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USING FINITE UNITARY GEOMETRY TO CONSTRUCT A CLASS OF PBIB DESIGNS

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Abstract

In this paper the author considers the n-dimensional ($n\geqslant 3$) unitary geometry over finite field $\mathbf{F}_{q^2}$, where q is a power of a prime. The author takesthe set of all one-dimensional non-isotropic subspaces as the set of treatments, and obtainsan association scheme and PBIB designs with q associate classes, whose parameters are also computed.

§ 1. Introduction

In [1] Wan Zhexian et al constructed many association schemes and PBIB designs using isotropic subspaces of finite geometries. Later, Wan Zhexian also constructed some association schemes and PBIB designs taking one-dimensional non-isotropic subspaces of finite geometries over some small fields as treatments (see [2]). In the present paper, we extend the unitary case in [2] from $\mathbf{F}_4$ to $\mathbf{F}_{q^2}$, and obtain an association scheme and PBIB designs with q associate classes whose parameters are also conputed.

Let $\mathbf{F}_{q^2}$ denote the finite field with q^2 elements, where q is a power of a prime. It is well known that $\mathbf{F}_{q^2}$ has an automorphism with order 2

$$a\rightarrow \bar{a}=a^q,$$

whose fixed subfield is $\mathbf{F}_q$. $\mathbf{F}_q^*$ and $\mathbf{F}_{q^2}^*$ denote the multiplication groups consisting of all non-zero elements of $\mathbf{F}_q$ and $\mathbf{F}_{q^2}$ respectively. An $n\times n$ matrix H over $\mathbf{F}_{q^2}$ is called a hermitian matrix if $\overline{H}'=H$. Let H be an $n\times n$ non-singular hermitian matrix. An $n\times n$ matrix T over $\mathbf{F}_{q^2}$ is called a unitary matrix with respect to H if $TH\overline{T}'=H$. All the unitary matrices form a group with respect to the matrix multiplication. This group is called the unitary group defined by H over $\mathbf{F}_{q^2}$, and denoted by $U_n(\mathbf{F}_{q^2}, H)$. The totality of n-tuple

$$(x_1, x_2, \cdots, x_n),\ x_i\in \mathbf{F}_{q^2},\ i=1, 2, \cdots, n$$

forms the n-dimensional unitary space over $\mathbf{F}_{q^2}$ and is denoted by $V_n(\mathbf{F}_{q^2})$. Let P be a m-dimensional subspace of $V_n(\mathbf{F}_{q^2})$, and we also use P to denote the $m\times n$ expression

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matrix with rank m of this subspace. The subspace P is called a (m, r)-type subspace if the rank of the matrix $PH\overline{P}'$ is r, and P is called a m-dimensional non-degenerate subspace if $r=m$. The vectors x and y are called orthogonal if $xH\overline{y}'=0$, and x is called isotropic if $xH\overline{x}'=0$. Hence a $(1, 1)$-type subspace is also called an one-dimensional non-isotropic subspace. We use $P^{\perp}$ to denote the orthogonal subspace of P.

Every $n\times n$ non-singular hermitian matrix H is congruent to the unit matrix $I^{(n)}$. Hence $U_n(\mathbf{F}_{q^2}, H)$ is isomorphism to the unitary group defined by $I^{(n)}$. Thus, in the following statements we always take $H=I^{(n)}$ and denote the unitary group by $U_n(\mathbf{F}_{q^2})$.

§ 2. Construction of association scheme and computation of parameters

(A) Construction of association scheme

Let $n\geqslant 3$ and take the set of all $(1, 1)$-type subspaces $V, V_1, V_2, \cdots$ of $V_n(\mathbf{F}_{q^2})$ as the set of treatments. We denote V_1 and V_2 to be the i-th associates of each other by $(V_1, V_2)=i$.

Thus, we define the associative relation as following:

$(V_1, V_2)=1$ if they are orthogonal, i. e.

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}\begin{pmatrix} \overline{V_1} \\ \overline{V_2} \end{pmatrix}'=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

$(V_1, V_2)=i(2\leqslant i\leqslant q-1)$ if

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}\begin{pmatrix} \overline{V_1} \\ \overline{V_2} \end{pmatrix}'=\begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix},$$

where the element i is the i-th $(2\leqslant i\leqslant q-1)$ element of $\mathbf{F}_q^*\backslash\{1\}$ in a fixed order;

$(V_1, V_2)=q$ if

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}\begin{pmatrix} \overline{V_1} \\ \overline{V_2} \end{pmatrix}'=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Now, we prove that it is an association scheme with q associate classes. For $2\leqslant i\leqslant q-1$, there exists a $\mu\in\mathbf{F}_{q^2}^*$ with $\mu\overline{\mu}=i$ (cf. [1] p. 31) such that

$$\begin{pmatrix} \mu^{-1} & \\ & \overline{\mu} \end{pmatrix}\begin{pmatrix} V_2 \\ V_1 \end{pmatrix}\begin{pmatrix} \overline{V_2} \\ \overline{V_1} \end{pmatrix}'\overline{\begin{pmatrix} \mu^{-1} & \\ & \overline{\mu} \end{pmatrix}}'=\begin{pmatrix} 1 & 1 \\ 1 & i \end{pmatrix}.$$

therefore $(V_2, V_1)=i(1\leqslant i\leqslant q)$ if $(V_1, V_2)=i$.

Next, by the transitivity of the unitary group ([1] p. 37 Theorem 3), we can see that $n_i(1\leqslant i\leqslant q)$ is independent of the choice of V.

Furthermore, suppose $(V_1, V_2)=i$ and $(V_3, V_4)=i$, then there exists $T\in U_n(\mathbf{F}_{q^2})$ such that

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_3 \\ V_4 \end{pmatrix} T.$$

([1] p. 38 Theorem 4), i. e.

$$V_1 = V_3 T, \ V_2 = V_4 T.$$

Hence p^i_{jk} $(1 \leqslant i,\ j,\ k \leqslant q)$ is independent of the choice of V_1 and V_2.

(B) Computation of the parameters of the association scheme

(1) Computation of v and $n_i (1 \leqslant i \leqslant q)$

(i)
$$v = N(1,\ 1;\ n) = \frac{q^n - (-1)^n}{q+1} q^{n-1},$$

where $N(m,\ r;\ n)$ denotes the number of $(m,\ r)$-type subspaces in $V_n(\mathbf{F}_{q^2})$. (see [1] ch. 3 § 4).

(ii) n_1: For any V, n_1 is the number of (1, 1) -type subspace in $V^{\perp}$. Because V is non-degenerate, so $V \cap V^{\perp} = \{0\}$ ([1] p. 35 Theorem 2). Thus

$$n_1 = N(1,\ 1;\ n-1) = \frac{q^{n-1} - (-1)^{n-1}}{q+1} q^{n-2}.$$

(iii) $n_i (2 \leqslant i \leqslant q-1)$: Let $V = \langle e \rangle$, where $e_1,\ e_2,\ \cdots,\ e_n$ is a basis of $V_n(\mathbf{F}_{q^2})$ satisfing $e_i \bar{e}'_j = \delta_{ij} (i,\ j = 1,\ 2,\ \cdots,\ n)$. If $(V,\ V_s) = i$, then we can suppose the (2, 2)-type subspace

$$B = \begin{pmatrix} V \\ V_s \end{pmatrix}$$

generated by V and V_s with $V_s = \langle e_1 + \lambda e_2 \rangle$, $\lambda \neq 0$. Thus

$$\begin{pmatrix} V \\ V_s \end{pmatrix} \overline{\begin{pmatrix} V \\ V_s \end{pmatrix}}' = \begin{pmatrix} 1 & 1 \\ 1 & 1+\lambda\bar{\lambda} \end{pmatrix}.$$

So, we have $1 + \lambda\bar{\lambda} = i$. But the equation $\lambda\bar{\lambda} = i - 1 \in \mathbf{F}^*_q$ has precisely $q+1$ solutions in $\mathbf{F}^*_{q^2}$ ([1] p. 31 Lemma 1), therefore, there are $q+1$ such V_s in B. We know that the number of the (2, 2)-type subspaces containing V in $V_n(\mathbf{F}_{q^2})$ is

$$\frac{N(2,\ 2;\ n) N(1,\ 1;\ 2,\ 2; n)}{N(1,\ 1;\ n)} = \frac{q^{n-1} - (-1)^{n-1}}{q+1} q^{n-2},$$

where $N(m_1,\ r_1;\ m,\ r;\ n)$ is the number of $(m_1,\ r_1)$-type subspaces which are contained some $(m,\ r)$-type subspace (see [1] ch. 3 § 5). So

$$n_i = (q^{n-1} - (-1)^{n-1}) q^{n-2}.$$

(iv) n_q: By the equality

$$v = n_1 + \cdots + n_{q-1} + n_q + 1,$$

we get n_q immediately.

(2) Computation of $p^i_{jk} (1 \leqslant i,\ j,\ k \leqslant q)$

(i) p^1_{11}: let $(V_1,\ V_2) = 1$ and $B = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$, then p^1_{11} is the number of (1, 1)-type subspaces in $B^{\perp}$, so

$$p_{11}^1=N(1,\ 1;\ n-2)=\frac{q^{n-2}-(-1)^{n-2}}{q+1}\,q^{n-3}.$$

(ii) $p_{1k}^1(2\leqslant k\leqslant q-1)$: let $(V_1,\ V_2)=1$, thus p_{1k}^1 is the number of V_s which satisfy $(V_1,\ V_s)=1$ and $(V_2,\ V_s)=k$. By the formula for n_k we have immediately

$$p_{1k}^1=(q^{n-2}-(-1)^{n-2})q^{n-3}.$$

(iii) $p_{jk}^1(2\leqslant j,\ k\leqslant q-1)$: let $V_1=\langle e_1\rangle$, $V_2=\langle e_2\rangle$. Now, we compute the number of V_s which satisfy $(V_1,\ V_s)=j$ and $(V_2,\ V_s)=k$. Suppose

$$B=\begin{pmatrix}V_1\\ V_2\end{pmatrix}.$$

If $V_s\in B$, then $V_s=\langle e_1+\lambda e_2\rangle$, $\lambda\neq 0$, thus λ satisfies

$$\begin{cases}\lambda\bar{\lambda}=j-1,\\ \lambda\bar{\lambda}=\dfrac{1}{k-1},\\ \lambda\neq 0.\end{cases}$$

It is clear that the number of those V_s in B is $q+1$ if $(j-1)(k-1)=1$, is 0 if $(j-1)\times(k-1)\neq 1$.

Suppose $V_s\notin B$. When V_s, V_1 and V_2 generate a (3, 3)-type subspace P, we can suppose $P=\begin{pmatrix}V_1\\ V_2\\ V_s\end{pmatrix}$ with $V_s=\langle e_1+\lambda_2e_2+\lambda_3e_3\rangle$, $\lambda_2\neq 0$, $\lambda_3\neq 0$. Thus λ_2, λ_3 satisfy

$$\begin{cases}\lambda_2\bar{\lambda}_2=\dfrac{j}{k},\\ \lambda_3\bar{\lambda}_3=\dfrac{(j-1)(k-1)-1}{k},\\ \lambda_2\neq 0,\ \lambda_3\neq 0.\end{cases}$$

Because the number of (3, 3)-type subspaces containing B in $V_n(\mathbf{F}_{q^2})$ is

$$\frac{N(3,\ 3;\ n)N(2,\ 2;\ 3,\ 3;\ n)}{N(2,\ 2;\ n)}=\frac{q^{n-2}-(-1)^{n-2}}{q+1}\,q^{n-3}.$$

therefore the number of those V_s is 0 if $(j-1)(k-1)=1$, is $(q+1)(q^{n-2}-(-1)^{n-2})q^{n-3}$ if $(j-1)(k-1)\neq 1$. When V_s, V_1 and V_2 generate *a* (3, 2)-type subspace, we can suppose $Q=\begin{pmatrix}V_1\\ V_2\\ V_s\end{pmatrix}$ and $V_s=\langle e_1+\lambda e_2+\mu\eta\rangle$, $\lambda\neq 0$, $\mu\neq 0$, where η is an isotropic vector in $B^\perp$. We know that the number of the (3, 2)-type subspaces containing B in $V_n(\mathbf{F}_{q^2})$ is

$$\frac{N(3,\ 2;\ n)N(2,\ 2;\ 3,\ 2;\ n)}{N(2,\ 2;n)}=\frac{(q^{n-2}-(-1)^{n-2})(q^{n-3}-(-1)^{n-3})}{q^2-1}.$$

Noticing the computation of V_s in B, we can know that the number of those V_s is

$$(q+1)(q^{n-2}-(-1)^{n-2})(q^{n-3}-(-1)^{n-3})$$

if $(j-1)(k-1)=1$, is 0 if $(j-1)(k-1)\neq 1$. Therefore

$$p_{jk}^1=\begin{cases}(q+1)[1+(q^{n-2}-(-1)^{n-2})(q^{n-3}-(-1)^{n-3})], & \text{if } (j-1)(k-1)=1;\\(q+1)(q^{n-2}-(-1)^{n-2})q^{n-3}, & \text{if } (j-1)(k-1)\neq 1.\end{cases}$$

(iv) $p_{jk}^i(2\leqslant i,\ j,\ k\leqslant q-1)$: Let $(V_1,\ V_2)=i$ and $V_1=\langle e_1\rangle$, $V_2=\langle e_1+\mu e_2\rangle$, where $\mu\in\mathbf{F}_{q^2}^*$ is a fixed element satisfying $1+\mu\bar{\mu}=i$. Now we compute the number of V_s which satisfies $(V_1,\ V_s)=j$ and $(V_2,\ V_s)=k$.

(a) If $V_s\in B=\begin{pmatrix}V_1\\V_2\end{pmatrix}$, we may suppose $V_s=\langle e_1+\lambda_2 e_2\rangle$. Thus λ_s satisfies

$$\begin{cases}(1+\bar{\mu}\lambda_2)(1+\mu\bar{\lambda}_2)=\dfrac{ij}{k},\\(1+\bar{\mu}\lambda_2)+(1+\bar{\lambda}\mu_2)=\dfrac{ij}{k}-ij+i+j.\end{cases}\tag{1}$$

Obviously, the number of the solution λ_2 is equal to the number of the solution x which satisfies equations

$$\begin{cases}x+\bar{x}=\dfrac{ij}{k}-ij+i+j,\\x\bar{x}=\dfrac{ij}{k}\end{cases}\tag{2}$$

in $\mathbf{F}_{q^2}$. Let $\mathbf{F}_{q^2}=\mathbf{F}_q(\theta)$, where θ satisfies an irreducible equation on $\mathbf{F}_q$

$$X^2+\alpha X+\beta=0,\ \alpha\neq 0.\tag{3}$$

Thus every $x\in\mathbf{F}_{q^2}$ is uniquely expressed as $x=\lambda+\nu\theta$, $\lambda,\ \nu\in\mathbf{F}_q$, and $\bar{x}=\lambda+\nu\bar{\theta}$. Put $a_{ijk}=\dfrac{ij}{k}$ and $b_{ijk}=\dfrac{ij}{k}-ij+i+j$, then the equations (2) become

$$\begin{cases}2\lambda-\alpha\nu=b_{ijk},\\\lambda^2-\alpha\nu\lambda+\beta\nu^2=a_{ijk}.\end{cases}\tag{4}$$

Let $q=p^h$. If $p\neq 2$, then (4) implies

$$b_{ijk}^2-4a_{ijk}=(\alpha^2-4\beta)\nu^2.$$

Because (3) is irreducible, so $\alpha^2-4\beta$ is an non-square element in $\mathbf{F}_q^*$. Hence the number of tne solutions $(\lambda,\ \nu)$ of (4) is 0 if $b_{ijk}^2-4a_{ijk}$ is a square in $\mathbf{F}_q^*$; is 1 if $b_{ijk}^2-4a_{ijk}=0$; is 2 if $b_{ijk}^2-4a_{ijk}$ is an non-square in $\mathbf{F}_q^*$.

If $p=2$, then from (4), we know that $\nu=\dfrac{b_{ijk}}{\alpha}$ and λ satisfies the equation over $\mathbf{F}_q$

$$\lambda^2+b_{ijk}\lambda+\frac{\beta b_{ijk}^2+\alpha^2 a_{ijk}}{\alpha^2}=0.\tag{5}$$

When $b_{ijk}=0$, (5) has a unique solution: when $b_{ijk}\neq 0$, put $y=\dfrac{\lambda}{b_{ijk}}$, then (5) becomes

$$y^2+y+c=0,\quad\text{where } c=\frac{\beta}{\alpha^2}+\frac{a_{ijk}}{b_{ijk}^2}.\tag{6}$$

Let $D(t)=t+t^2+t^4+\cdots+t^{2^{h-1}}$, obviously $D(t)^2=D(t)$, $\forall t\in\mathbf{F}_q$, so $D(c)=0$ or 1. If $D(c)=0$, we take $d\in\mathbf{F}_q^*$ such that $D(d)=1$, then

$$y=dc^2+(d+d^2)c^4+\cdots+(d+d^2+\cdots+d^{2^{h-2}})c^{2^{h-1}}$$

is a root of (6) and $y+1$ is the other. If $D(c)=1$, then (6) has no root (cf. [3] p. 3).

We observe that $D(t_1+t_2)=D(t_1)+D(t_2)$ and $D\left(\frac{\beta}{\alpha^2}\right)=1$. Thus the number of the solutions (λ, ν) of (4) is 0 if $D\left(\frac{a_{ijk}}{b_{ijk}^2}\right)=0$; is 1 if $b_{ijk}=0$; is 2 if $D\left(\frac{a_{ijk}}{b_{ijk}^2}\right)=1$.

We use the notation $\omega(i, j, k)$ to denote the number of the solutions (λ, ν) of (4), then the number of V_s in B is just $\omega(i, j, k)$.

(b) If $V_s\notin B$ and V_s, V_1 and V_2 generate a (3, 3)-type subspace P, then we can suppose $V_s=\langle e_1+\lambda_2e_2+\lambda_3e_3\rangle$, $\lambda_3\neq 0$. Thus λ_2, λ_3 satisfy

$$\begin{cases}(1+\bar{\mu}\lambda_2)(1+\mu\bar{\lambda}_2)=\frac{ij}{k},\\ (1+\bar{\mu}\lambda_2)+(1+\mu\bar{\lambda}_2)=\frac{ij}{k}-ij+i+j+(i-1)\lambda_3\bar{\lambda}_3.\\ \lambda_3\neq 0.\end{cases} \tag{7}$$

Because there exist exactly $q+1$ λ_2 which satisfy. the first equation of (7) in $\mathbf{F}_{q^2}$. Compare (7) with (1) and observe that $(i-1)\lambda_3\bar{\lambda}_3\neq 0$, then we can see that there are $q+1-\omega(i, j, k)$ λ_2 satisfing (7) in $\mathbf{F}_{q^2}$. Thus, the number of solutions (λ_2, λ_3) is $(q+1-\omega(i, j, k))(q+1)$. Referring to the corresponding calculation of p_{jk}^1, we can obtain the number of those V_s is $(q+1-\omega(i, j, k))(q^{n-2}-(-1)^{n-2})q^{n-3}$.

(c) If $V_s\notin B$, V_s, V_1 and V_2 generate a (3, 2)-type subspace, referring to the argument of p_{jk}^1, then we can see that the number of those V_s is

$$\omega(i, j, k)(q^{n-2}-(-1)^{n-2})(q^{n-3}-(-1)^{n-3}).$$

Therefore

$$\begin{aligned}p_{jk}^i=&\omega(i, j, k)[1+(q^{n-2}-(-1)^{n-2})(q^{n-3}-(-1)^{n-3})]\\ &+(q+1-\omega(i, j, k))(q^{n-2}-(-1)^{n-2})q^{n-3}.\end{aligned}$$

where $2\leqslant i, j, k\leqslant q-1$, in the case of char $\mathbf{F}_{q^2}\neq 2$

$$\omega(i, j, k)=\begin{cases}0, \text{ if } b_{ijk}^2-4a_{ijk} \text{ is square of } \mathbf{F}_q^*,\\ 1, \text{ if } b_{ijk}^2-4a_{ijk}=0,\\ 2, \text{ if } b_{ijk}^2-4a_{ijk} \text{ is non-square of } \mathbf{F}_q^*;\end{cases}$$

in the case of char $\mathbf{F}_{q^2}=2$

$$\omega(i, j, k)=\begin{cases}0, \text{ if } D\left(\frac{a_{ijk}}{b_{ijk}^2}\right)=0,\\ 1, \text{ if } b_{ijk}=0,\\ 2, \text{ if } D\left(\frac{a_{ijk}}{b_{ijk}^2}\right)=1,\end{cases}$$

where

$$a_{ijk}=\frac{ij}{k},\quad b_{ijk}=\frac{ij}{k}-ij+i+j$$

and

$$D(t)=t+t^2+t^4+\cdots+t^{2^{h-1}}$$

if $q=2^h$.

(v) The other parameters can be computed by the equalities

$$p^i_{jk}=p^i_{kj},\ i,\ j,\ k=1,\ 2,\ \cdots,\ q;$$

$$\sum_{k=1}^{q} p^i_{jk}=\begin{cases} n_i-1, \text{ if } i=j, \\ n_j, \text{ if } i\neq j, \end{cases} \quad i,\ j=1,\ 2,\ \cdots,\ q;$$

$$n_i p^i_{jk}=n_j p^j_{ik},\ i,\ j,\ k=1,\ 2,\ \cdots,\ q.$$

§ 3. PBIB designs with q associate classes

We take the set of all $(m,\ r)$-type subspaces $(r\leqslant m,$ and $n+r-2m\geqslant 0,\ n\geqslant 3)$ of $V_n(\mathbf{F}_{q^2})$ as the set of blocks, and define a treatment V_s ((1, 1)-type subspace) to be put in some block B_t if $V_s\subset B_t$. Then by transitivity theorem, we can prove that it is a PBIB design with q associate classes whose parameters $v,\ n_i,\ p^i_{jk}$ $(1\leqslant i,\ j,\ k\leqslant q)$ have been obtained in § 2. The other parameters are

$$b=N(m,\ r;\ n)=\frac{\prod\limits_{t=n+r-2m+1}^{n}(q^t-(-1)^t)}{\prod\limits_{t=1}^{r}(q^t-(-1)^t)\prod\limits_{t=1}^{m-r}(q^{2t}-1)}\ q^{r(n+r-2m)},$$

$$k=N(1,\ 1;m,\ r;n)=\frac{q^{2m}-1}{q^2-1}-\frac{q^{2(m-r)}-1}{q^2-1}-\frac{(q^r-(-1)^r)(q^{r-1}-(-1)^{r-1})}{q^2-1}\ q^{2(m-r)},$$

$$r=\frac{bk}{v}=\frac{q^{2m}-[(q^r-(-1)^r)(q^{r-1}-(-1)^{r-1})+1]q^{2(m-r)}}{(q-1)q^{n-1}}$$

$$\cdot\frac{\prod\limits_{t=n+r-2m+1}^{n-1}(q^t-(-1)^t)}{\prod\limits_{t=1}^{r}(q^t-(-1)^t)\prod\limits_{t=1}^{m-r}(q^{2t}-1)}\ q^{r(n+r-2m)},$$

$$\lambda_i=\frac{N(m,\ r;n)N(2,\ 2;m,\ r;n)}{N(2,\ 2;n)}$$

$$=\frac{\prod\limits_{t=n+r-2m+1}^{n-2}(q^t-(-1)^t)}{\prod\limits_{t=1}^{r-2}(q^t-(-1)^t)\prod\limits_{t=1}^{m-r}(q^{2t}-1)}\ q^{(r-2)(n+r-2m)}\ (1\leqslant i\leqslant q-1).$$

$$\lambda_q=\frac{N(m,\ r;\ n)N(2,\ 1;\ m,\ r;n)}{N(2,\ 1;n)}$$

$$=\frac{\prod\limits_{t=n+r-2m+1}^{n-3}(q^t-(-1)^t)}{\prod\limits_{t=1}^{r}(q^t-(-1)^t)\prod\limits_{t=1}^{m-r}(q^{2t}-1)}\ q^{(r-1)n+r^2-2rm+3}$$

$$\cdot\Big[\frac{(q^{2m}-1)(q^{2(m-1)}-1)}{q^3-q^2+q-1}-(q^r-(-1)^r)(q^{r-1}-(-1)^{r-1})q^{4m-2r-4}$$

$$-\frac{(q^r-(-1)^r)(q^{r-1}-(-1)^{r-1})(q^{r-2}-(-1)^{r-2})(q^{r-3}-(-1)^{r-3})}{q^3-q^2+q-1}\ q^{4(m-r)}$$

$$-\frac{(q^r-(-1)^r)(q^{r-1}-(-1)^{r-1})(q^{2(m-r)}-1)}{q-1}\ q^{2(m-r-1)}$$

$$-\frac{(q^{2(m-r)}-1)(q^{2(m-r-1)}-1)}{q^3-q^2+q-1}\Big].$$

For example, if we take the set of (2, 2)-type subspace of $V_n(\mathbf{F}_{q^2})$ as the set of blocks, then the parameters of the PBIB design with q associate classes are v, n_i, p^i_{jk} $(1\leqslant i, j, k\leqslant q)$ as in § 2, and

$$b=N(2,\ 2;n)=\frac{(q^n-(-1)^n)(q^{n-1}-(-1)^{n-1})}{(q^2-1)(q+1)}q^{2(n-2)},$$

$$k=N(1,\ 1;2,\ 2;n)=q(q-1),$$

$$r=\frac{bk}{v}=\frac{(q^{n-1}-(-1)^{n-1})}{q+1}q^{n-2},$$

$$\lambda_i=1(1\leqslant i\leqslant q-1),\ \lambda_q=0.$$

When $q=3$, we have a PBIB design with three associate classes, whose parameters are

$$v=\frac{1}{4}(3^n-(-1)^n\cdot 3^{n-1},$$

$$b=\frac{1}{32}(3^n-(-1)^n)(3^{n-1}-(-1)^{n-1})3^{2n-4},$$

$$k=6,$$

$$r=\frac{1}{4}(3^{n-1}-(-1)^{n-1})\cdot 3^{n-2},$$

$$n_1=\frac{1}{4}(3^{n-1}-(-1)^{n-1})\cdot 3^{n-2},$$

$$n_2=(3^{n-1}-(-1)^{n-1})\cdot 3^{n-2},$$

$$p^1_{11}=\frac{1}{4}(3^{n-2}-(-1)^{n-2})\cdot 3^{n-3},$$

$$p^1_{12}=(3^{n-2}-(-1)^{n-2})\cdot 3^{n-3},$$

$$p^1_{22}=4[1+(3^{n-2}-(-1)^{n-2})(3^{n-3}-(-1)^{n-3})],$$

$$p^2_{22}=2[1+(3^{n-2}-(-1)^{n-2})(3^{n-3}-(-1)^{n-3})+(3^{n-2}-(-1)^{n-2})\cdot 3^{n-3}],$$

$$\lambda_1=\lambda_2=1,\ \lambda_3=0.$$

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