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FELLER'S BOUNDARY IN ABSTRACT SPACES

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Abstract

In this paper, most of the Feller's boundaries are extended to abstract spaces, and a general expression of q-processes satisfing the Kolmogorov backward equation is obtained when $q(x)-q(x, A)$ is finite.

Introduction

The Feller's boundary theory on countable state spaces was established by W. Feller in 1956. It has played an important role in the constructions of Q-processes. In this paper, most of Feller's results are extended to abstract spaces. Firstly, the basic Lemma in[2, § 6]is extended to abstract spaces, but the proof is simpler than Feller's. Secondly, a general expression of q-processes satisfying the Kolmogorov backward equation is obtained and it is a preparation for constructing potential q-processes which satisfy the Kolmogorov backward equation when $q(x)-q(x, A)$ is finite. Finally, the criterion for honest q-processes mentioned above is established.

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§ 1. Operation on Lattices

Let $(E, \mathscr{E})$ be an abstract measurable space, and assume that all the singletons $\{x\}$ $(x\in E)$ belong to $\mathscr{E}$. We use the same notations as in [10, § 1], such as $b\mathscr{E}$, $b\,\mathscr{E}_+$, $r\,\mathscr{E}_+$, $\mathscr{L}_+$.

Let function $\pi(x, A)$ satisfy that

i) $\pi(\cdot, A)\in b\mathscr{E}_+$ for A, $\pi(x, \cdot)\in \mathscr{L}_+$ for x,

ii) $0\leqslant\pi(x, A)\leqslant 1$, $\forall x\in E$, $A\in\mathscr{E}$.

The operator π is defined by

$$\pi f(\cdot)=\int_E \pi(\cdot, dx)f(x) \quad (f\in b\mathscr{E}). \tag{1.1}$$

Clearly, for each fixed n, π^n is a bounded linear operator with norm less than or equal to 1.

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Put

$$\mathscr{B}=\{f\in b\mathscr{E}_{+}: \pi f=f,\ 0\leqslant f\leqslant 1\}, \tag{1.2}$$

$$\mathscr{B}^{*}=\{f\in b\mathscr{E}_{+}: \pi f\geqslant f,\ 0\leqslant f\leqslant 1\}, \tag{1.3}$$

$$\mathscr{B}_{*}=\{f\in b\mathscr{E}_{+}: \pi f\leqslant f,\ 0\leqslant f\leqslant 1\}. \tag{1.4}$$

Obviously, $\mathscr{B}$ is convex and closed.

Definition 1.1. *We call g the least upper bound of $\{f_\alpha\in b\mathscr{E},\ \alpha\in I\}$ in $\mathscr{B}$ if $\sup\limits_{\alpha\in I} f_\alpha\leqslant g\in\mathscr{B}$, and if $\sup\limits_{\alpha\in I} f_\alpha\leqslant \bar{g}\in\mathscr{B}$, then $g\leqslant\bar{g}$. We call h the greatest lower bound of $\{f_\alpha\in b\mathscr{E},\ \alpha\in I\}$ in $\mathscr{B}$ if $\inf\limits_{\alpha\in I} f_\alpha\geqslant h\in\mathscr{B}$, and if $\inf\limits_{\alpha\in I} f_\alpha\geqslant \bar{h}\in\mathscr{B}$, then $h\geqslant\bar{h}$. Here I is an arbitrary index set.*

Lemma 1.2. *For any fixed $f\in\mathscr{B}^{*}$ and $x\in E$, the limit*

$$g(x)=\lim_{n\to\infty}(\pi^n f)(x) \tag{1.5}$$

exists and g is the least upper bound of $\{f\}$ in $\mathscr{B}$.

Proof From (1.3) it is easy to prove that $\pi^n f\leqslant\pi^{n+1}f\leqslant 1$, $n=1, 2, \cdots$. Then $\lim\limits_{n\to\infty}(\pi^n f)(x)$ exists and we denote it by $g(x)$. By the dominated convergence theorem, $g\in\mathscr{B}$. If $g_1\in\mathscr{B}$, $g_1\geqslant f$, then by induction we have $g_1\geqslant\pi^n f$, $n=1, 2, \cdots$. Letting $n\to\infty$, from (1.5), we obtain $g_1\geqslant g$.

Similarly, we can prove

Lemma 1.3. *For any fixed $f\in\mathscr{B}_{*}$ and $x\in E$, the limit*

$$h(x)=\lim_{n\to\infty}(\pi^n f)(x)$$

exists and h is the greatest lower bound of $\{f\}$ in $\mathscr{B}$.

Theorem 1.4. *Let $f_1, f_2\in\mathscr{B}$. $\mathscr{B}$ contains a unique least upper bound $f_1\cup f_2$ of $\{f_1, f_2\}$ and a unique greatest lower bound $f_1\cap f_2$ of $\{f_1, f_2\}$.*

Proof For each x, put $f(x)=\max\{f_1(x), f_2(x)\}$. Clearly $f\in\mathscr{B}^{*}$ and the function g defined by (1.5) has the properties required by $f_1\cup f_2$. Put

$$h(x)=\min\{f_1(x), f_2(x)\},$$

then from Lemma 1.3 the assertion follows.

In the rest of the section, we assume that all the functions f, g, $f+g$, $f\cup g$ and so on, whenever they appear, belong to $\mathscr{B}$.

Using the method in [1, Lemma 4.1], it is not difficult to prove

Proposition 1.5. *If*

$$f_1+g_1=f_2+g_2=h, \tag{1.6}$$

then

$$f_1\cup f_2+g_1\cap g_2=h. \tag{1.7}$$

Particularly, we have

$$f_1+f_2=f_1\cup f_2+f_1\cap f_2. \tag{1.8}$$

Using the method in [1, Lemma 4.2], it is not difficult to prove

Proposition 1.6.

$$(f \cap g) \cup h \leqslant (f \cup h) \cap (g \cup h), \tag{1.9}$$
$$(f \cup g) \cap h \geqslant (f \cap h) \cup (g \cap h), \tag{1.10}$$
$$(f \cap g) + h \leqslant (f+h) \cap (g+h), \tag{1.11}$$
$$(f \cup g) + h \geqslant (f+h) \cup (g+h) \tag{1.12}$$
$$(f+g) \cap h \leqslant (f \cap h) + (g \cap h), \tag{1.13}$$
$$(f+g) \cup h \leqslant (f \cup h) + (g \cup h). \tag{1.14}$$

If $f \cap g = 0$, *then*

$$(f+g) \cap h = (f \cap h) + (g \cap h). \tag{1.15}$$

If $0 \leqslant \lambda \leqslant 1$, *then*

$$\lambda(f \cap g) = (\lambda f) \cap (\lambda g), \tag{1.16}$$
$$\lambda(f \cup g) = (\lambda f) \cup (\lambda g). \tag{1.17}$$

§ 2. Sojourn sets and sojourn solutions

Obviously $1 \in \mathscr{B}_*$. From Lemma 1.3 we know that for each $x \in E$ the limit

$$S_E(x) = \lim_{n \to \infty} (\pi^n 1)(x) \tag{2.1}$$

exists and S_E belongs to $\mathscr{B}$. From the fact that $0 \leqslant f \leqslant 1$ for every $f \in \mathscr{B}$, by induction we have $f \leqslant \pi^n 1$, $n = 1, 2, \cdots$. This proves

Lemma 2.1. *The function S_E defined by* (2.1) *is the maximal element of $\mathscr{B}$, that is, $S_E \in \mathscr{B}$ and $f \leqslant S_E$ for each $f \in \mathscr{B}$.*

If $\pi(\cdot, E) = 1$, then $S_E \equiv 1$. Conversely if $S_E \equiv 1$, then $1 \geqslant \pi^n 1 \downarrow 1 (n \to \infty)$, therefore $\pi(\cdot, E) \equiv 1$. This proves

Lemma 2.2. *$S_E \equiv 1$ if and only if $\pi(\cdot, E) = 1$.*

Now let A be an arbitrary set in $\mathscr{E}$ and let $\mathscr{E} \cap A = \{B \subset A : B \in \mathscr{E}\}$. Applying the above argument to the restriction π_A of π to $(A, \mathscr{E} \cap A)$ and letting $\pi_A^n(x, A) = \pi_A^n 1$ we know that the limit function

$$\rho_A(x) = \lim_{n \to \infty} \pi_A^n(x, A), \ (x \in A)$$

exists and satisfies

$$\rho_A = \pi_A \rho_A, \tag{2.2}$$
$$0 \leqslant \rho_A \leqslant 1. \tag{2.3}$$

Obviously ρ_A is the maximal element of $\mathscr{B}_A$, where

$$\mathscr{B}_A \triangleq \{f \in b(\mathscr{E} \cap A) : \pi_A f = f, \ 0 \leqslant f \leqslant 1\}.$$

Put

$$\sigma_A(x) = \begin{cases} \rho_A(x), & x \in A, \\ 0, & x \in E \setminus A, \end{cases} \tag{2.4}$$

For $x \in A$

$$\int \pi(x, dy)\sigma_A(y) = \int_A \pi(x, dy)\sigma_A(y) + \int_{E\setminus A} \pi(x, dy)\sigma_A(y) = \int_A \pi_A(x, dy)\sigma_A(y)$$
$$= \rho_A(x) = \sigma_A(x),$$

and for $x \in E\setminus A$

$$\int \pi(x, dy)\sigma_A(y) \geqslant 0 = \sigma_A(x),$$

therefore $\sigma_A \in \mathscr{B}^*$. Applying Lemma 1.2 to $f = \sigma_A$, we see that the limit function

$$S_A(x) = \lim_{n\to\infty} (\pi^n \sigma_A)(x) \quad (x \in E) \tag{2.5}$$

exists and is the least upper bound of $\{\sigma_A\}$ in $\mathscr{B}$. Partioulary $\sigma_E = S_E$.

Definition 2.3. *A set $A \in \mathscr{E}$ is called sojourn set if $S_A \not\equiv 0$ (or what amounts to the same, if $\rho_A \not\equiv 0$).*

Obviously, for A to be a sojourn set it is necessary and sufficient that there exists an $x \in A$ and an $\eta > 0$ such that

$$\pi_A^n(x, A) > \eta \tag{2.6}$$

for all n.

Definition 2.4. *Two sets A and B are equivalent if $S_A = S_B$.*

Definition 2.5. *The set of all functions S_A and 0 is be denoted by $\mathscr{C}$. The elements of $\mathscr{C}$ are referred to sojourn solutions.*

Using the method in [1, Lemma 5.1] it is not difficult to prove

Lemma 2.6. *If A and B are non-overlapping, then*

$$S_A \cap S_B = 0, \tag{2.7}$$

$$S_A \cup S_B = S_A + S_B = S_{A\cup B}. \tag{2.8}$$

Let A be a sojourn set and $B \supset A$, $A, B \in \mathscr{E}$. It is easy to prove that the limit function $\lim_{n\to\infty} \pi_B^n \bar{\rho}_A(x)$ $(x \in B)$ exists, where

$$\bar{\rho}_A(x) = \begin{cases} \rho_A(x), \text{ for } x \in A, \\ 0, \text{ for } x \in B\setminus A. \end{cases}$$

$$S_A^B(x) \triangleq \begin{cases} \lim_{n\to\infty} \pi_B^n \bar{\rho}_A(x), & x \in B, \\ 0, & x \in E\setminus B. \end{cases} \tag{2.9}$$

Obviously $S_A = S_A^E$, $\sigma_A = S_A^A$ and

$$\sigma_A \leqslant S_A^B \leqslant S_A. \tag{2.10}$$

Lemma 2.7. *We have*

$$S_A(x) = \lim_{n\to\infty} (\pi^n S_A^B)(x), \ (A \subset B, \ x \in E). \tag{2.11}$$

If $A \subset B$, $C \subset B$ and $S_A^B = S_C^B$, then $S_A = S_C$.

Proof Premultiply (2.10) by π^n. The right side remains unchanged, and (2.5) leads to this lemma.

Lemma 2.8. *Let $A \in \mathscr{E}$ and $f \in \mathscr{B}_*$. For each n and each $x \in A$ we have*

$$f(x) \geqslant \pi_A^n f(x) + \left[\sum_{\nu=0}^{n-1} \pi_A^\nu \int_{E\setminus A} \pi(\cdot, dy) f(y)\right](x). \tag{2.12}$$

If $f \in \mathscr{B}$, then the equality sign holds in (2.12).

Proof For $n=1$ the relation (2.12) reduces to $f \geqslant \pi f$. Assume that (2.12) holds for some n. Using the fact that $f \geqslant \pi f$, we have

$$\begin{aligned}\pi_A^n f(x) &\geqslant \left[\pi_A^n \int_E \pi(\cdot, dy) f(y)\right](x) \\ &= \pi_A^n \left[\int_A \pi(\cdot, dy) f(y) + \int_{E\setminus A} \pi(\cdot, dy) f(y)\right](x) \\ &= \pi_A^{n+1} f(x) + \left[\pi_A^n \int_{E\setminus A} \pi(\cdot, dy) f(y)\right](x). \end{aligned} \tag{2.13}$$

Substituting this into the first term of the right in (2.12), we get the assertion (2.12) with n replaced by $n+1$. When $f=\pi f$, each of the above inequalities is replaced by an equality, and the lemma is proved.

Choosing in particular $f \equiv 1$ we get

Corollary 1. *For $x \in A$*

$$\pi_A^n(x, A) + \left[\sum_{\nu=0}^{n-1} \pi_A^\nu \cdot \pi(\cdot, E\setminus A)\right](x) \leqslant 1. \tag{2.14}$$

If π is strictly stochastic, then the equality sign holds.

Letting $n \to \infty$ we get

Corollary 2. *For $x \in A$*

$$\sigma_A(x) + \left[\sum_{\nu=0}^{\infty} \pi_A^\nu \cdot \pi(\cdot, E\setminus A)\right](x) \leqslant 1. \tag{2.15}$$

Using the method in [1, Lemma 7.2], it is not difficult to prove

Lemma 2.9. *Let f be a bounded solution of the equation*

$$\pi g = g, \quad g \in b\mathscr{E}_+ \tag{2.16}$$

and let $\|f\| > 0$. For fixed $0 < \eta \leqslant \|f\|$ put

$$F_\eta = \{x \in E: f(x) \geqslant \|f\| - \eta\}. \tag{2.17}$$

Then F_η is a sojourn set. If

$$\delta/\eta < \varepsilon, \quad \delta > 0, \tag{2.18}$$

then

$$\sigma_{F_\eta}(x) > 1 - \varepsilon, \text{ for } x \in F_\delta. \tag{2.19}$$

Using the method in [1, Lemma 7.3] it is easy to prove

Lemma 2.10. *If A is a sojourn set, then $\|\sigma_A\| = 1$ and $\|S_A\| = 1$.*

Using the method in [1, Lemma 7.4], it is easy to prove

Lemma 2.11. *With the notations of Lemma 2.9 one has*

$$f \geqslant (\|f\| - \eta) S_{F_\eta}. \tag{2.20}$$

Using the method in [1, Theorem 1.8] it is easy to prove

Theorem 2.12. *For any sojourn set A and $0 < \eta < 1$ put*

$$A_\eta = \{x \in A: S_A(x) > 1 - \eta\} \tag{2.21}$$

and

$$\bar{A}_\eta=\{x\in A\colon \sigma_A(x)>1-\eta\}. \tag{2.22}$$

Then

$$S_A=S_{A_\eta}=S_{\bar{A}_\eta}.$$

Definition 2.13. *A sojourn set is called representative set if there exists some* $\eta>0$ *such that* $S_A(x)>1-\eta$ *for each* $x\in A$.

From Theorem 2.12 we have

Lemma 2.14. *Each sojourn set* A *contains an equivalent subset* $\bar{A}$ *which is representative.*

Using the methods in [1, Lemma 8.2] it is easy to prove

Lemma 2.15. *Let* A *be representative, and* $A\subset B$. *Then*

$$S_B=S_A+S_{B\setminus A}. \tag{2.23}$$

Using the methods in [1, Theorem 9.1] it is easy to prove

Theorem 2.16. *For an element* $f\in\mathscr{B}$ *to be a sojourn solution it is necessary and sufficient that*

$$f\cap(S_E-f)=0. \tag{2.24}$$

Equivalently it is necessary and sufficient that for any sojourn set C

$$f\geqslant tS_C,\ t>0 \text{ implies } f\geqslant S_C. \tag{2.25}$$

Using the methods in [1, Theorem 9.2] it is easy to prove

Theorem 2.17. *If* A *and* B *are sojourn sets, then*

$$S_{A\cap B}=S_A\cap S_B. \tag{2.26}$$

Using the methods in [1, Theorem 9.3], it is easy to prove

Theorem 2.18. *Let* X, Y *and* X_n *be sojourn solutions* (elements of $\mathscr{C}$), *then*

i) $X\cap Y\in\mathscr{C}$,

ii) $Y-X\in\mathscr{C}$ *provided* $X\leqslant Y$,

iii) $X\cup Y\in\mathscr{C}$,

iv) $X+Y\in\mathscr{C}$ *provided* $X\cap Y=0$,

v) *if either* $X_n\downarrow u$ *or* $X_n\uparrow u$, *then* $u\in\mathscr{C}$.

Using the methods in [1, Theorem 10], it is easy to prove

Theorem 2.19. *In order that an element* $X\in\mathscr{B}$ may *be a sojourn solution, it is necessary and sufficient that the relations*

$$X=tU+(1-t)V,\ 0<t<1,\ U,\ V\in\mathscr{B}, \tag{2.27}$$

imply $U=V=X$.

§ 3. The exit boundaries

Let $q(x)-q(x, A)$ $(x\in E,\ A\in\mathscr{E})$ be a totally stable q-pair [10, Definition 1.1]. For each $\lambda>0$ put

$$\pi(\lambda, x, A)=q(x, A)[\lambda+q(x)]^{-1}(x\in E,\ A\in\mathscr{E}) \tag{3.1}$$

and

$$\mathscr{B}_\lambda=\left\{f\in b\mathscr{E}_+: \int\pi(\lambda,\ \cdot,\ dy)f(y)=f(\cdot)0\leqslant f\leqslant 1\right\}. \tag{3.2}$$

Put

$$H=\{x\in E: q(x)>0\},$$
$$\pi(x,\ A)=\begin{cases} q(x,\ A)\backslash q(x),\ x\in H, \\ 0,\ x\in E\backslash H, \end{cases} \tag{3.3}$$

and

$$\mathscr{B}=\left\{f\in b\mathscr{E}_+: \int\pi(\cdot,\ dx)f(x)=f(\cdot),\ \ 0\leqslant f\leqslant 1\right\}. \tag{3.4}$$

Obviously, for each $\lambda>0$, $f\in\mathscr{B}_\lambda$

$$f(x)=\int q(x,\ dy)\,[\lambda+q(x)]^{-1}f(y)\leqslant\int q^{-1}(x)q(x,\ dy)f(y)\leqslant 1\ \ (x\in H)$$

$$f(x)=0=\int\pi(x,\ dy)f(y)\ \ (x\in E\backslash H).$$

Therefore $\mathscr{B}_\lambda\subset\mathscr{B}^*$. From Lemma 1.2 we know that there exists a function g such that

$$g(x)=\lim_{n\to\infty}\pi^n f(x),\ \forall x\in E \tag{3.5}$$

and g is the least upper bound of $\{f\}$ in $\mathscr{B}$. Clearly $\pi^{n+1}f\geqslant\pi^n f$, $n=1,\ 2,\ \cdots$.

Definition 3.1. *The canonical map of $\mathscr{B}_\lambda$ into $\mathscr{B}$ is the map which sends the element $f\in\mathscr{B}_\lambda$ into the element $g\in\mathscr{B}$ defined by* (3.5).

Lemma 3.2. *The canonical image f_0 of the element $f_\lambda\in\mathscr{B}_\lambda$ is given by*

$$f_0(\cdot)=f_\lambda(\cdot)+\lambda\sum_{a=0}^{\infty}\int_H\pi^a(\cdot,\ dy)\,[f_\lambda(y)/g(y)]. \tag{3.6}$$

Proof If $x\in E\backslash H$, then both sides of (3.6) vanish. We want to prove that for $x\in H$

$$f_\lambda(x)+\lambda\sum_{a=0}^{n-1}\int_H\pi^a(x,\ dy)f_\lambda(y)q^{-1}(y)=\int\pi^n(x,\ dy)f_\lambda(y). \tag{3.7}$$

From the fact that $f_\lambda\in\mathscr{B}_\lambda$, we get

$$f_\lambda(y)+[\lambda q^{-1}(y)]f_\lambda(y)=\int\pi(y,\ dz)f_\lambda(z). \tag{3.8}$$

Regarding $\pi^a(x,\ \cdot)$ as a measure in A and taking integrals for both sides of (3.8), we obtain

$$\int_H\pi^a(x,\ dy)f_\lambda(y)+\lambda\int_H\pi^a(x,\ dy)\,[f_\lambda(y)q^{-1}(y)]=\int_H\pi^{a+1}(x,\ dy)f_\lambda(y).$$

Summing for a from 0 to $n-1$, we obtain (3.7). Letting $n\to\infty$ in both sides of (3.7), we know that (3.6) holds for $x\in H$.

Lemma 3.3. *Let $f_0\in\mathscr{B}$. For each $\lambda>0$, put*

$$f_\lambda(\cdot)=f_0(\cdot)-\lambda\int p^{\min}(\lambda,\ \cdot,\ dy)f_0(y), \tag{3.9}$$

where $p^{\min}(\lambda, x, A)$ $(\lambda>0, x\in E, A\in\mathscr{E})$ *is defined by* [10, Lemma 1.4]. *Then* f_λ *is the greatest lower bound of* $\{f_0\}$ *in* $\mathscr{B}_\lambda$.

Proof From the fact that $f_0\in\mathscr{B}$, for each $x\in H$, we have

$$0\leqslant\int\pi(\lambda, x, dy)f_0(y)\leqslant\int\pi(x, dy)f_0(y)=f_0(x),$$

and for each $x\in E\setminus H$, we have

$$\int\pi(\lambda, x, dy)f_0(y)=0=f_0(x),$$

hence $\mathscr{B}\subset\mathscr{B}_{\lambda*}$. From Lemma 1.3 we know that the limit

$$\lim_{n\to\infty}\int\pi^n(\lambda, \cdot, dy)f_0(y) \tag{3.10}$$

exists and it is the greatest lower bound of $\{f_0\}$ in $\mathscr{B}_\lambda$. Put

$$\left.\begin{aligned}&P^0(\lambda, x, A)=\delta(x, A)[\lambda+q(x)]^{-1},\\&P^n(\lambda, x, A)=\sum_{a=0}^{n}\int\pi^a(\lambda, x, dy)P^0(\lambda, y, A),\\&n=1, 2, \cdots.\end{aligned}\right\} \tag{3.11}$$

It follows that

$$\begin{aligned}\delta(x, A)&=\int P^n(\lambda, x, dy)[\lambda+q(y)]\delta(y, A)-\int P^{n-1}(\lambda, x, dy)q(y, A)\\&=\int P^{n-1}(\lambda, x, dy)[\lambda+q(y)]\delta(y, A)\\&\quad+\int\pi^n(\lambda, x, dy)[\lambda+q(y)]^{-1}\int\delta(y, dz)[\lambda+q(z)]^{1}\delta(z, A)\\&\quad-\int P^{n-1}(\lambda, x, dy)q(y, A)\\&=\lambda P^{n-1}(\lambda, x, A)+\int_A P^{n-1}(\lambda, x, dy)q(y)+\pi^n(\lambda, x, A)\\&\quad-\int_H P^{n-1}(\lambda, x, dy)q(y, A).\end{aligned} \tag{3.12}$$

Regarding both sides as measures in A and taking integrals for f_0, from the fact that $f_0\in\mathscr{B}$ we have

$$\begin{aligned}f_0(x)=\lambda\int P^{n-1}(\lambda, x, dy)f_0(y)+\int_H P^{n-1}(\lambda, x, dy)q(y)f_0(y)\\+\int\pi^n(\lambda, x, dy)f_0(y)-\int_H P^{n-1}(\lambda, x, dy)q(y)\\\int q^{-1}(y)q(y, dz)f_0(z)=\lambda\int P^{n-1}(\lambda, x, dy)f_0(y)+\int\pi^n(\lambda, x, dy)f_0(y).\end{aligned} \tag{3.13}$$

Letting $n\to\infty$, from [4, Theorem 4.1] and [6, Appendix, Lemma 11] we obtain (3.9).

Definition 3.4. *The typical map of* $\mathscr{B}$ *into* $\mathscr{B}_\lambda$ *is the map which sends the element* $f_0\in\mathscr{B}$ *into the element* $f_\lambda\in\mathscr{B}_\lambda$ *defined by* (3.9).

Lemma 3.5. *Let* $f_0\in\mathscr{B}$. *Then the typical images* $f_\lambda(\lambda>0)$ *satisfy*

$$f_\lambda(\cdot)-f_\nu(\cdot)=(\nu-\lambda)\int P^{\min}(\lambda, \cdot, dy)f_\nu(y)=(\nu-\lambda)\int P^{\min}(\nu, \cdot, dy)f_\lambda(y)\ (\lambda, \nu>0). \tag{3.14}$$

Proof From (3.9) and the fact that $P^{\min}(\lambda, x, A)$ satisfy the resolvent equation, we get the first equality in (3.14). Exchanging λ and ν, we obtain the second equality in (3.14).

Corollary. *$f_\lambda \equiv 0$ for some $\lambda > 0$ is equivalent to $f_\lambda \equiv 0$ for each $\lambda > 0$. For each fixed $x \in E$, $f_\lambda(x)$ $(\lambda > 0)$ is a continuous function on $(0, \infty)$.*

Definition 3.6. *A function $f \in \mathscr{B}$ is called passive if for some, and consequently for each $\lambda > 0$*

$$f(x) = \lambda \int P^{\min}(\lambda, x, dy) f(y) \quad (x \in E). \tag{3.15}$$

Lemma 3.7. *Let $f_\lambda \in \mathscr{B}_\lambda$ and let $f_0 \in \mathscr{B}$ be the canonical image of f_λ. Then*

$$f_\lambda(\cdot) = f_0(\cdot) - \lambda \int P^{\min}(\lambda, \cdot, dy) f_0(y). \tag{3.16}$$

Proof Put

$$f'_\lambda(\cdot) = f_0(\cdot) - \lambda \int P^{\min}(\lambda, \cdot, dy)\, f_0(y).$$

From Lemma 3.3 we know that f'_λ is the greatest upper bound of $\{f_0\}$ in $\mathscr{B}_\lambda$. From (3.6) we get $f_\lambda \leqslant f_0$, hence $f_\lambda \leqslant f'_\lambda$. By induction

$$\int \pi^n(\cdot, dy) f_\lambda(y) \leqslant \int \pi^n(\cdot, dy) f'_\lambda(y) \leqslant f_0.$$

Letting $n \to \infty$ we obtain that f_0 is the common canonical image of f_λ and f'_λ. Thus $f'_\lambda - f_\lambda \in \mathscr{B}_\lambda$ and his canonical image is zero. From (3.6) we have $f_\lambda = f'_\lambda$.

Lemma 3.8. *For an $f_0 \in \mathscr{B}$ to be the canonical image of*

$$f_\lambda(\cdot) \triangleq f_0(\cdot) - \lambda \int P^{\min}(\lambda, \cdot, dy) f_0(y),$$

it is necessary and sufficient that there exists no passive non-null $g \in \mathscr{B}$ such that $g \leqslant f_0$.

Proof From Lemma 3.3 we know that f_λ is the greatest lower bound of $\{f_0\}$ in $\mathscr{B}_\lambda$. Let f'_0 be the canonical image of f_λ. From Lemma. 3.7 we have

$$(f_0 - f'_0)(\cdot) = \lambda \int P^{\min}(\lambda, \cdot, dy)\,(f_0 - f'_0)(y).$$

Clearly f'_0 is the least upper bound f_λ in $\mathscr{B}$, and $f_0 \geqslant f_\lambda$. Hence $0 \leqslant f_0 - f'_0 \leqslant f_0$. This proves that $f_0 - f'_0$ is passive. If there isn't any non-null passive g such that $g \leqslant f_0$, then $f_0 - f'_0 = 0$, thus $f_0 = f'_0$. Conversely, if there exists a passive non-null $g \in \mathscr{B}$ such that $g \leqslant f_0$, then we have

$$f_\lambda(\cdot) = (f_0 - g)(\cdot) - \lambda \int P^{\min}(\lambda, \cdot, dy)\,(f_0 - g)(y),$$

hence $f_0 - g \geqslant f_\lambda$. We will denote the canonical image of f_λ by f'_0. From the fact that f'_0 is the least upper bound of $\{f_\lambda\}$ in $\mathscr{B}$, we obtain that $f_0 - g \geqslant f'_0$. From the fact that $g \not\equiv 0$, we know that $f_0 \neq f'_0$, that is, f_0 is not the canonical image of f_λ.

Corollary. *The range of the canonical mapping from $\mathscr{B}_\lambda$ to $\mathscr{B}$ is independent of λ.*

Put

$$\overline{X}_\lambda(\cdot)=\lim_{n\to\infty}\pi^n(\lambda,\ \cdot,\ E). \tag{3.17}$$

We will denote the canonical image of $\overline{X}_\lambda$ by $\overline{X}$. From Corollary to Lemma 3.8, it is not difficult to prove that $\overline{X}$ is the maximal element in the range of cononical mapping, therefore it is independent of λ. We will denote the set of sojourn solution in $\mathscr{B}_\lambda(\mathscr{B})$ by $\mathscr{C}_\lambda(\mathscr{C})$.

Theorem 3.9.

i) *For each $\lambda>0$, the range of the canonical mapping of $\mathscr{B}_\lambda$ coinsides with*

$$\overline{\mathscr{B}}=\{f\in\mathscr{B}: f\leqslant\overline{X}\}. \tag{3.18}$$

ii) *$\overline{\mathscr{B}}(\overline{\mathscr{B}}\cap\mathscr{C})$ is the isomorphic lattice of $\mathscr{B}_\lambda(\mathscr{C}_\lambda)$.*

Proof i) Let f be an element in the range of the canonical mapping. From the fact that $\overline{X}$ is the maximal element of the range of the canonical mapping, we obtain that $f\in\overline{\mathscr{B}}$. Conversely, let $f\in\overline{\mathscr{B}}$ not be canoniosl image. From Lemma 3.8 we know that there exists a non-null passive g such that $g\leqslant f$, hence $g\leqslant\overline{X}$. This contradicts that $\overline{X}$ is a canonical image, therefore f is a canonical image.

ii) From (3.16) and i), it is easy to prove that the canonical mapping is a one to one mapping. Denoting the canonical images of $f^1_\lambda,\ f^2_\lambda\in\mathscr{B}_\lambda$ and $f^1_\lambda\cup f^2_\lambda$ by $f^1_0,\ f^2_0$ and f_0 respectively, we have

$$\begin{aligned}f_0(\cdot)&=\lim_{n\to\infty}\int\pi^n(\cdot,\ dy)\lim_{m\to\infty}\int\pi^m(\lambda,\ y,\ dz)\cdot[\max(f^1_\lambda,\ f^2_\lambda)](z)\\&\geqslant\lim_{n\to\infty}\int\pi^n(\cdot,\ dy)\lim_{m\to\infty}\int\pi^m(\lambda,\ y,\ dz)f^i_\lambda(z)\\&=\lim_{n\to\infty}\int\pi^n(\cdot,\ dy)f^i_\lambda(y)=f^i_0(\cdot),\ i=1,\ 2,\end{aligned}$$

hence

$$f_0\geqslant f^1_0\cup f^2_0. \tag{3.19}$$

Conversely we have

$$\int\pi^n(\lambda,\ \cdot,\ dy)\max\{f^1_\lambda,\ f^2_\lambda\}(y)\leqslant\int\pi^n(\cdot,\ dy)\max\{f^1_0,\ f^2_0\}(y).$$

Letting $n\to\infty$ we obtain

$$f^1_\lambda\cup f^2_\lambda\leqslant f^1_0\cup f^2_0.$$

From the fact that f_0 is the least upper bound of $\{f^1_\lambda\cup f^2_\lambda\}$ in $\mathscr{B}$, we obtain $f_0\leqslant f^1_0\cup f^2_0$. Denoting the canonical image of $f^1_\lambda\cap f^2_\lambda$ by f'_0, from (1.8) we obtain

$$f'_0=(f^1_0+f^2_0)-(f^1_0\cup f^2_0)=f^1_0\cap f^2_0.$$

From Theorem 2.16, (2.25), it is not difficult to prove that $\mathscr{C}\cap\overline{\mathscr{B}}$ is the isomorphic lattice of $\mathscr{C}_\lambda$.

Theorem 3.10. (Basic Lemma). *Let $f^t_0\in\overline{\mathscr{B}}(t\in T)$ be a family of non-null* sojourn *solutions. If for $t\neq s$ $f^t_0\cap f^s_0=0$, then for every fixed $t\in T$ and $\lambda>0$ there exist x^t_n, $n=1,\ 2,\ \cdots$, such that*

$$f_\lambda^s(x_n^t)\to\begin{cases}1, & s=t,\\ 0, & s\neq t,\end{cases}\quad n\to\infty, \tag{3.20}$$

where $f_\lambda^t\in\mathscr{C}_\lambda$ *is the typical image* of f_0^t.

Proof From Theorem 2.12, we know that for every $t\in T$ there exist $x_n^t\in E$, $n=1, 2, \cdots$, such that

$$f_n^t(x_n^t)>n/(n+1),\ n=1, 2, \cdots.$$

From Lemma 3.5, we know that $f_\lambda^t\leqslant f_{\lambda'}^t$ for $\lambda\geqslant\lambda'$. Hence for every fixed $\lambda>0$, for every positive integer $n>\lambda$ we have

$$1\geqslant f_\lambda^t(x_n^t)\geqslant f_n^t(x_n^t)>n/(n+1)\to 1,\ n\to\infty.$$

From Theorem 2.18 iv) and the fact that for $s\neq t$, $f_0^s\cap f_0^t=0$, we obtain that for $s\neq t$, $f_0^s+f_0^t$ is a non-null sojourn solution. Therefore for every $\lambda>0$ we have

$$0\leqslant f_\lambda^s(x_n^t)\leqslant 1-f_\lambda^t(x_n^t)\to 0,\ n\to\infty.$$

From the fact that $P^{\min}(\lambda, x, A)$ satisfy the resolvent equation, we have

$$Z_\lambda(\cdot)-Z_\nu(\cdot)+(\lambda-\nu)\int P^{\min}(\lambda, \cdot, dy)Z_\nu(y)=0, \tag{3.21}$$

where

$$Z_\lambda(\cdot)\triangleq 1-\lambda P^{\min}(\lambda, \cdot, E). \tag{3.22}$$

Hence for every fixed $x\in E$, $\lambda P^{\min}(\lambda, \cdot, E)\leqslant\lambda' P^{\min}(\lambda', \cdot, E)$ for $\lambda>\lambda'>0$. By the norm condition we have $1\geqslant\lambda P^{\min}(\lambda, x, E)\geqslant 0$, therefore the limit function

$$X^0(\cdot)=\lim_{\lambda\downarrow 0}\lambda P^{\min}(\lambda, \cdot, E) \tag{3.23}$$

exists.

Letting $\nu\to 0$ in (3.21), from (3.22), (3.23) and the dominated convergence theorem, it is easy to prove

Lemma 3.11.

$$\lambda\int P^{\min}(\lambda, \cdot, dy)X^0(y)=X^0(\cdot)\ (\lambda>0). \tag{3.24}$$

Lemma 3.12. *Let* $f_0\in\overline{\mathscr{B}}$, *for each* $\lambda>0$, *denote the typical image of* f_0 *in* $\mathscr{B}$ *by* f_λ. *Then*

$$f_\lambda\uparrow f_0\quad(\lambda\downarrow 0). \tag{3.25}$$

Proof From (3.14) we obtain $f_\lambda\uparrow$ $(\lambda\downarrow 0)$. From (3.9) we have

$$f_0'\triangleq\lim_{\lambda\downarrow 0}f_\lambda\leqslant f_0. \tag{3.26}$$

Obviously $0\leqslant f_0\leqslant 1$ and for $x\in E\setminus H$

$$\int\pi(x, dy)f_0'(y)=0=f_0'(x).$$

From the dominated convergence theorem and the fact that $f_\lambda\in\mathscr{B}_\lambda$, we have for $x\in H$

$$\int\pi(x, dy)f_0'(y)=\lim_{\lambda\downarrow 0}[\lambda+q(x)]q^{-1}(x)\cdot\int\pi(\lambda, x, dy)f_\lambda(y)=f_0'(x).$$

Hence $f_0'\in\mathscr{B}$ is an upper bound of $\{f_\lambda\}$ in $\mathscr{B}$. From (3.26) and the fact that f_0 is the least upper bound of $\{f_\lambda\}$ in $\mathscr{B}$, we obtain $f_0=f_0'$.

Lemma 3.13. *If* $\mu, \mu_n \in \mathscr{L}_+$, $n=1, 2, \cdots$, $\mu_n \uparrow \mu$, $f \in v\mathscr{E}_+$, *then*

$$\lim_{n\to\infty}\int \mu_n(dx)f(x) = \int \mu(dx)f(x).$$

Proof Put $\nu_0=\mu_0$, $\nu_n=\mu_n-\mu_{n-1}(n\geqslant 1)$. From [6, Appendix, Lemma 9] we have

$$\lim_{n\to\infty}\int \mu_n(dx)f(x) = \lim_{n\to\infty}\sum_{p=0}^{n}\int \nu_p(dx)\,f(x) = \int\sum_{p=0}^{\infty}\nu_p(dx)f(x) = \int \mu(dx)f(x).$$

Put

$$X_\lambda^\Delta(\cdot) = \int P^{\min}(\lambda, \cdot, dy)\,[q(y)-q(y, E)]. \tag{3.27}$$

Theorem 3.14. $0\leqslant X_\lambda^\Delta\leqslant 1\,(\lambda>0)$ *and the limit function*

$$X^\Delta(\cdot) = \lim_{\lambda\downarrow 0} X_\lambda^\Delta(\cdot) \tag{3.28}$$

exists. Furthermore

$$\overline{X}_\lambda(x) + \lambda P^{\min}(\lambda, x, E) + \int P^{\min}(\lambda, x, dy)\,[q(y)-q(y, E)] = 1$$

$$(\lambda>0,\ x\in E), \tag{3.29}$$

$$\overline{X} + X^0 + X^\Delta = 1. \tag{3.30}$$

Proof From (3.11) we have

$$\lambda P^n(\lambda, x, E) + \int P^n(\lambda, x, dy)\,[q(y)-q(y, E)] + \pi^{n+1}(\lambda, x, E)$$

$$= \pi^{n+1}(\lambda, x, E) + \sum_{p=0}^{n}\int \pi^p(\lambda, x, dy)\,[\lambda+q(y)]^{-1}.$$

$$\int \delta(y, dz)\,[\lambda+q(z)] - \int \sum_{p=0}^{n}\pi^p(\lambda, x, dy)\,[\lambda+q(y)]^{-1}q(y, E)$$

$$= \sum_{p=0}^{n+1}\pi^p(\lambda, x, E) - \sum_{p=1}^{n+1}\pi^p(\lambda, x, E) = \delta(x, E) = 1.$$

Letting $n\to\infty$, from [3, Theorem 4.1], Lemma 3.13 and (3.17) we obtain (3.29). Letting $\lambda\downarrow 0$, from Lemma 3.12 and (3.23) we obtain (3.30) and (3.28). From (3.29) we know that $0\leqslant X_\lambda^\Delta\leqslant 1$.

Lemma 3.15.

$$X_\lambda^\Delta(\cdot) - X_\mu^\Delta(\cdot) = (\mu-\lambda)\int P^{\min}(\lambda, \cdot, dy)\,X_\mu^\Delta(y) \quad (\lambda, \mu>0), \tag{3.31}$$

$$X_\lambda^\Delta(\cdot) = X^\Delta(\cdot) - \lambda\int P^{\min}(\lambda, \cdot, dy)\,X^\Delta(y). \tag{3.32}$$

Proof From the fact that $P^{\min}(\lambda, x, A)$ satisfy the resolvent equation, we have

$$X_\lambda^\Delta(\cdot) - X_\mu^\Delta(\cdot) + (\lambda-\mu)\int P^{\min}(\lambda, \cdot, dy)\,X_\mu^\Delta(y)$$

$$= \int [P^{\min}(\lambda, \cdot, dy) - P^{\min}(\mu, \cdot, dy) + (\lambda-\mu)\int P^{\min}(\lambda, \cdot, dz)P^{\min}(\mu, z, dy)]$$

$$\cdot\,[q(y)-q(y, E)] = 0 \quad (\lambda, \mu>0).$$

This proves (3.31). Letting $\mu\downarrow 0$ on both sides of (3.31), from (3.28) and the dominated convergence theorem we obtain (3.32).

Theorem 3.16. *If $f\in\mathscr{C}_\lambda$, $x_n\in E$, $n=1, 2$, such that*

$$f(x_n)\to 1,\ n\to\infty, \tag{3.33}$$

then

$$P^{\min}(\lambda,\ x_n,\ E)\to 0,\ n\to\infty. \tag{3.34}$$

Proof From Lemma 2.1 we know that if (3.33) holds, then $\overline{X}_\lambda(x_n)\to 1$ $(n\to\infty)$. The assertion follows from (3.29).

§ 4. Expression of q-processes satisfying the kolmogorov backward equation when $q(x)-q(x, A)$ is finite

Definition 4.1. *A q-pair $q(x)-q(x, A)$ is called finite (m) if for some and consequently for each $\lambda>0$, the dimension of $\mathscr{B}_\lambda$ is finite(m).*

Lemma 4.2 (Substitution Theorem). *Let $f_1, \cdots, f_m$ be a family of linearly independent real functions and let $\alpha_1, \cdots, \alpha_m$ be a family of real numbers.*

i) *If $g=\sum_{i=1}^{m}\alpha_i f_i$ and $\alpha_k\neq 0$, then $f_1,\cdots,f_{k-1}, g, f_{k+1},\cdots, f_m$ are linearly independent.*

ii) *For $1\leqslant i,\ k\leqslant m,\ i\neq k, f_1,\ \cdots,\ f_i-f_k,\ \cdots,\ f_k,\ \cdots,\ f_m$ are linearly independent.*

Lemma 4.3. *If $f_1, f_2\in\mathscr{C}_\lambda(\lambda>0)$, then*

$$(f_1\cap f_2)\cap(f_i-f_1\cap f_2)=0\quad(i=1,\ 2) \tag{4.1}$$

If $f,\ g,\ h\in\mathscr{C}_\lambda,\ f\leqslant g,\ g\cap h=0$, then $f\cap h=0$.

Proof For each $i=1$, 2, we can take A_i such that there exists a representative set B such that $B\subset A_1\cap A_2$ and $f_1\cap f_2=S_B$. By Lemmas 2.15 and 2.16, we have

$$(f_1\cap f_2)\cap(f_i-f_1\cap f_2)=S_B\cap S_{A_i\setminus B}=0.$$

The second assertion is obvious.

Theorem 4.4. *If the dimension m of $\mathscr{B}_\lambda$ is finite, then there exists a family of sojourn solutions $X_\lambda^1,\ \cdots,\ X_\lambda^m$ such that they are a basis for $\mathscr{B}_\lambda$, $X_\lambda^i\cap X_\lambda^j=0$ for $i\neq j$ and*

$$\overline{X}_\lambda=\sum_{i=1}^{n}X_\lambda^i. \tag{4.2}$$

Proof Let $f_1,\ \cdots,\ f_n(n\leqslant m)$ be a basis for $\mathscr{C}_\lambda$. From the fact that $\mathscr{B}_\lambda$ is a bounded closed set in a finitely dimensional subspace of the Banach space $b\ \mathscr{E}$, we obtain that $\mathscr{B}_\lambda$ is compact and convex. By Theorem 2.19 and Krein-Milman theorem, we know that $\mathscr{B}_\lambda$ is the closure of the convex hull of $\mathscr{C}_\lambda$. Therefore each elemant of $\mathscr{B}_\lambda$ can be expressed by a linear combination of $f_1,\ \cdots,\ f_n$. This proves $n\geqslant m$, hence $n=m$.

We want to prove that without loss of generality we can assume that $f_i\cap f_j=0$ for $i\neq j$.

First, we want to prove that under proper adjustment we can assume $f_1 \cap f_2 = 0$. If $f_1 \cap f_2 \neq 0$, then there exist real numbers $\alpha_1, \cdots, \alpha_m$ such that at least one of them is non-null and $f_1 \cap f_2 = \sum_{i=1}^{m} \alpha_i f_i$. If $\alpha_1 \neq 0$, then substitute $f_1 \cap f_2$, $f_2 - f_1 \cap f_2$ for f_1, f_2 respectively. By Lemma 4.2, we know that $f_1 \cap f_2$, $f_2 - f_1 \cap f_2$, $f_3, \cdots, f_m$ are linearly independent. From Lemma 4.3, we have $(f_1 \cap f_2) \cap (f_1 - f_1 \cap f_2) = 0$. By the same reasons, we know that if $\alpha_1 = 0$ but $\alpha_2 \neq 0$, then the assertion is proved by substituting $f_1 - f_1 \cap f_2$, $f_1 \cap f_2$ for f_1, f_2 respectively. If $\alpha_1 = \alpha_2 = 0$, then there exists some $3 \leqslant k \leqslant m$ such that $\alpha_k \neq 0$, The assertion is proved by substituting $f_1 - f_1 \cap f_2$, $f_1 \cap f_2$, f_2 for f_1, f_2, f_k respectively.

Secondly we want to prove that if there exist $f_1, \cdots, f_n (n < m)$ such that $f_i \cap f_j = 0$ for $i \neq j$, $i, j \leqslant n$, then under proper adjustment we can assume that $f_i \cap f_j = 0$ for $i \neq j$, $i, j \leqslant n+1$. If $f_1 \cap f_{n+1} \neq 0$, then there exist real numbers $\alpha_1, \cdots, \alpha_m$ such that at least one of them is non-null and $f_1 \cap f_{n+1} = \sum_{i=1}^{m} \alpha_i f_i$ If α_1, α_{n+1} are not all zero, we can adjust them referring to the case of $f_1 \cap f_2 = 0$. If $\alpha_1 = \alpha_{n+1} = 0$, then there exists some $n+2 \leqslant k \leqslant m$ such that $\alpha_k \neq 0$. If it is not, then $f_1 \cap f_{n+1} = \sum_{i=2}^{n} \alpha_i f_i$. Put $M = \max_{i=2,\cdots,n} \{|\alpha_i|, 1\}$. Obviously $M^{-1}[f_1 \cap f_{n+1}] \leqslant \sum_{i=2}^{n} f_i$. From (1.16), (1.15) and the fact that $f_i \cap f_j = 0$ for $i \neq j$, $i, j \leqslant n$, we have

$$0 \leqslant M^{-1}(f_1 \cap f_{n+1}) \leqslant f_1 \cap \left(\sum_{i=2}^{n} f_i\right) = \sum_{i=2}^{n} f_1 \cap f_i = 0.$$

This contradicts the fact that $f_1 \cap f_{n+1} \neq 0$. Substitute $f_1 - f_1 \cap f_{n+1}$, $f_1 \cap f_{n+1}$, f_{n+1} for f_1, f_{n+1}, f_k respectively. Under the above adjustment, f_1 does not increase and $f_2, \cdots, f_n$ remain fixed. Hence, under substituting $r \leqslant n$ for 1, the assertion remain true. Thus passing through the finite steps of adjustment, we can arrive at our purpose. We denote these sojourn solutions by $X_\lambda^1, \cdots, X_\lambda^m$.

Particularly there exist real numbers $\alpha_1, \cdots, \alpha_m$ such that

$$\bar{X}_\lambda(\cdot) = \sum_{i=1}^{m} \alpha_i X_\lambda^i(\cdot). \tag{4.3}$$

By Basic Lemma, for fixed positive integer $i \leqslant m$ we can take $x_n^i \in E$, $n = 1, 2$, such that

$$X_\lambda^j(x_n^i) \to \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases} \quad (n \to \infty). \tag{4.4}$$

Substituting x_n^i for x in (4.3) and letting $n \to \infty$, we obtain $\alpha_i = 1$. (4.2) is proved.

For fixed $\lambda > 0$, we can take $X_\lambda^1, \cdots, X_\lambda^m$ satisfying Theorem 4.4. Let $X_0^1, \cdots, X_0^m$ be their canonical images respectively. We denote the typical images of $X_0^1, \cdots, X_0^m$ in $\mathscr{B}_v$ by $X_v^1, \cdots, X_v^m$ respectively. From Theorem 3.9 they are sojourn solutions satisfying Theorem 4.4.

Theorem 4.5. *Suppose that $q(x)-q(x, A)$ is finite(m). $P(\lambda, x, A)$ $(\lambda>0, x\in E, A\in\mathscr{E})$ is a* B *q-process*[11.§ 1] *if and only if*

$$P(\lambda, x, A)=P^{\min}(\lambda, x, A)+\sum_{i=1}^{m} X_\lambda^i(x)\xi_\lambda^i(A) \quad (\lambda>0, x\in E, A\in\mathscr{E}), \tag{4.5}$$

where $\xi_\lambda^i\in\mathscr{L}_+(i=1, \cdots, m)$ satisfy resolvent condition

$$\xi_\lambda^i(A)-\xi_\nu^i(A)=(\nu-\lambda)\int\xi_\lambda^i(dx)P(\nu, x, A) \quad (\lambda, \nu>0, A\in\mathscr{E}) \tag{4.6}$$

and norm condition

$$\lambda\xi_\lambda^i(E)\leqslant 1. \tag{4.7}$$

A B *q-process is honest if and only if $q(x)-q(x, A)$ is conservative and the equality in* (4.7) *holds for some $\lambda>0$ and each $i=1, \cdots, m$.*

Proof i) If $P(\lambda, x, A)$ is a B q-process, then $P(\lambda, x, A)-P^{\min}(\lambda, \cdot, A)\in\mathscr{B}_\lambda$ for $\lambda>0$, $A\in\mathscr{E}$. From Theorem 4.4, we know that there exist real numbers $\xi_\lambda^i(A)$, $i=1, \cdots, m$ such that (4.5) holds. By Basic Lemma we can take y_n^i satisfying (4.4). For fixed i, substituting y_n^i for x in (4.5) and letting $n\to\infty$, by Vitali-Hahn-Saks theorem we know $\xi_\lambda^i\in\mathscr{L}_+$.

ii) Substituting $P(\lambda, x, A)$ in (4.5) for $P(\lambda, x, A)$ in the resolvent equation, from the fact that $P^{\min}(\lambda, x, A)$ satisfies the resolvent equation, $X_\lambda^1, \cdots, X_\lambda^m$ are linearly independent and for each i, X_λ^i is coordinated, we obtain that $P(\lambda, x, A)$ in (4.5) satisfies the resolvent equation if and only if $\xi_\lambda^1, \cdots, \xi_\lambda^m$ satisfy (4.6).

iii) By Basic Lemma and Theorem 3.16, we know that $P(\lambda, x, A)$ in (4.5) satisfies the norm condition if and only if (4.7) holds.

Combining i)–iii), we know that $P(\lambda, x, A)$ is a B q-process if and only if (4.5) $\sim$(4.7) hold.

iv) Suppose that $q(x)-q(x, A)$ is conservative and $\lambda\xi_\lambda^i(E)=1$ for some $\lambda>0$, each $i=1, \cdots, m$. By Lemma 4.4 and (3.29), we have

$$\lambda P(\lambda, x, E)=\lambda P^{\min}(\lambda, x, E)+\sum_{i=1}^{m} X_\lambda^i(x)=\lambda P^{\min}(\lambda, x, E)+\overline{X}_\lambda(x)=1\ (x\in E).$$

Conversely, suppose that the B q-process is honest. From [11, Theorem 1.6] we know that $q(x)-q(x, A)$ is conservative. Obviously for each $\lambda>0$ we have

$$\lambda P^{\min}(\lambda, x, E)+\lambda\sum_{i=1}^{m} X_\lambda^i(x)\xi_\lambda^i(E)=1.$$

Taking x_n^i, $n=1, 2, \cdots, i=1, \cdots, m$ satisfying (4.4), substituting x_n^i for x in the above equality, letting $n\to\infty$, from Basic Lemma and Theorem 3.16 we obtain $\lambda\xi_\lambda^i(E)=1$ for each $i=1, \cdots, m$.

Definition 4.6. *We call that Markov process $P(t, x, A)$ is honest for $x_0\in E$ if*

$$P(t, x_0, E)=1 \quad (\forall t\geqslant 0). \tag{4.9}$$

From the properties of Laplace transform, we know that this is equivalent to that for some and consequently for each $\lambda>0$ the Laplace transform $P(\lambda, x, A)$ of $P(t, x, A)$

satisfies

$$\lambda P(\lambda, x_0, E)=1 \tag{4.10}$$

Put

$$D(I)=\{x\in E: X_0^i(x)>0, \forall i\in I, X_0^j(x)=0, \forall j\notin I\}, I\subset\{1, \cdots, m\}, \tag{4.11}$$

$$D(\lambda, I)=\{x\in E: X_\lambda^i(x)>0, \forall i\in I, X_\lambda^j(x)=0, \forall j\notin I\}, \lambda>0, \tag{4.12}$$

Lemma 4.7. *For any fixed $I\subset\{1, \cdots, m\}$, in order that $x\in D(I)$ it is necessary and sufficient that there exists $\lambda>0$ (λ dependent of x) such that*

$$x\in D(\mu, I), \forall\mu<\lambda. \tag{4.13}$$

Proof By Corollary to Lemma 3.5 and Lemma 3.12, we know that for fixed $i=1, \cdots, m$, $x\in E$ $X_\lambda^i(x)$ $(\lambda>0)$ is continunons on $[0, \infty)$, and $X_\lambda^i(x)\uparrow(\lambda\downarrow)$. Suppose $x\in D(I)$, then there exists $\lambda>0$ such that

$$0<X_\mu^i(x)\leqslant X_0^i(x), \ \mu<\lambda, \ i\in I,$$

$$X_\mu^i(x)=0, \ \mu>0, \ i\notin 1.$$

This proves that there exists $\lambda>0$ such that (4.13) holds. From the fact that $X_\lambda^i(x)\uparrow X_0^i(x)$, we know that the sufficiency holds.

Lemma 4.8. *Suppose the conservative q-pair $q(x)-q(x, A)$ is finite(m). For any fixed $I\subset\{1, \cdots, m\}$, $\mu>0$ and $x\in D(\mu, I)$, we have that a q-process $P(\lambda, x, A)$ is honest for x if and only if*

$$\mu\xi_\lambda^i(E)=1, \forall i\in I. \tag{4.14}$$

Proof Let (4.14) holds, From Theorem 4.5 and (3.29), we have that for each $x\in D(\mu, I)$

$$\mu P(\mu, x, E)=\mu P^{\min}(\mu, x, E)+\mu\sum_{i=1}^m X_\mu^i(x)\xi_\mu^i(E)=\mu P^{\min}(\mu, x, E)+\sum_{i\in I}X_\mu^i(x)$$

$$=\mu P^{\min}(\mu, x, E)+\overline{X}_\mu(x)=1.$$

Conversely, let $P(\lambda, x, A)$ be honest for Some $x\in D(\mu, I)$. By (4.12), (3.29) and (4.5) we know that

$$\mu P^{\min}(\mu, x, E)+\sum_{i\in I}X_\mu^i(x)=1=\mu P(\mu, x, E)=\mu P^{\min}(\mu, x, E)+\sum_{i\in I}X_\mu^i(x)\mu\xi_\lambda^i(E).$$

Hence

$$\sum_{i\in I}X_\mu^i(x)=\sum_{i\in I}X_\mu^i(x)\mu\xi_\mu^i(E).$$

From (4.7) and the fact that $X_\mu^i(x)>0 (\forall i\in I)$, we obtain (4.14).

Theorem 4.9. *Suppose $q(x)-q(x, A)$ is finite(m) and conservative, $P(\lambda, x, A)$ is a q-process, $I\subset\{1, \cdots, m\}$, then*

i) *either $P(\lambda, x, A)$ is honest for each $x\in D(I)$* or *$P(\lambda, x, A)$ is dishonest for each $x\in D(I)$.*

ii) *put*

$$C(I)=\bigcup_{J\subset I}D(J), \tag{4.15}$$

$P(\lambda, x, A)$ is honest for each $x\in C(I)$ if and only if $P(\lambda, x, A)$ is honest for each

$x \in D(I)$.

Proof i) Taking fixed $y \in D(I)$, from Lemma 4.7 we know that for any fixed $z \in D(I)$, there exists some $\lambda > 0$ such that

$$y,\ z \in D(\mu,\ I),\quad \forall \mu \leqslant \lambda. \tag{4.16}$$

Suppose $P(\lambda,\ x,\ A)$ is honest for y. From Lemma 4.8, we have

$$\mu \xi^{1}_{\mu}(E) = 1,\ i \in I,\ \mu < \lambda.$$

By Lemma 4.8, we know that $P(\lambda,\ x,\ A)$ is honest for z. Similarly, if $P(\lambda,\ x,\ A)$ is dishonest for y, then for each $\mu < \lambda$ there exist $i \in I$ such that $\mu \xi^{i}_{\mu}(E) < 1$. From Lemma 4.8 and (4.16), we know that $P(\lambda,\ x,\ A)$ is dishonest in z.

ii) Necessity is obvious, we want to prove the sufficiency part. Let $P(\lambda,\ x,\ A)$ be honest for each $x \in D(I)$. From Lemma 4.7 and Lemma 4.8, we know that for any fixed $x \in D(I)$, $\phi \neq J \subset I$ there exists some $\lambda > 0$ such that

$$\mu \xi^{i}_{\mu}(E) = 1,\ i \in J,\ \mu < \lambda. \tag{4.17}$$

For any fixed $y \in C(I)$, there exists uniquely $J \subset I$ such that $y \in D(J)$. By Lemma 4.7, we know that there exists some $\nu > 0$ such that

$$y \in D(\mu,\ J),\ \forall 0 < \mu \leqslant \min\{\lambda,\ \nu\},$$

From (4.17) and Lemma 4.8, we know that $P(\lambda,\ x,\ A)$ is honest for y.

Corollary. *(Criterion on honesty) Suppose $q(x) - q(x,\ A)$ is finite(m). For any fixed non-empty $D(I)$ defined by (4.1) we take $x \in D(I)$, then a B q-process $P(\lambda,\ x,\ A)$ is honest if and only if $q(x) - q(x,\ A)$ is conservative and $P(\lambda,\ x,\ A)$ is honest for every x chosen above.*

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