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A CRITERION OF EXISTENCE OF A PERIODIC SOLUTION FOR *n*-TH ORDER SYSTEMS OF AUTONOMOUS DIFFERENTIAL EQUATIONS ($n\geqslant 3$)

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Abstract

There have been many results[1-7] on seeking a periodic solution for systems of autonomous differential equations by using the torus principl. The finest one amony them is still Liapunov's method of rotating functions proposed by B. B. Nemytskii[3,4]. However, this method is limited in the case that the absolute values of derivatives of rotating functions are greater than some positive constants. This paper summarizes the methods proposed in[3-5] and gives a more careful criterion such that derivatives of rotating functions might be alternative functions. The author provides a sufficient condition of existence of a periodic solution for systems of autonomous differential equations by using Liapunov's method of rotating functions under more general circumstances.

§ 1. Introduction

There have been many results[1-7] on seeking a periodio solution for systems of autonomous differential equations by using the torus principle[8]. It ought to be said that the finest one among them is still Liapunov's method of rotating functions proposed by B. B. Nemytskii in [3] and [4]. However, the method of B. B. Nemytskii is limited to the case that derivatives of rotating functions are either greater than some positive constants or smaller than some negative constants. In this papre we have summarized the methods proposed in [3—5] and given a more careful criterion such that derivatives of rotating functions might be alternative functions. Thus. we have provided a sufficient condition of existence of a periodic solution for systems of autonomous differential equations by using Liapunov's method of rotating functions under more general circumstances.

Suppose

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$$\frac{dx_i}{dt}=f_i(x_1, \cdots, x_n) \quad (i=1, 2, \cdots, n) \tag{1}$$

is an n-th order autonomous system defined in some neighborhood of a bounded and closed toroidal domain $\bar{G}(x)\subset E$ (n-dimensional Euclidean space), and the right hand side of (1) satisfies conditions which guarentee the existence, uniqueness and continuity of all solutions of (1).

§ 2. Preliminaries

1. In E^n we call a bounded piecewise smooth surface dividing the space into two connected domains (the interior and exterior) a torus [9] if the interior can be simply covered by an $(n-1)$-parameter family of closed paths free from double points. By a toroidal domain[4,9] we mean the interior of a torus or sphere from which finitely many domains may have been removed.

In E^n an $(n-1)$-dimensional double-faced manifold S is called unilaterial conductive[5] if every field vector starting at any point on S either is directed toward a fixed side of S, or is tangent to S (at this time, the tangent vector is not zero) when the field vector is very small in magnitude.

2. A function

$$V=\frac{F_1(x_1, \cdots, x_n)}{F_2(x_1, \cdots, x_n)}$$

is called a Liapunov rotating function[3,4] with respect to a closed toroidal domain $\bar{G}(x)$ if the following conditions are satisfied:

i) $F_1(x_1, \cdots, x_n)$ and $F_2(x_1, \cdots, x_n)$ are continuous differentiable in $\bar{G}(x)$;

ii) $F_1(x_1, \cdots, x_n)-\mathrm{tg}\varphi\cdot F_2(x_1, \cdots, x_n)=0$ $(0\leqslant\varphi\leqslant 2\pi)$ is an equation of a surface pencil whose axis is out of $\bar{G}(x)$ and whose surfaces have common points each other only on the pencil axis;

iii) Every surface of pencil $F_1-\mathrm{tg}\varphi\, F_2=0$ crosses $\bar{G}(x)$ with two disjoint closed connected domains. One of them corresponds to $\varphi_0(\leqslant\pi)$, and another corresponds to $\varphi_0+\pi$. When φ continuously varies, the corresponding surface of the pencil continuously varies, too;

iv) Every point in $\bar{G}(x)$ belongs to one and only one surface of the pencil.

If the property about angles is not too emphasized, then the equation of the pencil may be written as $F_1-CF_2=0$. At this time, when C continuously varies throughout the following intervals

$$(-\infty, 0), (0, +\infty), (-\infty, 0) \text{ and } (0, +\infty),$$

we obtain all surfaces of the pencil.

Consider the derivative of $V(x_1, \cdots, x_n)$ by means of (1), that is

$$\frac{dV}{dt}=\frac{\sum_{i=1}^{n}\left(\frac{\partial F_1}{\partial x_i}F_2-\frac{\partial F_2}{\partial x_i}F_1\right)f_i}{F_2^2(x_1,\ \cdots,\ x_n)} \tag{2}$$

Since along trajectories of (1) we have $\frac{F_1}{F_2}=C(t)$, (2) can be rewritten as

$$\frac{dV}{dt}=\frac{\sum_{i=1}^{k}\left(\frac{\partial F_1}{\partial x_i}F_2-\frac{\partial F_2}{\partial x_i}F_1\right)f_i}{F_1^2}C^2(t)=\Gamma(x_1,\ \cdots,\ x_n)C^2(t). \tag{3}$$

Besides, we have to consider values of $\frac{dV}{dt}$ on surface $V=0$, i. e., surface $F_1=0$. Then we have

$$\left.\frac{dV}{dt}\right|_{C=0}=\frac{\sum_{i=1}^{n}\frac{\partial F_1}{\partial x_i}f_i}{F_2}=B(x_1,\ \cdots,\ x_n). \tag{4}$$

§ 3. Narration and Proof of Theorems

To simplify our illustration, we use the following notations: By $\bar{S}_1$ we denote the section of Liapunov layer-surface $C=0$ with $\bar{G}(x)$ corresponding to $\varphi=0$, by $\bar{S}_3$ denote the section corresponding to $\varphi=\pi$; By $\bar{S}_2$ denote the section of layer-surface $C=+\infty(-\infty)$ with $\bar{G}(x)$ corresponding to $\varphi=\frac{\pi}{2}$, by $\bar{S}_4$ denote the section corresponding to $\varphi=\frac{3\pi}{2}$. Domain $\bar{G}-\sum_{i=1}^{4}\bar{S}_i$ is divided into four components. By g_i we denote the component which is located between $\bar{S}_i$ and $\bar{S}_{i+1}(S_5=S_1)$.

Theorem 1. *Suppose $\bar{G}(x)$ having no singular point is a bounded and closed toroidal domain whose boundary is inward unilaterial conductive manifold, and there is a Liapunov rotating function with respect to $\bar{G}(x)$. If*

i) *On sections $\bar{S}_1$, $\bar{S}_3$ and $\bar{S}_2$, $\bar{S}_4$, i. e., on sections of surfaces $F_1=0$ and $F_2=0$ with $\bar{G}(x)$, functions B and Γ have no zero-point, respectively;*

ii) *Any trajectory of (1) passes through zero-points of Γ at most finite number-times;*

iii) *There is a section $\bar{S}_0$ that is homeomorphic to an $(n-1)$-dimensional ball and is out of contact with the vector field defined by (1), then there is at least one closed trajectory which is not contractible.*

Before establishing the theorem let us first state a lemma which can be easily proved

Lemma. *If functions Γ and B have no zero-point on a Liapunov layer-surface-section $\bar{S}$, then S is an unilaterial conductive manifold out of contact with the vector field.*

Proof of Theorem 1.

For any fixed component $g_i(i=1, 2, 3, 4)$ we can investigate any half-trajectory $f(P, t\geqslant 0)$ which has points out of g_i, $\forall P\in g_i$.

Without lossing generality, we assume $P\in g_1$. For cases $P\in g_i(i=2, 3, 4)$ the proofs are all the same.

By the assumptions, the trajectory $f(P, t)$ passes through zero-points of Γ at most finite number- times. Now suppose these zero-points all lie on arc $\widehat{MN}$. Moreover, we may assume these zero-points all lie in the interior of $\widehat{MN}$ and points M and N do not lie on the sections $\bar{S}_1$ and $\bar{S}_2$. By $C=C_M$ and $C=C_N$ we denote, respectively, the sections on which M and N lie. By removing $\widehat{MN}$ from trajectory $f(P, t)$, the rest in g_1 is two arcs, and on the two arcs Γ has no zero-point any more. On the two arcs, by (3), $C(t)$ is single-valued and strictly monotone function of t, i. e., t is a single-valued function of C on the two arcs. Hence, the related expression $\frac{dV}{dt}=\frac{dC}{dt}$ can be rewritten as

$$dt(C)=\frac{dC}{\frac{dV}{dt}}. \tag{5}$$

Assume $C_M<C_N$. Since there is no singular point in g_1, C is strictly decreasing and finally reach O as it extends from M along the arc and strictly increasing and finilly reach $+\infty$, as it extends from N along the arc.

In what follows we come to show that the period, denoted by T_1, consumed by a particle passing through the arc of $f(P, t)$ which is located between M and $\bar{S}_1$ and the period, denoted by T_2, consumed by a particle passing through the arc of $f(P, t)$ located between N and $\bar{S}_2$, are all finite.

From (5) we have

$$T_1=\left|\int_0^{C_M} dt\right|\leqslant\int_0^{C_M}\frac{dC}{\left|\frac{dV}{dt}\right|}\leqslant\frac{C_M}{L_1}<+\infty,$$

where L_1 is a positive constant. Using (2), we know $\frac{dV}{dt}$ is continuous, therefore, L_1 may be taken as the infimum of $\left|\frac{dV}{dt}\right|$ defined on the arc of $f(P, t)$ located between $\bar{S}_1$ and $\bar{S}_M$. Clearly $L_1\neq 0$, otherwise, either B has at least a zero-point on S_1 or Γ has at least a zero-point on the arc of $f(P, t)$ located between $\bar{S}_1$ and $\bar{S}_M$. All these contradict the previous assumptions.

Similarly, we have

$$T_2=\left|\int_{C_N}^{+\infty} dt\right|\leqslant\int_{C_N}^{+\infty}\frac{dC}{\left|\frac{dV}{dt}\right|}=\int_{C_N}^{+\infty}\frac{dC}{|\Gamma|C}\leqslant\frac{1}{L_2}\int_{C_N}^{+\infty}\frac{dC}{C^2}=\frac{1}{L_2C_N}<+\infty,$$

where L_2 is a positive constant, and it may be taken as the infimum of $|\Gamma|$ defined on the arc of $f(P, t)$ located between $\bar{S}_N$ and $\bar{S}_2$. Obviously, $L_2\neq 0$, otherwise, Γ has at least a zero-point either on $\bar{S}_2$ or on the arc of $f(P, t)$ located between $\bar{S}_N$ and S_2. All these contradict the previous assumptions.

By T_3 we denoted the period consumed by a particle passing through $\widehat{MN}$. Evidently, $T_3<+\infty$.

Hence, the half-trajectory $f(P, t\geqslant 0)$ certainly has points out of g_1 when $t>T_1+T_2+T_3$.

Let Q be any point which lies on $\bar{S}_1$. By Lemma, $\bar{S}_1$ is a unilaterial conductive manifold out of contact with the vector field. Therefore, $f(Q, t\geqslant 0)$ must go into g_1 or g_4 at once. For explicitness, we assume the direction of unilaterial conductivity of $\bar{S}_1$ is pointed to that as $f(Q, t\geqslant 0)$ penetrates $\bar{S}_1$ from g_4 to g_1. On the basis of the previous establishment, after $f(Q, t\geqslant 0)$ goes into g_1 it will come out. we have already known $f(Q, t\geqslant 0)$ does not penetrate the boundary of $\bar{G}(x)$ and escape from $\bar{G}(x)$ since the boundary of $\bar{G}(x)$ is an inward unilaterial conductive manifold. Thus, $f(Q, t\geqslant 0)$ only escapes from g_1 through $\bar{S}_1$ or $\bar{S}_2$. Since $\bar{S}_1$ is an unilaterial conductive manifold whose direction of conductivity, by the previous assumption, is pointed to that as $f(Q, t\geqslant 0)$ penetrates $\bar{S}_1$ from g_4 to g_1, $f(Q, t\geqslant 0)$ only goes into g_2 through $\bar{S}_2$. After $f(Q, t\geqslant 0)$ goes into g_2, it will come out, and by the same reason, it will penetrate $\bar{S}_3$ and go into g_3. Finilly, $f(Q, t\geqslant 0)$ will go into g_4 and penetrate $\bar{S}_1$ at some point, denoted by $O(Q)$, on $\bar{S}_1$. It can be seen that the function $O(Q)$ is a mapping from $\bar{S}_1$ into itself. The solution of (1), being of continuity and of uniqueness and $\bar{S}_1$ being out of contact with the vector field definee by (1), the mapping is single-valued and continuous. By the well known theorem of Brouwer, there is at least one fixed point R, i. e., $f(R, t)$ is a closed trajectory which is not contractible. The proof of Theorem1 is completed.

It is not difficult to see that Theorem 1 can be extended as follows:

Theorem 2. *Suppose $\bar{G}(x)$ having no singular point is a bounded and closed toroidal domain whose boundary is an inward unilaterial conductive manifold, and there is a Liapunov rotating function with respect to $\bar{G}(x)$. If*

i) *on sections $\bar{S}_1$, $\bar{S}_3$ and $\bar{S}_2$, $\bar{S}_4$,* i. e., *on sections of surfaces $F_1=0$ and $F_2=0$ with $\bar{G}(x)$, functions B and Γ have no zero-points, respectively;*

ii) *any trajectory of (1) passes through zero-point of Γ at most in a finite period;*

iii) *there is a section $\bar{S}_0$ that is homeomorphic to an $(n-1)$-dimensional ball and out of contact with the vector field defined by (1), then there is at least one closed trajectory which is not contractible.*

§ 4. Examples

Example 1. As an application of the theorems in § 3, let us consider the following system:

$$\left.\begin{aligned}\frac{dx}{dt}&=\varepsilon y(R^2+r^2-x^2)+xy^2+yz^2,\\\frac{dy}{dt}&=-\varepsilon x\left(\frac{R^2+r^2}{2}+y^2\right)-x^2y-xz^2,\\\frac{dz}{dt}&=-z+\frac{a}{2},\end{aligned}\right\}\tag{6}$$

which is defined in a toroidal domain $\overline{G}(x): r^2\leqslant x^2+y^2\leqslant R^2, -a\leqslant z\leqslant a$, where constants $R>r>0$, $a>2R$. Now we show there is at least one periodic solution of (6) which is not contractible.

1. Consider the behavior of trajectories on the boundary of $\overline{G}(x)$.

i) Since $\left.\frac{dz}{dt}\right|_{z=-a}=\frac{3a}{2}>0, \left.\frac{dz}{dt}\right|_{z=a}=-\frac{a}{2}<0$, the vector field defined by (6) is pointed upward on the lower bottom and pointed downward on the upper bottom.

ii) Let $V^*=x^2+y^2$, then we have

$$\frac{1}{2}\frac{dV^*}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt}=\varepsilon xy\left[\frac{R^2+r^2}{2}-(x^2+y^2)\right].$$

We can see $\frac{dV^*}{dt}\geqslant 0$ as $x^2+y^2=r^2$; $\frac{dV}{dt}\leqslant 0$ as $x^2+y^2=R^2$. That is, on the inside cylinder of $\overline{G}(x)$, the vector field is pointed to that direction along which V^* is not decreasing; on the outside cylinder of $\overline{G}(x)$ the vector field is pointed to that direction along which V^* is not increasing.

To sum up, the boundary of $\overline{G}(x)$ is an inward unilateral conductive manifold.

2. There is no singular point in $\overline{G}(x)$. In fact, assume the contrary, that is, assume

$$\left.\begin{aligned}&xy^2+\varepsilon y(R^2+r^2-x^2)+yz^2=0,\\&-x^2y-\varepsilon x\left(\frac{R^2+r^2}{2}+y^2\right)-xz^2=0,\\&-z+\frac{a}{2}=0,\end{aligned}\right\}\tag{7}$$

then from the preceding two equations of (7) we obtain

$$xy(x^2+y^2)+\varepsilon(R^2+r^2)\left(y^2+\frac{x^2}{2}\right)+(x^2+y^2)z^2=0,\text{[1)]}$$

namely

$$z^2=-xy-\frac{\varepsilon(R^2+r^2)\left(\frac{x^2}{2}+y^2\right)}{x^2+y^2}.$$

Thus, in $\overline{G}(x)$ we have

$$z^2\leqslant\frac{x^2+y^2}{2}\leqslant\frac{R^2}{2}<R^2,$$

i. e., $|z|<R$. By the third equation of (7) we know $z=\frac{a}{2}>R$. As a consequence, (7) is not consistent.

1). In fact, it is the zero-point surface of Γ, see (9).

3. We try to construct a Liapunov rotating function

$$V=\frac{F_1(x,\ y,\ z)}{F_2(x,\ y,\ z)}=\frac{x}{y}.$$

Then

$$\Gamma=\frac{xy(x^2+y^2)+\varepsilon(R^2+r^2)\left(\frac{x^2}{2}+y^2\right)+(x^2+y^2)z^2}{x^2}, \tag{8}$$

$$B=xy+\varepsilon(R^2+r^2-x^2)+z^2.$$

It is clearly $B=\varepsilon(R+r)+z^2$ as $x=0$; $\Gamma=\frac{\varepsilon}{2}(R^2+r^2)+z^2$ as $y=0$. In other words, B has no zero-point on plane $F_1=0\,(x=0)$, and Γ has no zero-point on plane

$$F_2=0\,(y=0).$$

4. It can be seen from (8) that the zero-point surface of Γ is

$$xy(x^2+y^2)+\varepsilon(R^2+r^2)\left(\frac{x^2}{2}+y^2\right)+(x^2+y^2)z^2=0. \tag{9}$$

Obviously, the intersection of surface (9) and $\overline{G}(x)$ is not empty provided x, y have distinct signs and ε is small enough.

We have already proved in 2 that in $\overline{G}(x)$, for any point on zero-point surface (9) of Γ, $|z|<R$ holds for ever. On the other hand, from the third equation of (6) we know that any trajectory of (6) tends to $z=\frac{a}{2}>R$ as the period increases. Therefore, for any trajectory of (6), after lasting some finite period, it has no intersection point any more with zero-point surface (9) of Γ.

Hence, by Theorem 2, there is at least one periodic solution in $\overline{G}(x)$ which is not contractible.

Example 2. As an application of our theorems to a simple case, we come to discuss Nemytskii's original example[1], too.

Suppose

$$\left.\begin{aligned}\frac{ux}{dt}&=y+\varepsilon x^2P(x,\ y),\\ \frac{dy}{dt}&=-x+\varepsilon x^2Q(x,\ y),\\ \frac{dz}{dt}&=-z+R(x,\ y)\end{aligned}\right\} \tag{10}$$

is a system defined in a toroidal domain $\overline{G}(x)$: $r^2\leqslant x^2+y^2\leqslant R^2$, $0\leqslant z\leqslant a$. If P, Q and R are continuous and satisfy Lipschitz condition in $r^2\leqslant x^2+y^2\leqslant R^2$, and in $\overline{G}(x)$ the following conditions are satisfied:

i) $xP+yQ>0$ as $x^2+y^2=r^2$;

ii) $xP+yQ<0$ as $x^2+y^2=R^2$;

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1) In [4] there is some wrong with selection and calculation of Γ.

iii) $0<R(x, y)<a$,

then there is at least one periodic solution in $\overline{G}(x)$ when ε is small enough.

Similarly with Example 1, on the boundary of $\overline{G}(x)$ the vector field defined by (10) is pointed inward. Construct a Liapunov rotating function

$$V=\frac{F_1(x, y, z)}{F_2(x, y, z)}=\frac{x}{y}.$$

Then

$$\Gamma=\frac{x^2+y^2}{x^2}+\varepsilon(yP-xQ),$$

$$B=1+\frac{\varepsilon x^2P}{y}.$$

On $F_1=0(x=0)$, $B=1$, i. e., B has no zero-point on $F_1=0$. When $x\neq0$, we have an estimate

$$|\Gamma|\geqslant\frac{r^2}{R^2}-\varepsilon|yP-xQ|,$$

since $x^2+y^2\geqslant r^2$, $x^2\leqslant R^2$. Moreover, $yP-xQ$ is bounded in $\overline{G}(x)$. Therefore, $|\Gamma|>0$ provided ε is enough small. In other words, Γ has no zero-point in $\overline{G}(x)$ when ε is enough small, i. e., in $\overline{G}(x)$ any trajectory of (10) does not pass the zero-point surface of Γ. By Theorem 1, there is at least one periodic solution.

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References

[1] Fridrichs, K. O., On non-linear vibrations of the third order, in: Studies in non-linear vibration theory, *Inst. Math. Mech. New York, Univ*, (1946), 65—103.

[2] Rauch, L. L., Oscillations of a third order non-linear autonomous system, in: Contributions to the theory of non-linear oscillations, *Annals of Math. Studies no.*, **20**(1950), 39—88.

[3] Немыцкий, В. В., Метод вращающих функций Ляпунов для разыскания колебательных режимов, *ДАН, СССР*, **97**: 1(1954), 33—36.

[4] Немыикий, В. В., О Некоторых методах качественного исследования «в бальшом»многомерных автономных систем, *Труды Моск. об-ва.* **5**(1956), 455—482.

[5] Блинчевский, В.С., Существование периодического рещения у одной автономной системы и дифференциальныхуравнений, *Матем, сб.,.*, **50**(92): 1(1960).

[6] Vaisbord, E. M., On the existence of a periodic solution of a non-linear differential equation of the third order, *Mat. Sb.*, **56**(1962), 43—58.

[7] Korolov, V. V., On the uniqueness of a limit cycle for highr-diemensional autonomous systems, Dopovidi Akad. Nauk Ukrain. RSR(1964), 1430—1433.

[8] Noldus, E., A counterpart of Popov's theorem for the existence of periodic solutions, *Int. J. Control*, **13**: 4(1971), 705—719.

[9] Reissig, R., Sansone, G. and Conti, R., Non-linear higher order differential equations, Noordhoff international publishing, Leyden, (1974), 66—68.