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SINGULAR PERTURBATIONS FOR QUASILINEAR HYPERBOLIC EQUATIONS

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Abstract

This paper deals with the following mixed problem for Quasilinear hyperbolic equations

$$L_\varepsilon u_\varepsilon \equiv \varepsilon \frac{\partial^2 u_\varepsilon}{\partial t^2} - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j} + b_0(t) \frac{\partial u_\varepsilon}{\partial t} + \sum_{i=1}^{n} b_i(x, t, u_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} + c(x, t, u_\varepsilon) = f(x, t),$$

$$u_\varepsilon|_{t=0} = \varphi(x),$$

$$\left.\frac{\partial u_\varepsilon}{\partial t}\right|_{t=0} = \psi(x),$$

$$u_\varepsilon|_F = \chi(x, t),$$

The M order uniformly valid asymptotic solutions are obtained and there errors are estimated.

In this paper we investigate the mixed initial-boundary value problem for a hyperbolic partial differential equation as follows

$$L_\varepsilon u_\varepsilon \equiv \varepsilon \frac{\partial^2 u_\varepsilon}{\partial t^2} - \sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j} + b_0(t) \frac{\partial u_\varepsilon}{\partial t} + \sum_{i=1}^{n} b_i(x, t, u_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} + c(x, t, u_\varepsilon) = f(x, t), \tag{1}$$

$$u_\varepsilon|_{t=0} = \varphi(x), \tag{2}$$

$$\left.\frac{\partial u_\varepsilon}{\partial t}\right|_{t=0} = \psi(x), \tag{3}$$

$$u_\varepsilon|_F = \chi(x, t), \tag{4}$$

where ε is a positive and small parameter; $(x, t) \in G$ and $G = \Omega \times [0, T]$ is a cylindrical domain; F is the lateral surface of G; $x = (x_1, x_2, \cdots, x_n)$. Let all the coefficients and all the data a_{ij}, b_i, c, f, φ, ψ, χ be sufficiently smooth with respect to all the arguments. Suppose the coefficients of the equation satisfy (i) $a_{ij} = a_{ji} (i = 1, \cdots, n)$ and $\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geqslant \gamma \sum_{i=1}^{n} \xi_i^2$, $\forall \xi = (\xi_1, \cdots, \xi_n) \in R^n$, where $\gamma > 0$ is a given real number; (ii) $b_0(t) \geqslant \alpha > 0$, where α is a given real number. Suppose further that all the initial data are $M+1$–order compatible with the boundary condition.

Let the formal arymptotic solution be

$$u_N = W_N + V_N = \sum_{n=0}^{N} \varepsilon^n w_n + \sum_{n=0}^{N} \varepsilon^{n+1} v_n, \tag{5}$$

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where W_N is the external solution and V_N is boundary layer correcting term. we have

$$b_i(x,\ t,\ W_N)=\sum_{j=0}^{\infty}\varepsilon^i b_{ij},$$

where $b_{i0}=b_i(x,\ t,\ w_0)$, $b_{i1}=\frac{\partial b}{\partial u_\varepsilon}(x,\ t,\ w_0)w_1$ and

$$b_{in}=\frac{\partial b_i}{\partial u_\varepsilon}(x,\ t,\ w_0)w_n+\bar{b}_{in}(x,\ t,\ w_0,\ w_1,\ \cdots,\ w_{n-1}).$$

The similar expension

$$c(x,\ t,\ W_N)=\sum_{j=0}^{\infty}\varepsilon^j c_j$$

can be also obtained.

By comparing the coefficients for the zero-order power of ε, we obtain a problem for w_0 as follows

$$b_0(t)\ \frac{\partial w_0}{\partial t}-\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2 w_0}{\partial x_i\partial x_j}+\sum_{j=1}^{n}b_j(x,\ t,\ w_0)\frac{\partial w_0}{\partial x_j}+c(x,\ t,\ w_0)=f(x,\ t), \tag{6}$$

$$w_0|_{t=0}=\varphi(x), \tag{7}$$

$$w_0|_F=\chi(x,\ t). \tag{8}$$

In the same way, by comparing the coefficients for ε^N, the equation satisfied by w_N is the following

$$\begin{aligned}&b_0(t)\ \frac{\partial w_N}{\partial t}-\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2 w_N}{\partial x_i\partial x_j}+\sum_{j=1}^{n}b_j(x,\ t,\ w_0)\ \frac{\partial w_N}{\partial x_j}\\&\quad+\Big[\sum_{j=1}^{n}\frac{\partial b_j}{\partial u_\varepsilon}(x,\ t,\ w_0)\frac{\partial w_0}{\partial x_j}+\frac{\partial c}{\partial u_\varepsilon}(x,\ t,\ w_0)\Big]w_N\\&=-\frac{\partial^2 w_{N-1}}{\partial t^2}+P_{N-1}\Big(x,\ t,\ w_0,\ \cdots,\ w_{N-1},\ \frac{\partial w_0}{\partial t},\ \cdots,\ \frac{\partial w_{N-1}}{\partial t},\ \frac{\partial w_0}{\partial x_j},\ \cdots,\ \frac{\partial w_{N-1}}{\partial x_j}\Big),\end{aligned} \tag{9}$$

where P_{N-1} is a known function which can be determined by induction. The boundary condition and initial data for w_N will be determined with the boundary layer correcting terms later. It is eary to see that except w_0 each w_N satisfies only a second-order linear equation, but w_0 satisfies a second-order quasilinear equation.

The boundary layer correcting terms must be constructed near $t=0$, since in general w_0 doesn't satisfy the second initial condition.

We consider t as a double variable $t=(\bar{t},\ \tilde{t})$ near $t=0$, where $\bar{t}=t$, $\tilde{t}=\frac{M(t)}{\varepsilon}$ and $M(t)$, a function to be chosen later on, satisfies $M(0)=0$; $M(t)>0$, $t>0$.

Substituting (5) into (1), we get

$$\begin{aligned}L_\varepsilon(W_N+V_N)=L_\varepsilon W_N+\varepsilon\ \frac{\partial^2 V_N}{\partial t^2}-\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2 V_N}{\partial x_i\partial x_j}+b_0(t)\frac{\partial V_N}{\partial t}\\+\Big[\sum_{i=1}^{n}b_i(x,\ t,\ W_N+V_N)\frac{\partial(W_N+V_N)}{\partial x_i}-\sum_{i=1}^{n}b_i(x,\ t,\ W_N)\ \frac{\partial W_N}{\partial x_i}\Big]\\+[c(x,\ t,\ W_N+V_N)-c(x,\ t,\ W_N)].\end{aligned}$$

Moreover, we have

$$b_i(x,\ t,\ W_N+V_N)\frac{\partial(W_N+V_N)}{\partial x_i}-b_i(x,\ t,\ W_N)\frac{\partial W_N}{\partial x_i},$$
$$=\sum_{j=1}^{2(N^2+N)}\varepsilon^j\bar{b}_{ij}+\Big(\sum_{j=0}^{(N+1)^2}\varepsilon^j\bar{\bar{b}}_{ij}\Big)\frac{\partial V_N}{\partial x_i},$$

where

$$\bar{b}_{i1}=\frac{\partial b_i}{\partial u_\varepsilon}(x,\ t,\ w_0)v_0\frac{\partial w_0}{\partial x_i},$$
$$\bar{b}_{i2}=\frac{\partial b_i}{\partial u_\varepsilon}(x,\ t,\ w_0)v_0\frac{\partial w_1}{\partial x_i}+\Big(\frac{\partial^2 b_i}{\partial u_i^2}w_1v_0+\frac{\partial b_i}{\partial u_\varepsilon}v_1\Big)\frac{\partial w_0}{\partial x_i},$$
$$\bar{\bar{b}}_{i0}=b_i(x,\ t,\ w_0),$$
$$\bar{\bar{b}}_{i1}=\frac{\partial b_i}{\partial u_\varepsilon}(x,\ t,\ w_0)(w_1+v_0),$$
$$\bar{\bar{b}}_{i2}=\frac{\partial b_i}{\partial u_\varepsilon}(x,\ t,\ w_0)(w_2+v_1)+\frac{1}{2}\frac{\partial^2 b_i}{\partial u_\varepsilon}(w_1+v_0)^2.$$

In the same way we obtain

$$c(x,\ t,\ W_N+V_N)-c(x,\ t,\ W_N)=\sum_{i=1}^{2N^2+N}\varepsilon^i\bar{c}_i,$$

where

$$\bar{c}_1=\frac{\partial c}{\partial u_\varepsilon}(x,\ t,\ w_0)v_0,$$
$$\bar{c}_2=\frac{\partial c}{\partial u_\varepsilon}v_1+\frac{\partial^2 c}{\partial u_\varepsilon^2}w_1v_0.$$

Therefore

$$\begin{aligned}L_\varepsilon(W_N+V_N)=&\Big\{\varepsilon\Big[\varepsilon^{-2}M_t^2\frac{\partial^2}{\partial\tilde{t}^2}+\varepsilon^{-1}\Big[2M_t\frac{\partial^2}{\partial t\partial\tilde{t}}+M_{tt}\frac{\partial}{\partial\tilde{t}}\Big)+\frac{\partial^2}{\partial t^2}\Big]\\&-\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}+b_0(t)\Big(\varepsilon^{-1}M_t\frac{\partial}{\partial\tilde{t}}+\frac{\partial}{\partial t}\Big)+\sum_{j=0}^{(N+1)^2}\varepsilon^j\bar{\bar{b}}_{ij}\Big)\frac{\partial}{\partial x_i}\Big\}V_N\\&+\sum_{i=1}^{n}\sum_{j=1}^{2(N^2+N)}\varepsilon^j\bar{b}_{ij}+\sum_{i=1}^{2N^2+N}\varepsilon^j\bar{c}_j+f+O(\varepsilon^N)\\=&\varepsilon^{-1}[K_0+\varepsilon K_1+\cdots+\varepsilon^{2N^2+2N+1}K_{2N^2+2N+1}]V_N+\sum_{j=1}^{2(N^2+N)}\varepsilon^jD_j+f\\&+O(\varepsilon^N),\end{aligned}$$

where

$$D_j=\sum_{i=1}^{n}\bar{b}_{ij}+\bar{c}_j.$$

For convenience, we still use t to denote $\tilde{t}$. Then we obtain

$$K_0=M_t^2\frac{\partial^2}{\partial\tilde{t}^2}+b_0(t)M_t\frac{\partial}{\partial\tilde{t}},$$
$$K_1=2M_t\frac{\partial^2}{\partial t\partial\tilde{t}}+M_{tt}\frac{\partial}{\partial\tilde{t}}-\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}+b_0(t)\frac{\partial}{\partial t}+\sum_{i=1}^{n}b_i(x,\ t,\ w_0)\frac{\partial}{\partial x_i},$$
$$K_2=\frac{\partial^2}{\partial t^2}+\sum_{i=1}^{n}\bar{\bar{b}}_{i1}\frac{\partial}{\partial x_i}.$$

Generally speaking, for each $l\leqslant 2N^2+2N+1$

$$K_l=\sum_{i=1}^{n}\bar{b}_{i,l-1}\frac{\partial}{\partial x_i}.$$

Let $K_0v_0=0$ and $M_t=b_0(t)$, i. e. $M(t)=\int_0^t b_0(t)dt$, we have

$$v_0=P_0(x,\ t)e^{-\tilde{t}}=P_0(x,\ t)e^{-\frac{1}{\varepsilon}\int_0^t b_0(t)dt}.$$

Since $b_0(t)\geqslant\alpha>0$, v_0 is the boundary layer correcting term with $P_0(x, t)$ to be determined. v_1 is determined from $K_0v_1=-K_1v_0-D_1$. To avoid the long-time term, the following second-order linear equation must be satisfied by P_0

$$\mathscr{L}_2P_0\equiv b_0(t)\frac{\partial P_0}{\partial t}+\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2 P_0}{\partial x_i\partial x_j}-\sum_{i=1}^{n}b_i(x,\ t,\ w_0)\frac{\partial P_0}{\partial x_i}$$
$$+\Big(\frac{\partial b_0(t)}{\partial t}-\sum_{i=1}^{n}\frac{\partial b_i}{\partial u_s}(x,\ t,\ w_0)\frac{\partial W_0}{\partial x_i}+\frac{\partial c}{\partial u_s}(x,\ t,\ w_0)\Big)P_0=0$$

and the initial condition for P_0 will be clearified later. Hence

$$v_1=P_1(x,\ t)e^{-\tilde{t}},$$

where P_1 will also be determined later and v_1 is also a boundary layer correcting term v_2 is determined from the equation $K_0v_2=-K_1v_1-K_2v_0-D_2$. In the same reason, P_1 satisfies the following equation

$$\mathscr{L}_2P_1=\Big[\frac{\partial^2 P_0}{\partial t^2}+\sum_{i=1}^{n}\frac{\partial b_i}{\partial u_s}(x,\ t,\ w_0)w_1\frac{\partial P_0}{\partial x_i}+\Big(\sum_{i=1}^{n}\frac{\partial b_i}{\partial u_s}(x,\ t,\ w_0)\frac{\partial w_1}{\partial x_i}$$
$$+\sum_{i=1}^{n}\frac{\partial^2 b_i}{\partial u_s^2}(x,\ t,\ w_0)w_1\frac{\partial w_0}{\partial x_i}+\frac{\partial^2 c}{\partial u_s^2}(x,\ t,\ w_0)\Big)P_0\Big]\equiv g_{21}(x,\ t).$$

The initial condition for $P_1(x, t)$ is to be determined. Hence v_2 satisfies

$$K_0v_2=-\sum_{i=1}^{n}\frac{\partial b_i}{\partial u_s}\frac{\partial w_0}{\partial x}P_0\frac{\partial P_0}{\partial x_i}e^{-2\tilde{t}}\equiv g_2(x,\ t)e^{-2\tilde{t}}$$

and

$$v_2=\bar{v}_2+\tilde{v}_2=P_2(x,\ t)e^{-\tilde{t}}+P_{22}(x,\ t)e^{-2\tilde{t}},$$

where $P_2(x, t)$ is a function to be determined and $\bar{v}_2=P_{22}(x, t)e^{-2\tilde{t}}$, a special solution for above nonhomegeneou equation, is completely defined.

Generally, v_{l+1} satisfies

$$K_0v_{l+1}=-\sum_{i=1}^{l+1}K_iv_{l+1-i}-\bar{D}_{l+1}=[\mathscr{L}_2P_l-g_{l+1,1}(x,\ t)]e^{-\tilde{t}}+\sum_{i=2}^{l+1}g_{l+1,i}(x,\ t)e^{-i\tilde{t}},$$

where $g_{l+1,i}(x, t)\ (i=1, 2, \cdots, l+1)$ are known functions obtained by induction. To avoid the long-time term, we set

$$\mathscr{L}_2P_l=g_{l+1,1}(x,\ t).$$

The initial condition is to be determined too. Thus P_l can be solved, and then we obtain

$$v_l(x,\ t)=\bar{v}_l+\tilde{v}_l=P_l(x,\ t)e^{-\tilde{t}}+\sum_{i=2}^{l}P_{li}(x,\ t)e^{-i\tilde{t}}$$
$$=P_l(x,\ t)e^{-\frac{1}{\varepsilon}\int_0^t b_0(t)dt}+\sum_{i=2}^{l}P_{li}(x,\ t)e^{-\frac{i}{\varepsilon}\int_0^t b(t)dt}.$$

Now we deduce the conditions stisfied by w_i and v_i. Since

$$u_\varepsilon|_{t=0}=\varphi(x),\quad \frac{\partial u_\varepsilon}{\partial t}\Big|_{t=0}=\psi(x),\quad u_\varepsilon|_F=\chi(x,\ t),$$

we obtain

$$w_0|_{t=0}=\varphi(x),\ w_1|_{t=0}=-v_0|_{t=0},\ \cdots,\ w_{N+1}|_{t=0}=-v_N|_{t=0},$$
$$w_0|_F=\chi(x,\ t),\ w_1|_F=-v_0|_F,\ \cdots,\ w_{N+1}|_F=-v_N|_F,$$
$$P_0(x,\ 0)=\frac{-1}{b_0(0)}\Big[\psi(x)-\frac{\partial w_0}{\partial t}(0,\ x)\Big],$$
$$P_l(x,\ 0)=\Big(\frac{\partial v_{l-1}}{\partial t}+\frac{\partial w_l}{\partial t}\Big)\Big|_{t=0}-\sum_{i=2}^{l}iP_{li}(x,\ 0).$$

Thus, we can obtain w_l, v_l by induction. From now on, we shall concentrate our attension on the estimation for the remainder.

Set $U_M=W_{M+1}+V_M=\sum_{j=0}^{M+1}\varepsilon^j w_j+\sum_{j=0}^{M}\varepsilon^{j+1}v_j$, it is easy to see that

$$L_\varepsilon U_M=f+O(\varepsilon^{M+1})\equiv f+\varepsilon^{M+1}f_1(x,\ t),\quad U_M|_{t=0}=\varphi(x),$$
$$\frac{\partial U_M}{\partial t}\Big|_{t=0}=\psi(x)+\varepsilon^{M+1}h(x),\quad U_M|_F=\chi(x,\ t),$$

where $f_1(x,\ t)=O(1)$, $h(x)=O(1)$. Let $u_\varepsilon=U_M+R$, then

$$L_\varepsilon(U_M+R)=L_\varepsilon(U_M)+\varepsilon\frac{\partial^2R}{\partial t^2}-\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2R}{\partial x_i\partial x_j}+b_0(t)\frac{\partial R}{\partial t}$$
$$+\sum_{i=1}^{n}b_i(U_M+R)\frac{\partial R}{\partial x_i}+\sum_{i=1}^{n}[b_i(U_M+R)-b_i(U_M)]\frac{\partial U_M}{\partial x}$$
$$+[c(U_M+R)-c(U_M)]=f(x,\ t).$$

Hence

$$F_{\varepsilon,U_M}(R)\equiv\varepsilon\frac{\partial^2R}{\partial t^2}-\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2R}{\partial x_i\partial x_j}+b_0(t)\frac{\partial R}{\partial t}+\sum_{i=1}^{n}b_i(U_M+R)\frac{\partial R}{\partial x_i}$$
$$+\sum_{i=1}^{n}[b_i(U_M+R)-b_i(U_M)]\frac{\partial U_M}{\partial x_i}+[c(U_M+R)-c(U_M)]=-\varepsilon^{M+1}f_1,$$
$$R(x,\ t)|_{t=0}=0,\quad \frac{\partial R}{\partial t}\Big|_{t=0}=-\varepsilon^{M+1}h(x),\quad R|_F=0.$$

Let $R^*=R+\varepsilon^{M+1}th(x)$, $U_M^*+R^*=U_M+R$. Then

$$F_{\varepsilon,U_M^*}(R^*)=\varepsilon^{M+1}f_2(x,\ t)\equiv\bar g(x,\ t),\ F_{\varepsilon,U_M^*}(0)=0,$$

and R^* satisfies the homogeneous initial and boundary conditions. Let

$$M=\Big\{u;\ u\in W_2^2(G),\ u(x,\ 0)=0,\ \frac{\partial u}{\partial x}(x,\ 0)=0,\ u|_F=0\Big\},$$
$$|u|_\varepsilon=\|u\|_2+\Big\|\frac{\partial u}{\partial x}\Big\|_2+\varepsilon^{\frac12}\Big\|\frac{\partial u}{\partial t}\Big\|_2,\quad B=\{v;\ v\in L_2(G),\ |v|=\|v\|_2\}.$$

Then F is a nonlinear mapping from M to B. Let $F_{\varepsilon,U_M^*}(u)=L(u)+N(u)$, where $L(u)$ is the linearlized mapping. Then

$$L(u)=\varepsilon\frac{\partial^2u}{\partial t^2}-\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2u}{\partial x_i\partial x_j}+b_0(t)\frac{\partial u}{\partial t}+\sum_{i=1}^{n}b_i(U_M^*)\frac{\partial u}{\partial x_i}$$
$$+\sum_{i=1}^{n}\frac{\partial b_i}{\partial u_\varepsilon}(U_M^*)\frac{\partial U_M^*}{\partial x_i}u+\frac{\partial c}{\partial u_\varepsilon}(U_M^*)u,$$

$$N(u)=\sum_{i=1}^{n}\frac{\partial b_i}{\partial u_s}(U_M^*+\theta_i u)\frac{\partial u}{\partial x_i}u+\frac{\partial^2 c}{\partial u_s^2}(U_M^*+\theta_{00}u)u^2+\sum_{i=1}^{n}\frac{\partial^2 b_i}{\partial u_s^2}(U_M^*+\bar{\theta}_i u)u^2.$$

Owing to the estimation of solutions of the mixed problem for hyperbolic equations[2], we have $|u|_s\leqslant c_1\|f\|_2$, where c_1 is a constant independent of s.

Hence L is an one-to-one mapping from M *to* B and its continuous inverse exists. Thus, We have

$$|L^{-1}v|_s\leqslant c_1\|v\|_2.$$

For $F_{s,U_M^*}(u)=L(u)+N(u)=\bar{g}$, set $L(u)=v$, then $v+N(L^{-1}v)=\bar{g}$, i. e.

$$v=\bar{g}-N(L^{-1}v)=T(v).$$

For the nonlinear part of F, it is easy to see that

$$\|N(u_1)-N(u_2)\|_2\leqslant m(\rho)|u_1-u_2|_s.$$

where $m(\rho)=c_2\rho$, c_2 is independent of s and $|u_i|_s<\rho$.

By choosing $\rho_0=\frac{1}{2c_1c_2}$ and denoting $\|\bar{g}_2\|=\frac{1}{2c_1}\rho$, for $\rho<\rho_0$, we consider ball

$$\Omega_B(c^{-1}\rho)=\{v\,|\,v\in B,\ \|v\|_2\leqslant c_1^{-1}\rho\}.$$

If $v\in\Omega_B(c_1^{-1},\rho)$, then

$$\|T(v)\|_2\leqslant c_1^{-1}\rho,$$

$$\|T(v_1)-T(v_2)\|_2=\|N(L^{-1}(v_1))-N(L^{-1}(v_2))\|\leqslant m(\rho)|L^{-1}(v_2)-L^{-1}(v_1)|_s$$

$$\leqslant m(\rho)c_1\|v_2-v_1\|_2\leqslant\frac{1}{2}\|v_2-v_1\|_2.$$

Therefore, T is a contraction mapping from $\Omega_B^{-1}(c_1^{-1}\rho)$ to itself. By the fixed point theorem, we conclude that there exists a unique element v^* in $\Omega_B(c^{-1}\rho)$ such that

$$v^*=T(v^*)=\bar{g}-N(L^{-1}(v^*)).$$

By taking R^* such that $L(R^*)=v^*$, it is easy to see that R^* satisfies

$$L(R^*)+N(R^*)=\bar{g},\quad |R^*|_s\leqslant c_1\|v^*\|_2=O(s^{M+1}).$$

Since $R=R^*-s^{M+1}th(x)$, we obtain $|R|_s=O(s^{M+1})$. Finally, we hav

Theorem. *Let the coefficients of* (1) a_{ij}, b_i, c *and the deta* f, φ, ψ, χ *be sufficiently smooth functions with respect to all the arguments. Suppose that* $a_{ij}=a_{ji}$, $a_{ij}\xi_i\xi_j\geqslant\gamma\sum_{i=1}^{n}\xi_i^2$, $b_0(t)\geqslant\alpha>0$ *for some positive constant numbers* γ *and* α *and the initial data are* $m+1$-*order compatible with the boundary condition. Then the mixed problem for quasilinear hyperbolic equations admits a unique solution* u_s *for which the following asymptotic expension holds* $u_s=\sum_{i=0}^{M+1}s^iw_i+\sum_{i=0}^{M}s^{i+1}v_i+R$, *where* w_0 *is obtained by the mixed problem* (6) ~ (8) *for quasilinear equations,* w_i *by linear equation* (9) *and corresponding initial and boundary conditions, and* v_j *is the boundary layer correcting term. Furthemore, the remander estimation* $|R|_s=O(s^{M+1})$ *is satisfied.*

References

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