac{1}{2\pi i}\int_L\frac{\omega^-[\beta(\tau)]d\tau}{\tau-z},\ z\in D^-. \tag{34}$$

记 $\omega(z)$ 的反函数 $\varphi(\zeta)$, 对原问题(34)进行变量代换, 引入新的分片正则广义解析函数

$$w_1^+(\zeta)=w^+[\varphi(\zeta)],\ \zeta\in E^+,\ w_1^-(\zeta)=w^-[\varphi(\zeta)],\ \zeta\in E^-. \tag{35}$$

它满足下述方程和边值条件

$$\frac{\partial w_1}{\partial\bar\zeta}+\hat{B}(\zeta)\overline{w_1}=0,\ \zeta\in E.$$

$$\hat{a}_1(t)\frac{\partial w_1^+}{\partial t}+\hat{b}_1(t)\overline{\frac{\partial w_1^+}{\partial t}}+\hat{a}_0(t)w_1^+(t)+\hat{b}_0(t)\overline{w_1^+(t)}$$
$$=\hat{c}_1(t)\frac{\partial w_1^-}{\partial t}+\hat{d}_1(t)\overline{\frac{\partial w_1^-}{\partial t}}+\hat{c}_0(t)w_1^-+\hat{d}_0(t)\overline{w_1^-(t)}+\hat{f}(t),\ t\in\Gamma, \tag{36}$$

此处 $\hat{B}(\zeta)=\overline{\varphi'(\zeta)}B[\varphi(\zeta)]$ 分别在 E^+、E^- 上属于 C_α, 且在无穷远处有估计式

$$|\hat{B}(\zeta)|\leqslant\frac{K}{|\zeta|^{1+\varepsilon}},\ K>0,\ \varepsilon>0.$$

$$\begin{aligned}
&\hat{a}_1(t)=\frac{a_1[\alpha(\sigma(t)]}{\varphi'(t)},\\
&\hat{a}_0(t)=a_0[\alpha(\sigma(t))],\\
&\hat{b}_1(t)=\frac{b_1[\alpha(\sigma(t))]}{\overline{\varphi'(t)}},\\
&\hat{b}_0(t)=b_0[\alpha(\sigma(t))],\\
&\hat{c}_1(t)=c_1[\sigma(t)]\omega^{-1}[\sigma(t)],\\
&\hat{d}_1(t)=d_1[\sigma(t)]\overline{\omega^{-1}[\sigma(t)]},\\
&\hat{c}_0(t)=c_0[\sigma(t)],\\
&\hat{d}_0(t)=d_0[\sigma(t)],\\
&\hat{f}(t)=f[\sigma(t)],\ t\in\Gamma,
\end{aligned} \tag{37}$$

此处 $\sigma(t)$ 为 $\omega^-(t)$ 的反函数. 利用广义解析函数的预解表达式

$$w_1(\zeta)=\Phi(\zeta)+\iint_E\hat{\Gamma}_1(\zeta,t)\Phi(t)dT_t+\iint_E\hat{\Gamma}_2(\zeta,t)\overline{\Phi(t)}dT_t, \tag{38}$$

$$\Phi^+(\zeta)=-\frac{1}{2\pi i}\int_\Gamma\mu(\tau)\ln\left(1-\frac{\zeta}{\tau}\right)d\tau+c_1+ic_2,\ \zeta\in E^+, \tag{39}$$

$$\Phi^-(\zeta)=-\frac{1}{2\pi i}\int_\Gamma\frac{\mu(\tau)}{\tau}\ln\left(1-\frac{\tau}{\zeta}\right)d\tau+c_3+ic_4,\ \zeta\in E^-. \tag{40}$$

于是我们有

$$\left.\begin{aligned}
&w_1^+(t)=\int_\Gamma \overset{(1)}{n}_1(t,\tau)\mu(\tau)d\tau+\int_\Gamma \overset{(1)}{n}_2(t,\tau)\overline{\mu(\tau)d\tau}+\sum_{i=1}^4 c_i\hat{w}_i^+(t),\\
&\frac{\partial w_1^+}{\partial t}=\frac{1}{2}\mu(t)+\frac{1}{2\pi i}\int_\Gamma\frac{\mu(\tau)d\tau}{\tau-t}+\int_\Gamma \overset{(2)}{n}_1(t,\tau)\overline{\mu(\tau)d\tau}+\int_\Gamma \overset{(2)}{n}_2(t,\tau)\overline{\mu(\tau)d\tau}\\
&\qquad+\sum_{i=1}^4 c_i\frac{\partial\hat{w}_i^+}{\partial t},\\
&w_1^-(t)=\int_\Gamma \overset{(3)}{n}_1(t,\tau)\mu(\tau)d\tau+\int_\Gamma \overset{(3)}{n}_2(t,\tau)\overline{\mu(\tau)d\tau}+\sum_{i=1}^4 c_i\hat{w}_i^-(t),\\
&\frac{\partial w_1^-}{\partial t}=-\frac{1}{2}\frac{\mu(t)}{t}+\frac{1}{2\pi it}\int_\Gamma\frac{\mu(\tau)d\tau}{\tau-t}+\int_\Gamma \overset{(4)}{n}_1(t,\tau)\mu(\tau)d\tau\\
&\qquad+\int_\Gamma \overset{(4)}{n}_2(t,\tau)\overline{\mu(\tau)d\tau}+\sum_{i=1}^4 c_i\frac{\partial\hat{w}_i^-}{\partial t},
\end{aligned}\right\}\quad(41)$$

$\overset{(k)}{n}_i(t,\tau)$, $k=1\sim4$, $i=1$、2 为确定的 Fredholm 正则核.

将(41)代入(36)经整理后可得

$$\begin{aligned}
K\mu\equiv{}&\alpha_1(t)\mu(t)+\alpha_2(t)\overline{\mu(t)}+\beta_1(t)\frac{1}{\pi i}\int_\Gamma\frac{\mu(\tau)d\tau}{\tau-t}+\beta_2(t)\overline{\frac{1}{\pi i}\int_\Gamma\frac{\mu(\tau)d\tau}{\tau-t}}\\
&+\int_\Gamma K_1(t,\tau)\mu(\tau)d\tau+\int_\Gamma K_2(t,\tau)\overline{\mu(\tau)d\tau}=f(t)+\sum_{i=1}^4 c_ig_i(t),\ t\in\Gamma.
\end{aligned}\quad(42)$$

其中

$$\alpha_1(t)=\frac{1}{2}\left\{\hat{a}_1(t)+\frac{\hat{c}_1(t)}{t}\right\},\qquad \alpha_2(t)=\frac{1}{2}\left\{\hat{b}_1(t)+\frac{\hat{d}_1(t)}{\bar{t}}\right\},$$

$$\beta_1(t)=\frac{1}{2}\left\{\hat{a}_1(t)-\frac{\hat{c}_1(t)}{t}\right\},\qquad \beta_2(t)=\frac{1}{2}\left\{\hat{b}_1(t)-\frac{\hat{d}_1(t)}{\bar{t}}\right\},$$

$K_i(t,\tau)$为确定的正则核，$g_i(t)$为确定的 H 类函数.

方程(42)的共轭齐次方程为

$$\begin{aligned}
K'\varphi\equiv{}&\alpha_1(t)\varphi(t)+\overline{t'^2(s)\alpha_2(t)\varphi(t)}-\frac{1}{\pi i}\int_\Gamma\frac{\beta_1(\tau)\varphi(\tau)d\tau}{\tau-t}-\frac{1}{\pi i}\int_\Gamma\frac{\overline{\tau'^2(\sigma)\beta_2(\tau)\varphi(\tau)d\tau}}{\tau-t}\\
&+\int_\Gamma K_1(\tau,t)\varphi(\tau)d\tau+\int_\Gamma K_2(\tau,t)\overline{\tau'^2(\sigma)\varphi(\tau)}d\tau=0,\ t\in L,
\end{aligned}\quad(43)$$

对方程(42)来说，条件(31)成立，此方程是正则的. 指数 $\varkappa=\operatorname{In}d\left\{\dfrac{\det(P-Q)}{\det(P+Q)}\right\}$起着重要的作用，其中

$$P=\begin{bmatrix}\alpha_1(t), & \alpha_2(t)\\ \overline{\alpha_2(t)}, & \overline{\alpha_1(t)}\end{bmatrix},\qquad Q=\begin{bmatrix}\beta_1(t), & -\beta_2(t)\\ \overline{\beta_2(t)}, & -\overline{\beta_1(t)}\end{bmatrix},$$

$\det(P-Q)=\dfrac{1}{t}[\overline{\hat{a}_1(t)}\hat{c}_1(t)-\hat{b}_1(t)\overline{\hat{d}_1(t)}]$, $\det(P+Q)=\dfrac{1}{\bar{t}}[\hat{a}_1(t)\overline{\hat{c}_1(t)}-\overline{\hat{b}_1(t)}\hat{d}_1(t)]$. 当 $\overline{\hat{a}_1(t)}\hat{c}_1(t)-\hat{b}_1(t)\overline{\hat{d}_1(t)}\neq0(t\in\Gamma)$，则问题(36)是正则的，于是

$$\varkappa=-2+2\operatorname{In}d_\Gamma\{\overline{\hat{a}_1(t)}\hat{c}_1(t)-\hat{b}_1(t)\overline{\hat{d}_1(t)}\}=-2+2\varkappa_1.\quad(44)$$

注意到关系式(37)可得 $\varkappa_1=\operatorname{In}d_L\{\overline{a_1[\alpha(t)]}c_1(t)-b_1[\alpha(t)]\overline{d_1(t)}\}$. 由(31)推出正则条件是满足的，因此奇异积分方程(42)、(43)的 Noether 理论成立.

定理 8 记 k、k' 分别为方程 $K\mu=0$, $K'\varphi=0$ 的解数，则下述关系式成立

$$k-k'=-2+2\varkappa_1\quad(45)$$

定理 9 非齐次方程 $K\mu=f(t)+\sum\limits_{k=1}^4 c_kg_k(t)$ 可解的充要条件是

$$\mathrm{Re}\int_{\Gamma}\left\{f(t)+\sum_{i=1}^{4}c_i g_i(t)\right\}\varphi_j(t)dt=0,\quad j=1、2\cdots k', \tag{46}$$

$\varphi_j(t)$为方程 $K'\varphi=0$ 的解. 记 $\mathrm{Re}\int_{\Gamma}g_i(t)\varphi_j(t)dt=g_{ij}$. $\|g_{ij}\|$的秩为 r,

利用上述理论，可以建立边值问题(30)的可解性条件和解个数的精确结果.

定理 10　齐次问题(30)恒有 $4+k-r$ 个线性无关解，且其个数不小于 $2(1+\varkappa_1)$.

定理 11　在正则情况下，当 $\mathrm{In}\,d_\Gamma G(t)>-1$ 时，齐次问题总有非零解，问题无非零解的情况仅可能在 $\mathrm{In}\,d_L G(t)\leqslant 1$ 的情况. 此处 $G(t)=\overline{a_1[\alpha(t)]}c_1(t)-b_1[\alpha(t)]\overline{d_1(t)}$. 本定理在几何及壳体理论中有着重要的价值.

类似的可以建立与§1 中相当的其他定理. 且这些结果更为精确.

参 考 文 献

[1] Векуа, И. Н., Обобщенные аналитически функции, физматгиз. 1959.
[2] Михайлов, Л. Г., *Изв. Высш. Уч. Зав. Матем*, **5** (1960), 99—109.
[3] Михайлов, Л. Г., *Учен. Зап. Тадж. Ун-Та.* **10** (1957).
[4] Мусхелишвичи, н. и., Сингулярные интегральные уравнения, ФМ, 1962.
[5] 袁益让，山东大学学报，**2**(1965)，1—26.
[6] Симонко И. Б., *Изв. АНСССР, Сер, Мат,* **28**: 2 (1964).

THE GENERALIZED RIEMANN–HASEMAN PROBLEM

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ABSTRACT

In this paper we study the generalized Riemann–Haseman problem which was given by Vekua, I. N.

Problem (***R–H***). Find a sectionally generalized holomorphic function $w(z)=\{w^+(z),\ w^-(z)\}$ such that

$$\frac{\partial w}{\partial \bar z}+B(z)\bar w=0,\ z\in E.$$

Here $B(z)\in C_\alpha^{n-1}(D^++L)$, $B(z)\in C_\alpha^{n-1}(D^-+L)$, $L\in C_\alpha^{n-1}$, $0<\alpha\leqslant 1$, $|B(z)|\leqslant\frac{K}{|z|^{1+\varepsilon}}$ $(z\to\infty)$, $K>0$, $\varepsilon>0$; $w(z)$ Satisfies the boundary condition

$$\sum_{k=0}^{n}\left\{a_k(t)\frac{\partial^k w^+}{\partial t^k}+b_k\overline{\frac{\partial^k w^+}{\partial t^k}}\right\}_{t=\alpha(z)}-\sum_{k=0}^{n}\left\{c_k(t)\frac{\partial^k w^-}{\partial t^k}+d_k(t)\overline{\frac{\partial^k w^-}{\partial t^k}}\right\}=f(t),\ t\in L,$$

Where $a_k(t)$、$b_k(t)$、$c_k(t)$、$d_k(t)$、$f(t)\in H$; $\alpha(t)$ is a mapping of L into itself, $\alpha'(t)\neq 0$ and $\alpha[\alpha(t)]\equiv t$.

We study the conditions of the solubility and the number of linearly independent solutions.