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SINGULAR INTEGRALS IN SEVERAL COMPLEX VARIABLES (II)——HADAMARD PRINCIPAL VALUE ON A SPHERE

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Abstract

Hadamard introduced the concept of finite parts of divergent integrals. i.e. Hadamard principal value, when he researched the Cauchy problems of the hyperbolic type partial differential equations. In this paper, the authors try to generalize this concept to the singular integrals on a sphere of several complex variables space C^n. The Hadamard principal value of higher order singular integral

$$\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1}\frac{f(u)\dot{u}}{(1-v\bar{u}')^{n+\frac{1}{2}}}$$

is defined and the corresponding Plemelj formula is obtained.

§ 0. Introduction

Hadamard introduced the concept of finite parts of divergent integrals, when he researched the Cauchy problems of the hyperbolic type partial differential equations. In the case of one dimension, the integral

$$\int_a^b \frac{f(x)dx}{(x-u)^{n+1}} \tag{0.1}$$

is singular, where $a<u<b$, $n\geq 0$. As $n=0$, the Cauchy principal value of this integral is defined as

$$\lim_{\varepsilon\to 0^+}\left\{\int_a^{u-\varepsilon}\frac{f(x)}{x-u}dx+\int_{u+\varepsilon}^b\frac{f(x)}{x-u}dx\right\}$$

if the above limit exists. As $n>0$, the integral (0.1) is divergent. For given $f(x)$, take $g(x)$, such that

$$\lim_{t\to u^-}\int_a^t\frac{f(x)-g(x)}{(x-u)^{n+1}}dx$$

exists. Hadamard called the limit value the finite part of this divergent integral. He researched the singular integral

$$\int_a^b\frac{f(x)}{(x-u)^{n+\frac{1}{2}}}dx$$

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and gave many important properties of the finite part. Using these results he also solved the Cauchy problem of hyperbolic type partial differential equations[1].

According to Hadamard's idea, Fox[2] regarded the finite part of divergent singular integral (0.1) as a generalization of Cauchy principal value of singular integral, and called it Hadamard principal value. He defined the Hadamard principal value of the divergent integral (0.1) as follow:

$$P\int_a^b \frac{f(x)}{(x-u)^{n+1}}\,dx=\lim_{\varepsilon\to 0^+}\left\{\int_a^{u-\varepsilon}\frac{f(x)}{(x-u)^{n+1}}\,dx+\int_{u+\varepsilon}^b \frac{f(x)}{(x-u)^{n+1}}\,dx-H_n(u,\ \varepsilon)\right\},$$

where

$$H_0(u,\ \varepsilon)=0,\ H_n(u,\ \varepsilon)=\sum_{i=0}^{n-1}\frac{f^{(i)}(u)}{i!}\left\{\frac{1-(-1)^{n-i}}{(u-i)\varepsilon^{n-i}}\right\}.$$

It is clear, when $n=0$, it is just the Cauchy principal value.

Generalize this idea to the complex plane, Hadamard principal value becomes a generalization of the Cauchy principal value of Cauchy type integral. Fox obtained the corresponding Plemelj formula and solved the boundary value problem and singular integral equations with the aid of these tools.

This paper generalizes the concept of Hadamard principal value to the unit ball ——the simplest and the most essential irreducible domain. We shall give the Hadamard principal value of following singular integral

$$\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1}\frac{f(u)\dot{u}}{(1-v\bar{u}')^{n+\frac{1}{2}}}$$

and derive the corresponding Plemelj formula by means of this principal value.

It is not hard to see, these results can be generalized to the strictly pseudoconvex domain Ω. For the Cauchy-Fantapiee kernel

$$K(w,\ z)=C_n g^{-n}\omega\wedge dz_1\wedge\cdots\wedge dz_n,\ w\in\Omega,\ z\in b\Omega,$$

where

$$C_n=(-1)^{\frac{n(n-1)}{2}}(n-1)!(2\pi i)^{-n},$$

$$g(w,\ z)=\sum_{i=1}^{n}(z_i-w_i)g_i(w,\ z),$$

$$\omega=\sum_{j=1}^{n}(-1)^{j-1}g_j\bar{\partial}g_1\wedge\cdots\wedge[\bar{\partial}g_j]\wedge\cdots\wedge\bar{\partial}g_n.$$

we can discuss the Hadamard principal value of singular integral

$$\int_{b\Omega}f(z)C_n g^{-(n+\frac{1}{2})}\omega\wedge dz_1\wedge\cdots\wedge dz_n,\ w\in b\Omega,\ z\in b\Omega.$$

When the kernel is Henkin-Ramirez kernel or Stein-Kerzman kernel, we can give the corresponding Hadamard principal value according to the method of [4].

The notations in this paper are similar to the paper [4].

§ 1. Some Lemmas

Lemma 1. *Suppose* $c\in\left[0,\ \frac{\pi}{2}\right]$ *we have*

$$J=\int_{-(\pi-c)}^{-c}\frac{d\theta}{(1-re^{i\theta})^{n+\frac{1}{2}}}+\int_{c}^{\pi-c}\frac{d\theta}{(1-re^{i\theta})^{n+\frac{1}{2}}}=2\mathrm{Im}(B_1-B_2),$$

where

$$B_1=\sum_{k=1}^{n}\frac{2}{2k-1}\frac{1}{(1+re^{-ic})^{k-\frac{1}{2}}}+\log\left(1-\sqrt{1+re^{-ic}}\right)-\log\left(1+\sqrt{1+re^{-ic}}\right),$$

$$B_2=\sum_{k=1}^{n}\frac{2}{2k-1}\frac{1}{(1-re^{ic})^{k-\frac{1}{2}}}+\log\left(1-\sqrt{1-re^{-ic}}\right)-\log\left(1+\sqrt{1-re^{ic}}\right).$$

Proof It is clear that

$$J=2\,\mathrm{Re}\int_{c}^{\pi-c}\frac{d\theta}{(1-re^{i\theta})^{n+\frac{1}{2}}},$$

put $z=re^{i\theta}$, then

$$\int_{c}^{\pi-c}\frac{d\theta}{(1-re^{i\theta})^{n+\frac{1}{2}}}=\frac{1}{i}\int_{re^{ic}}^{-re^{-ic}}\frac{dz}{z(1-z)^{n+\frac{1}{2}}},$$

Substitute

$$\frac{1}{z(1-z)^{n+\frac{1}{2}}}=\sum_{k=1}^{n}\frac{1}{(1-z)^{k+\frac{1}{2}}}+\frac{1}{z(1-z)^{\frac{1}{2}}}$$

into the above equality, we obtain

$$J=2\mathrm{Im}(B_1-B_2).$$

Lemma 2. *Suppose* $\alpha>0,\ \beta>0,\ \gamma=\frac{\alpha}{\beta}$. *Let*

$$A_n=-\frac{2^{n+\frac{1}{2}}\sqrt{\alpha}\,(n-1)}{\pi(2n-1)}\,\mathrm{Im}\left(\int_0^1\frac{x^{n-2}dx}{(x-i\gamma\sqrt{1-x^2})^{n-\frac{1}{2}}}\right)$$

then

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\dot{u}}{(1-\bar{u}_n)^{n+\frac{1}{2}}}-\frac{A_n}{\sqrt{\varepsilon}}\right\}=1,$$

where

$$\sigma_\varepsilon=\{u:u\bar{u}'=1,\ \alpha^2(1-|u_n|^2)^2+4\beta^2(\mathrm{Im}\,u_n)^2>\varepsilon^2\}.$$

Proof Let

$$c=\arcsin\frac{\sqrt{\varepsilon^2-\alpha^2(1-r^2)^2}}{2\beta r},$$

$$\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\dot{u}}{(1-\bar{u}_n)^{n+\frac{1}{2}}}=\frac{1}{\omega_{2n-1}}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}\dot{v}\left\{\int_{-(\pi-c)}^{-c}\frac{d\theta}{(1-re^{i\theta})^{n+\frac{1}{2}}}+\int_{c}^{\pi-c}\frac{d\theta}{(1-re^{i\theta})^{n+\frac{1}{2}}}\right\}$$

$$+\frac{1}{\omega_{2n-1}}\int_{v\bar{v}'>\frac{\varepsilon}{\alpha}}\dot{v}\int_{-\pi}^{\pi}\frac{d\theta}{(1-re^{i\theta})^{n+\frac{1}{2}}}=I_1+I_2.$$

By Lemma 1, I_1 is equal to

$$\frac{2}{\omega_{2n-1}}\,\mathrm{Im}\Big\{\Big(\sum_{k=1}^{n}J_k+J_0-J_0'\Big)-\Big(\sum_{k=1}^{n}H_k+H_0-H_0'\Big)\Big\},$$

where

$$J_k=\frac{2}{2k-1}\int_{v\bar v'<\frac{\varepsilon}{\alpha}}\frac{\dot v}{(1+re^{-ic})^{k-\frac12}},\quad J_0=\int_{v\bar v'<\frac{\varepsilon}{\alpha}}\log(1-\sqrt{1+re^{-ic}})\dot v,$$

$$J_0'=\int_{v\bar v'<\frac{\varepsilon}{\alpha}}\log(1+\sqrt{1+re^{-ic}})\dot v,\quad H_k=\frac{2}{2k-1}\int_{v\bar v'<\frac{\varepsilon}{\alpha}}\frac{\dot v}{(1-re^{ic})^{k-\frac12}},$$

$$H_0=\int_{v\bar v'<\frac{\varepsilon}{\alpha}}\log(1-\sqrt{1-re^{ic}})\dot v,\quad H_0'=\int_{v\bar v'<\frac{\varepsilon}{\alpha}}\log(1+\sqrt{1-re^{ic}})\dot v.$$

Since

$$1+re^{-ic}=1+(2\beta)^{-1}(\sqrt{4\beta^2r^2-\varepsilon^2+\alpha^2(1-r^2)^2}-i\sqrt{\varepsilon^2-\alpha^2(1-r^2)^2}),$$

using the polar coordinate and letting $1-r^2=s^2$, we have

$$J_k=\frac{4\pi^{n-1}}{(2k-1)\Gamma(n-1)}\int_0^{\sqrt{\varepsilon\alpha^{-1}}}s^{2n-3}\{1+(2\beta)^{-1}[\sqrt{4\beta^2(1-s^2)+\alpha^2s^4-\varepsilon^2}$$
$$-i\sqrt{\varepsilon^2-\alpha^2s^4}]\}^{-(k-\frac12)}ds.$$

Put $\eta=\varepsilon\alpha^{-1}$, $s=\sqrt{\eta}\,t$. Then J_k equals

$$\frac{4\pi^{n-1}(2\beta)^{k-\frac12}}{(2k-1)\Gamma(n-1)}\eta^{n-1}\int_0^1 t^{2n-3}[2\beta+\sqrt{4\beta^2(1-\eta t^2)+\alpha^2\eta^2(t^4-1)}$$
$$-i\sqrt{\alpha^2\eta^2(1-t^4)}]^{-(k-\frac12)}dt.$$

Since the absolute value of the integrand does not exceed $(4\beta^2)^{-1}$, so

$$\lim_{\varepsilon\to0}J_k=0,\quad(k=1,\,2,\,\cdots,\,n).$$

By the same reason, we have

$$\lim_{\varepsilon\to0}\mathrm{Im}(J_0)=\lim_{\varepsilon\to0}\mathrm{Im}(J_0')=0.$$

H_k may be written as

$$\frac{4\pi^{n-1}(2\beta)^{k-\frac12}}{(2k-1)\Gamma(n-1)}\eta^{n-1}\int_0^1 t^{2n-3}[2\beta-\sqrt{4\beta^2(1-\eta t^2)+\alpha^2\eta^2(t^4-1)}$$
$$-i\sqrt{\alpha^2\eta^2(1-t^4)}]^{-(k-\frac12)}dt.$$

Let $P=\beta t^2-i\alpha\sqrt{1-t^4}$, and $Q=(4\beta)^{-1}[\alpha^2-(\alpha^2-\beta^2)t^4]$. Then

$$[2\beta-\sqrt{4\beta^2(1-\eta t^2)+\alpha^2\eta^2(t^4-1)}-i\alpha\eta\sqrt{1-t^4}]^{k-\frac12}$$
$$=\eta^{k-\frac12}\Big[P^{k-\frac12}+\Big(k-\frac12\Big)QP^{k-\frac32}\eta+O(\eta^2)\Big],$$

so

$$H_k=\frac{4\pi^{n-1}(2\beta)^{k-\frac12}}{(2k-1)\Gamma(n-1)}\eta^{n-k-\frac12}\int_0^1 t^{2n-3}\Big[P^{k-\frac12}+\Big(k-\frac12\Big)QP^{k-\frac32}\eta+O(\eta^2)\Big]^{-1}dt.$$

When $k<n-\dfrac12$, we have

$$\lim_{\varepsilon\to 0} H_k=0 \quad (k=1,\ 2,\ \cdots,\ n-1).$$

On the other hand, we have

$$\lim_{\varepsilon\to 0}\operatorname{Im}(H_0)=\lim_{\varepsilon\to 0}\operatorname{Im}(H_0')=0.$$

Now we calculate H_n

$$H_n=\frac{4\pi^{n-1}(2\beta)^{n-\frac{1}{2}}}{(2n-1)\Gamma(n-1)}\eta^{-\frac{1}{2}}\int_0^1 t^{2n-3}\left[P^{n-\frac{1}{2}}+\left(n-\frac{1}{2}\right)QP^{n-\frac{3}{2}}\eta+O(\eta^2)\right]^{-1}dt.$$

It is easy to see, A_n may be written as

$$A_n=\frac{-2}{\omega_{2n-1}}\frac{4\pi^{n-1}(2\beta)^{n-\frac{1}{2}}\sqrt{\alpha}}{(2n-1)\Gamma(n-1)}\operatorname{Im}\left(\int_0^1 t^{2n-3}[\beta t^2-i\alpha\sqrt{1-t^4}]^{-(n-\frac{1}{2})}\,dt\right),$$

then

$$-\frac{2}{\omega_{2n-1}}\operatorname{Im}(H_n)-\frac{1}{\sqrt{\varepsilon}}A_n$$

$$=\frac{-2}{\omega_{2n-1}}\frac{4\pi^{n-1}(2\beta)^{n-\frac{1}{2}}}{(2n-1)\Gamma(n-1)\sqrt{\alpha}}\sqrt{\varepsilon}\operatorname{Im}\left\{\int_0^1\frac{-t^{2n-3}\left[\left(n-\frac{1}{2}\right)P^{n-\frac{3}{2}}Q+O(\eta)\right]dt}{P^{n-\frac{1}{2}}\left[P^{n-\frac{1}{2}}+\left(n-\frac{1}{2}\right)P^{n-\frac{3}{2}}Q\eta+O(\eta^2)\right]}\right\}.$$

So we have

$$\lim_{\varepsilon\to 0}\left\{\frac{-2}{\omega_{2n-1}}\operatorname{Im}(H_n)-\frac{1}{\sqrt{\varepsilon}}A_n\right\}=0.$$

For I_2, we notice

$$(1-re^{i\theta})^{-(n+\frac{1}{2})}=\sum_{q=0}^{\infty}\frac{\Gamma\left(n+q+\frac{1}{2}\right)}{\Gamma(q+1)\Gamma\left(n+\frac{1}{2}\right)}r^q e^{iq\theta},$$

so

$$\int_{-\pi}^{\pi}\frac{d\theta}{(1-re^{i\theta})^{n+\frac{1}{2}}}=2\pi,$$

and then $\lim\limits_{\varepsilon\to 0} I_2=1$. Summing up the above, we have

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\dot{u}}{(1-\bar{u}_n)^{n+\frac{1}{2}}}-\frac{A_n}{\sqrt{\varepsilon}}\right\}=1.$$

The proof of Lemma 2 is complete.

Lemma 3.

$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\operatorname{Im}(\bar{u}_n)\dot{u}}{(1-\bar{u}_n)^{n+\frac{1}{2}}}=-\frac{2n+1}{4ni},$$

$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\operatorname{Re}(1-\bar{u}_n)}{(1-\bar{u}_n)^{n+\frac{1}{2}}}\dot{u}=\frac{2n-1}{4n}.$$

Proof

$$\frac{1}{\omega_{2n-1}}\int_{\sigma_e}\frac{\operatorname{Im}(\bar{v}_n)}{(1-\bar{v}_n)^{n+\frac{1}{2}}}\dot{u}$$
$$=\frac{2i}{\omega_{2n-1}}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}\left\{\operatorname{Im}\int_c^{\pi-c}\frac{r\sin\theta d\theta}{(1-re^{i\theta})^{n+\frac{1}{2}}}\right\}\dot{v}+\frac{1}{\omega_{2n-1}}\int_{v\bar{v}'\geq\frac{\varepsilon}{\alpha}}\dot{v}\int_{-\pi}^{\pi}\frac{r\sin\theta d\theta}{(1-re^{i\theta})^{n+\frac{1}{2}}}=I_1+I_2.$$

We calculate I_1 and I_2 respectively. Since

$$\frac{1}{z^2(1-z)^{n+\frac{1}{2}}}=\sum_{k=1}^{n}\frac{n+1-k}{(1-z)^{k+\frac{1}{2}}}+\frac{1}{z^2(1-z)^{\frac{1}{2}}}+\frac{n}{z(1-z)^{\frac{1}{2}}}$$

so

$$\begin{aligned}\int_c^{\pi-c}\frac{r\sin\theta d\theta}{(1-re^{i\theta})^{n+\frac{1}{2}}}=&\left\{\frac{r^2-1}{2n-1}\,\frac{1}{(1+re^{-ic})^{n-\frac{1}{2}}}+\sum_{k=1}^{n-1}\frac{n+1-k}{2k-1}\,\frac{r^2}{(1+re^{-ic})^{k-\frac{1}{2}}}\right.\\&\left.+\frac{r\sqrt{1+re^{-ic}}}{2e^{-ic}}-\frac{2n+1}{4}\,r^2\log\frac{1+\sqrt{1+re^{-ic}}}{1-\sqrt{1+re^{-ic}}}\right\}\\&-\left\{\frac{r^2-1}{2n-1}\,\frac{1}{(1-re^{ic})^{n-\frac{1}{2}}}+\sum_{k=1}^{n-1}\frac{n+1-k}{2k-1}\,\frac{r^2}{(1-re^{ic})^{k-\frac{1}{2}}}\right.\\&\left.-\frac{r\sqrt{1-re^{ic}}}{2e^{ic}}-\frac{2n+1}{4}\,r^2\log\frac{1+\sqrt{1-re^{ic}}}{1-\sqrt{1-re^{ic}}}\right\}\end{aligned}\tag{1.1}$$

and

$$I_1=\frac{2i}{\omega_{2n-1}}\left\{\operatorname{Im}\left(M_n+\sum_{k=1}^{n-1}M_k+M_0-M_0'\right)-\operatorname{Im}\left(N_n+\sum_{k=1}^{n-1}N_k-N_0-N_0'\right)\right\}.$$

Where

$$M_n=\frac{1}{2n-1}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}(r^2-1)(1+re^{-ic})^{-(n-\frac{1}{2})}\dot{v},$$

$$M_k=\frac{n+1-k}{2k-1}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}r^2(1+re^{-ic})^{-(k-\frac{1}{2})}\dot{v}\ (k=1,\ \cdots,\ n-1).$$

$$M_0=\frac{1}{2}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}re^{ic}\sqrt{1+re^{-ic}}\,\dot{v},$$

$$M_0'=\frac{2n+1}{4}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}r^2\log\frac{1+\sqrt{1+re^{-ic}}}{1-\sqrt{1+re^{-ic}}}\,\dot{v},$$

$$N_n=\frac{1}{2n-1}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}(r^2-1)(1-re^{ic})^{-(n-\frac{1}{2})}\dot{v},$$

$$N_k=\frac{n+1-k}{2k-1}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}r^2(1-re^{ic})^{-(k-\frac{1}{2})}\dot{v}(k=1,\ \cdots,\ n-1),$$

$$N_0=\frac{1}{2}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}re^{-ic}\sqrt{1-re^{ic}}\,\dot{v},$$

$$N_0'=\frac{2n+1}{4}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}r^2\log\frac{1+\sqrt{1-re^{ic}}}{1-\sqrt{1-re^{ic}}}\,\dot{v}.$$

It is easy to prove that

$$\lim_{\varepsilon\to0}M_k=0\quad(k=1,\ \cdots,\ n).$$

Since the absolute values of M_0, N_0 are bounded, so

$$\lim_{\varepsilon\to0}M_0=\lim_{\varepsilon\to0}N_0=0.$$

Clearly

$$\lim_{\varepsilon\to 0}\operatorname{Im}(M_0')=\lim_{\varepsilon\to 0}\operatorname{Im}(N_0')=0.$$

Similar to the proof of

$$\lim_{\varepsilon\to 0} H_k=0 \quad (k=1, \cdots, n-1)$$

in Lemma 2, we can prove

$$\lim_{\varepsilon\to 0}\operatorname{Im}(N_k)=0 \quad (k=1, \cdots, n-1).$$

N_n is equal to

$$\frac{-2\pi^{n-1}(2\beta)^{n-\frac{1}{2}}}{(2n-1)\Gamma(n-1)}\int_0^1 \eta^n t^{2n-1}\left[2\beta-\sqrt{4\beta^2(1-\eta t^2)+\alpha^2\eta^2(t^4-1)}-i\sqrt{\alpha^2\eta^2(1-t^4)}\right]^{-\left(n-\frac{1}{2}\right)}dt$$

$$=\frac{-2\pi^{n-1}(2\beta)^{n-\frac{1}{2}}}{(2n-1)\Gamma(n-1)}\sqrt{\eta}\int_0^1 t^{2n-1}\left[\beta t^2-i\alpha\sqrt{1-t^4}+O(\eta)\right]^{-\left(n-\frac{1}{2}\right)}dt.$$

so $\lim_{\varepsilon\to 0} N_n=0$. Hence $\lim_{\varepsilon\to 0} I_1=0$.

Since

$$\int_{-\pi}^{\pi}\frac{r\sin\theta d\theta}{(1-re^{i\theta})^{n+\frac{1}{2}}}=\frac{1}{2i}\sum_{q=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}+q\right)}{\Gamma(q+1)\Gamma\left(n+\frac{1}{2}\right)}r^{q+1}\int_{-\pi}^{\pi}\left[e^{i(q+1)\theta}-e^{i(q-1)\theta}\right]d\theta$$

$$=i\left(n+\frac{1}{2}\right)\pi r^2.$$

so

$$\lim_{\varepsilon\to 0} I_2=-\frac{2n+1}{4ni}.$$

Summing up the above, we obtain

$$\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\operatorname{Im}(\bar{u}_n)}{(1-\bar{u}_n)^{n+\frac{1}{2}}}\dot{u}=-\frac{2n+1}{4ni}.$$

Since $\lim_{\varepsilon\to 0}\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\dot{u}}{(1-\bar{u}_n)^{n-\frac{1}{2}}}=1$, and

$$\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\operatorname{Re}(1-\bar{u}_n)}{(1-\bar{u}_n)^{n+\frac{1}{2}}}\dot{u}=\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\dot{u}}{(1-\bar{u}_n)^{n-\frac{1}{2}}}+\frac{i}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\operatorname{Im}(\bar{u}_n)}{(1-\bar{u}_n)^{n+\frac{1}{2}}}\dot{u},$$

another equality of this Lemma holds.

Lemma 4.

$$\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\operatorname{Re}(u_k)}{(1-\bar{u}_n)^{n+\frac{1}{2}}}\dot{u}=0,\quad \frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\operatorname{Im}(u_k)}{(1-\bar{u}_n)^{n+\frac{1}{2}}}\dot{u}=0 \quad (k=1, \cdots, n-1).$$

Proof We only need to prove the case $k=1$. Set $\bar{u}_n=re^{i\theta}$, $u_1=v_1, \cdots, u_{n-1}=v_{n-1}$, we have

$$\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\operatorname{Re}u_1}{(1-\bar{u}_n)^{n+\frac{1}{2}}}\dot{u}=\frac{1}{\omega_{2n-1}}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}\operatorname{Re}v_1\dot{v}\left\{\int_{-(\pi-c)}^{-c}\frac{d\theta}{(1-re^{i\theta})^{n+\frac{1}{2}}}+\int_c^{\pi-c}\frac{d\theta}{(1-re^{i\theta})^{n+\frac{1}{2}}}\right\}$$

$$+\frac{1}{\omega_{2n-1}}\int_{v\bar{v}'>\frac{\varepsilon}{\alpha}}\operatorname{Re}v_1\dot{v}\int_{-\pi}^{\pi}\frac{d\theta}{(1-re^{i\theta})^{n+\frac{1}{2}}}=I_1+I_2.$$

It is easy to see

$$I_1=\frac{2}{\omega_{2n-1}}\operatorname{Im}\left\{\left(\sum_{k=1}^{n}J_k+J_0-J_0'\right)-\left(\sum_{k=1}^{n}H_k+H_0-H_0'\right)\right\},$$

where

$$J_k=\frac{2}{2k-1}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}(\operatorname{Re}v_1)\,(1+re^{-ic})^{-\left(k-\frac{1}{2}\right)}\,\dot{v},$$

$$J_0=\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}(\operatorname{Re}v_1)\log\left(1-\sqrt{1+re^{-ic}}\right)\dot{v},$$

$$J_0'=\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}(\operatorname{Re}v_1)\log\left(1+\sqrt{1+re^{-ic}}\right)\dot{v},$$

$$H_k=\frac{2}{2k-1}\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}(\operatorname{Re}v_1)\,(1-re^{ic})^{-\left(k-\frac{1}{2}\right)}\,\dot{v},$$

$$H_0=\int_{v\bar{v}_,<\frac{\varepsilon}{\alpha}}(\operatorname{Re}v_1)\log\left(1-\sqrt{1-re^{ic}}\right)\dot{v},$$

$$H_0'=\int_{v\bar{v}'<\frac{\varepsilon}{\alpha}}(\operatorname{Re}v_1)\log\left(1++\sqrt{1-re^{ic}}\right)\dot{v}.$$

Using the spherical polar coordinate and putting $v=(x_1, x_2, \cdots, x_{2n-2})$, we have

$$J_k=\frac{2}{2k-1}\int_0^{\sqrt{\varepsilon\alpha^{-1}}}\frac{s^{2n-2}ds}{(1+re^{-ic})^{k-\frac{1}{2}}}\int_0^{\pi}\cos\varphi_1\sin^{2n-4}\varphi_1 d\varphi_1\int_0^{\pi}\sin^{2n-5}\varphi_2\,d\varphi_2\cdots\int_0^{2\pi}d\varphi_{2n-3}=0$$

$$(k=1, \cdots, n).$$

Similarly $J_0=J_0'=H_k=H_0=H_0'=0$, so $I_1=0$. Accordingly, $I_2=0$ It follows that

$$\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{\operatorname{Re}\dot{u}_1}{(1-\bar{u}_n)^{n+\frac{1}{2}}}\,\dot{u}=0.$$

We can prove the another equality of this Lemma by the same method.

Lemma 5. *Suppose $f(u_1, \cdots, u_n)=f(x_1+iy_1, \cdots, x_n+iy_n)$ is a real function defined on the unit ball $\bar{B}=\{u:u\bar{u}'\leqslant 1\}$. If*

$$\frac{\partial f}{\partial x_j}\in\operatorname{Lip}\alpha_j,\ \frac{\partial f}{\partial y_j}\in\operatorname{Lip}\beta_j\ (j=1, \cdots, n),$$

then

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$$I_{x_k}=\omega_{2n-1}^{-1}\int_{u\bar{u}'=1}\Big[f(0, \cdots, 0, x_k+iy_k, \cdots, x_n+iy_n)-f(0, \cdots, 0, iy_k, \cdots, x_n+iy_n)$$
$$-\frac{\partial f}{\partial x_k}(p_n)x_k\Big](1-\bar{u}_n)^{-\left(n+\frac{1}{2}\right)}\dot{u},$$

$$I_{y_k}=\omega_{2n-1}^{-1}\int_{u\bar{u}'=1}\Big[f(0, \cdots, 0, iy_k, x_{k+1}+iy_{k+1}, \cdots, x_n+iy_n)$$
$$-f(0, \cdots, 0, x_{k+1}+iy_{k+1}, \cdots, x_n+iy_n)-\frac{\partial f}{\partial y_k}(p_n)y_k\Big](1-\bar{u}_n)^{-\left(n+\frac{1}{2}\right)}\dot{u},$$

$$I_{x_n}=\omega_{2n-1}^{-1}\int_{u\bar{u}'=1}\Big[f(0, \cdots, 0, x_n)-f(0, \cdots, 0, 1)-\frac{\partial f}{\partial x_n}(p_n)(x_n-1)\Big](1-\bar{u}_n)^{-\left(n+\frac{1}{2}\right)}\dot{u},$$

$$I_{y_n}=\omega_{2n-1}^{-1}\int_{u\bar{u}'=1}\Big[f(0, \cdots, 0, x_n+iy_n)-f(0, \cdots, 0, x_n)-\frac{\partial f}{\partial y_n}(p_n)y_n\Big](1-\bar{u}_n)^{-\left(n+\frac{1}{2}\right)}\dot{u}$$

exist, where $p_n=(0, \cdots, 0, 1)$, $k=1, \cdots, n-1$.

(ii) *Let* $z\in B$. *Replacing* $(1-\bar{u}_n)^{-(n+\frac{1}{2})}$ *by* $(1-z\bar{u}')^{-(n+\frac{1}{2})}$ *in the above integrals, denote these new integrals by* $I_{x_k}(z)$, $I_{y_k}(z)$, $I_{x_n}(z)$, $I_{y_n}(z)$ *respectively. Then*

$$(K-\lim_{z\to p_n}) I_{x_k}(z)=I_{x_k},\quad (K-\lim_{z\to p_n}) I_{y_k}(z)=I_{y_k}\ (k=1, \cdots, n).$$

The concept of K limit was introduced by A. Koranyi[5]: Let

$$D_\alpha(v)=\left\{z: z\bar{z}'<1,\ v\bar{v}'=1,\ |1-z\bar{v}'|<\frac{\alpha}{2}(1-z\bar{z}'),\ \alpha>1\right\}.$$

If

$$\lim_{z\to v, z\in D_\alpha(v)} f(z)=\lambda.$$

holds for all $\alpha>1$, we say that f has the K limit λ at the point v.

Proof It is clear that

$$\left|f(0, \cdots, 0, x_k+iy_k, \cdots, x_n+iy_n)-f(0, \cdots, 0, iy_k, \cdots, x_n+iy_n)-\frac{\partial f}{\partial x_k}(p_n)x_k\right|$$

$$\leqslant K|1-\bar{u}_n|^{\frac{1}{2}(\alpha_k+1)},$$

where K is a constant. By Lemma 5.4.3 of [6], when $z\in D_\alpha(p_n)$, $u\bar{u}'=1$, then

$$|1-z\bar{u}'|\geqslant\frac{1}{4\alpha}|1-\bar{u}_n|.$$

Let $g_k(z, u)$ be the integrand of $I_{x_k}(z)$, then

$$|g_k(z, u)|\leqslant M|1-\bar{u}_n|^{-(n-\frac{\alpha_k}{2})}.$$

So $\lim_{z\to p_n} g_k(z, u)$ is integrable on $u\bar{u}'=1$, i. e. I_{x_k} exists, and

$$\lim_{z\to p_n}\int_{u\bar{u}'=1} g_k(z, u)\dot{u}=\int_{u\bar{u}'=1}\lim_{z\to p_n} g_k(z, u)\dot{u}.$$

It follows that $(K-\lim_{z\to p_n}) I_{x_k}(z)=I_{x_k}$. The proofs of other parts of this lemma are similar.

§ 2. Hadamard principal value

Now we can prove the following

Theorem 1. *If, with* I_{x_k}, I_{y_k} *defined by Lemma* 5, f *satisfies the same conditions in Lemma* 5, *then we have*

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{f(u)\dot{u}}{(1-\bar{u}_n)^{n+\frac{1}{2}}}-\frac{f(p_n)}{\sqrt{\varepsilon}}A_n\right\}$$

$$=\sum_{k=1}^{n}(I_{x_k}+I_{y_k})+\frac{1}{2n}\frac{\partial f}{\partial u_n}(p_n)-\frac{\partial f}{\partial \bar{u}_n}(p_n)+f(p_n). \tag{2.1}$$

Proof f can be written as

$$f(u)=\sum_{k=1}^{n-1}\Big[f(0,\cdots,0,x_k+iy_k,\cdots,x_n+iy_n)-f(0,\cdots,0,iy_k,\cdots,x_n+iy_n)$$
$$-\frac{\partial f}{\partial x_k}(p_n)x_k\Big]+\sum_{k=1}^{n-1}\Big[f(0,\cdots,0,iy_k,\cdots,x_n+iy_n)$$
$$-f(0,\cdots,0,x_{k+1}+iy_{k+1},\cdots,x_n+iy_n)-\frac{\partial f}{\partial y_k}(p_n)y_k\Big]$$
$$+\sum_{k=1}^{n-1}\frac{\partial f}{\partial x_k}(p_n)x_k+\sum_{k=1}^{n-1}\frac{\partial f}{\partial y_k}(p_n)y_k$$
$$+\Big[f(0,\cdots,0,x_n+iy_n)-f(0,\cdots,0,x_n)-\frac{\partial f}{\partial y_n}(p_n)y_n\Big]+\frac{\partial f}{\partial y_n}(p_n)y_n$$
$$+\Big[f(0,\cdots,0,x_n)-f(p_n)-\frac{\partial f}{\partial x_n}(p_n)(x_n-1)\Big]+\frac{\partial f}{\partial x_n}(p_n)(x_n-1)+f(p_n),$$

then

$$\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon}\frac{f(u)\dot u}{(1-\bar u_n)^{n+\frac12}}-\frac{f(p_n)}{\sqrt{\varepsilon}}A_n$$
$$=\sum_{k=1}^{n-1}\omega_{2n-1}^{-1}\int_{\sigma_\varepsilon}\Big[f(0,\cdots,0,x_k+iy_k,\cdots,x_n+iy_n)-f(0,\cdots,0,iy_k,\cdots,x_n+iy_n)$$
$$-\frac{\partial f}{\partial x_k}(p_n)x_k\Big](1-\bar u_n)^{-(n+\frac12)}\dot u+\sum_{k=1}^{n-1}\omega_{2n-1}^{-1}\int_{\sigma_\varepsilon}\Big[f(0,\cdots,0,iy_k,\cdots,x_n+iy_n)$$
$$-f(0,\cdots,0,x_{k+1}+iy_{k+1},\cdots,y_n+iy_n)-\frac{\partial f}{\partial y_k}(p_n)y_k\Big](1-\bar u_n)^{-(n+\frac12)}\dot u$$
$$+\sum_{k=1}^{n-1}\frac{\partial f}{\partial x_k}(p_n)\omega_{2n-1}^{-1}\int_{\sigma_\varepsilon}x_k(1-\bar u_n)^{-(n+\frac12)}\dot u$$
$$+\sum_{k=1}^{n-1}\frac{\partial f}{\partial y_k}(p_n)\omega_{2n-1}^{-1}\int_{\sigma_\varepsilon}y_k(1-\bar u_n)^{-(n+\frac12)}\dot u$$
$$+\omega_{2n-1}^{-1}\int_{\sigma_\varepsilon}\Big[f(0,\cdots,0,x_n+iy_n)-f(0,\cdots,0,x_n)-\frac{\partial f}{\partial y_n}(p_n)y_n\Big](1-\bar u_n)^{-(n+\frac12)}\dot u$$
$$+\frac{\partial f}{\partial y_n}(p_n)\omega_{2n-1}^{-1}\int_{\sigma_\varepsilon}y_k(1-\bar u_n)^{-(n+\frac12)}\dot u$$
$$+\omega_{2n-1}^{-1}\int_{\sigma_\varepsilon}\Big[f(0,\cdots,0,x_n)-f(p_n)-\frac{\partial f}{\partial x_n}(p_n)(x_n-1)\Big](1-\bar u_n)^{-(n+\frac12)}\dot u$$
$$+\frac{\partial f}{\partial x_n}(p_n)\omega_{2n-1}^{-1}\int_{\sigma_\varepsilon}(x_n-1)(1-\bar u_n)^{-(n+\frac12)}\dot u$$
$$+f(p_n)\Big\{\omega_{2n-1}^{-1}\int_{\sigma_\varepsilon}(1-\bar u_n)^{-(n+\frac12)}\dot u-A_n\varepsilon^{-\frac12}\Big\}.$$

By Lemma 2, 3, 4, 5, we obtain the equality (2.1) immediately.

Theorem 2. *Suppose v is an arbitrary point on the sphere S and f satisfies the same condition as in Lemma 5, then we have*

$$\lim_{\varepsilon\to0}\Big\{\omega_{2n-1}^{-1}\int_{\sigma_\varepsilon(v)}f(u)(1-v\bar u')^{-(n+\frac12)}\dot u-\frac{A_n}{\sqrt{\varepsilon}}f(v)\Big\}$$
$$=\sum_{k=1}^{n}(J_{x_k}+J_{y_k})+\frac{1}{2n}\sum_{j=1}^{n}\frac{\partial f}{\partial u_j}(v)v_j-\sum_{j=1}^{n}\frac{\partial f}{\partial\bar u_j}(v)\bar v_j+f(v).\qquad(2.2)$$

where $\sigma_\varepsilon(v)=\{u:u\bar u'=1,\ \alpha^2(1-|v\bar u'|^2)^2+4\beta^2(\mathrm{Im}(v\bar u'))^2>\varepsilon^2\}$, J_{x_k}, J_{y_k} are the integrals which are obtained by substituting $f(u)$ into $f(u\bar u')$ in the integrals I_{x_k}, I_{y_k}, and U is the

$n\times n$ matrix such that $vU=p_n$.

Proof Let $uU=w$. Then the left side of (2.2) is

$$\omega_{2n-1}^{-1}\int_{\sigma_\varepsilon} f(w\overline{U}')\,(1-\overline{w}_n)^{-(n+\frac{1}{2})}\,\dot{w}-\frac{A_n}{\sqrt{\varepsilon}}f(p_n\overline{U}').$$

by Theorem 1, its limit is

$$\sum_{k=1}^{n}(J_{x_k}+J_{y_k})+\frac{1}{2n}\frac{\partial f(w\overline{U}')}{\partial w_n}(p_n\overline{U}')-\frac{\partial f(w\overline{U}')}{\partial \overline{w}_n}(p_n\overline{U}')+f(p_n\overline{U}') \tag{2.3}$$

as $\varepsilon\to 0$. Let $\overline{U}'=(\alpha_{ij})$. Then $(v_1,\cdots,v_n)=(\alpha_{n1},\cdots,\alpha_{nn})$. and

$$\frac{\partial f(w\overline{U}')}{\partial w_n}=\sum_{j=1}^{n}\frac{\partial f}{\partial u_j}v_j,\quad \frac{\partial f(w\overline{U}')}{\partial w_n}=\sum_{j=1}^{n}\frac{\partial f}{\partial \overline{u}_j}\overline{v}_j.$$

From (2.3) we obtain (2.2).

We call the limit (2.2) Hadamard principal value of singular integral

$$\omega_{2n-1}^{-1}\int_{u\overline{u}'=1}\frac{f(u)\dot{u}}{(1-v\overline{u}')^{n+\frac{1}{2}}}$$

and denote it by

$$P\frac{1}{\omega_{2n-1}}\int_{u\overline{u}'=1}\frac{f(u)\dot{u}}{(1-v\overline{u}')^{n+\frac{1}{2}}}=\lim_{\varepsilon\to 0}\left\{\frac{1}{\omega_{2n-1}}\int_{\sigma_\varepsilon(v)}\frac{f(u)\dot{u}}{(1-v\overline{u}')^{n+\frac{1}{2}}}-\frac{A_n}{\sqrt{\varepsilon}}f(v)\right\}.$$

Since A_n depends on σ_ε and there are many ways to define σ_ε, so there are many ways to define the Hadamard principal value. But we can see from (2.2) that the Hadamard principal value are the same, it does not depend on the choice of σ_ε.

§ 3. Plemelj formula

Lemma 6.

$$\lim_{z\to p_n}\omega_{2n-1}^{-1}\int_{u\overline{u}'=1}\frac{x_n-1}{(1-z\overline{u}')^{n+\frac{1}{2}}}\dot{u}=\frac{1-2n}{4n},$$

$$\lim_{z\to p_n}\omega_{2n-1}^{-1}\int_{n\overline{u}'=1}\frac{y_n}{(1-z\overline{u}')^{n+\frac{1}{2}}}\dot{u}=-\frac{2n+1}{4n}i.$$

Proof Fix $z\in B$. Suppose $z\overline{z}'=\rho^2$, there is a unitary matrix U, such that $zU=\rho p_n$. Let $u=w\overline{U}'$, $\overline{U}'=(\alpha_{ij})$. Then

$$x_n=\operatorname{Re}u_n=\frac{1}{2}\sum_{j=1}^{n}(\alpha_{jn}w_j+\overline{\alpha}_{jn}\overline{w}_j),$$

so

$$\frac{1}{\omega_{2n-1}}\int_{u\overline{u}'=1}\frac{(x_n-1)\dot{u}}{(1-z\overline{u}')^{n+\frac{1}{2}}}=\frac{1}{2\omega_{2n-1}}\left\{\alpha_{nn}\int_{w\overline{w}'=1}\frac{w_n\dot{w}}{(1-\rho\overline{w}_n)^{n+\frac{1}{2}}}+\overline{\alpha}_{nn}\int_{w\overline{w}'=1}\frac{\overline{w}_n\dot{w}}{(1-\rho\overline{w}_n)^{n+\frac{1}{2}}}\right.$$
$$\left.-2\int_{w\overline{w}'=1}\frac{\dot{w}}{(1-\rho\overline{w}_n)^{n+\frac{1}{2}}}\right\}. \tag{3.1}$$

Since

$$\int_{-\pi}^{\pi}\frac{re^{-i\theta}\,d\theta}{(1-\rho re^{i\theta})^{n+\frac{1}{2}}}=(2n+1)\pi\rho r^2,$$

so

$$\frac{1}{2\omega_{2n-1}}\int_{w\bar{w}'=1}\frac{w_n\dot{w}}{(1-\rho w_n)^{n+\frac{1}{2}}}=\frac{1}{2\omega_{2n-1}}\int_{0<v\bar{v}'<1}\dot{v}\int_{-\pi}^{\pi}\frac{re^{-i\theta}\,d\theta}{(1-\rho re^{i\theta})^{n+\frac{1}{2}}}=\frac{2n+1}{4n}\rho.$$

Using the same method, we have

$$\int_{w\bar{w}'=1}\frac{\bar{w}_n\dot{w}}{(1-\rho\bar{w}_n)^{n+\frac{1}{2}}}=0,\quad \frac{1}{\omega_{2n-1}}\int_{w\bar{w}'=1}\frac{\dot{w}}{(1-\rho\bar{w}_n)^{n+\frac{1}{2}}}=1.$$

It is not hard to see, when $\rho\to1$, $\alpha_{nn}\to1$, $\alpha_{kn}\to0(k=1,\cdots,n-1)$. From (3.1), we obtain

$$\lim_{z\to p_n}\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1}\frac{(x_n-1)\dot{u}}{(1-z\bar{u}')^{n+\frac{1}{2}}}=\frac{1-2n}{4n}.$$

The proof of the other equality is similar.

Lemma 7.

$$\lim_{z\to p_n}\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1}\frac{\operatorname{Re}u_k}{(1-z\bar{u}')^{n+\frac{1}{2}}}\dot{u}=0,$$

$$\lim_{z\to p_n}\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1}\frac{\operatorname{Im}u_k}{(1-z\bar{u}')^{n+\frac{1}{2}}}\dot{u}=0\quad(k=1,\cdots,n-1).$$

Proof The proof of this lemma is similar to Lemma 6.

Theorem 3. *(Plemelj formula) Suppose f satisfies the condition in Lemma 5, then we have*

$$(K-\lim_{z\to p_n})\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1}\frac{f(u)\dot{u}}{(1-z\bar{u}')^{n+\frac{1}{2}}}=P\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1}\frac{f(u)\dot{u}}{(1-\bar{u}_n)^{n+\frac{1}{2}}}$$

From this theorem, we can obtain immediately the following

Theorem 4. *Suppose $f=f_1+if_2$ is a complex function defined on the unit ball $\bar{B}$, f_1, f_2 satisfy the conditions in Lemma 5, then*

$$(K-\lim_{z\to v})\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1}\frac{f(u)\dot{u}}{(1-z\bar{u}')^{n+\frac{1}{2}}}=P\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1}\frac{f(u)\dot{u}}{(1-v\bar{u}')^{n+\frac{1}{2}}},$$

where v is an arbitrary point on the sphere S, and

$$P\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1}\frac{f(u)\dot{u}}{(1-v\bar{u}')^{n+\frac{1}{2}}}$$
$$=P\frac{1}{\omega_{2n-1}}\int_{u\bar{u}'=1}\frac{f_1(u)\dot{u}}{(1-v\bar{u}')^{n+\frac{1}{2}}}+P\frac{i}{\omega_{2n-1}}\int_{u\bar{u}'=1}\frac{f_2(u)\dot{u}}{(1-v\bar{u}')^{n+\frac{1}{2}}}.$$

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