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SOME GENERAL RESULTS ON THE FIRST BOUNDARY VALUE PROBLEM FOR QUASILINEAR DEGENERATE PARABOLICE QUATIONS

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Abstract

In this paper, the authors investigate the first boundary value problem for equations of the form

$$Lu=\frac{\partial u}{\partial t}-\frac{\partial}{\partial x_i}\left(a^{ij}(u, x, t)\frac{\partial u}{\partial x_j}\right)-\frac{\partial f^i(u, x, t)}{\partial x_i}=g(u, x, t)$$

with $a^{ij}(u, x, t)\xi_i\xi_j\geqslant 0.$

An existence theorem of solution in $BV_{1,\frac{1}{2}}(Q_T)$ is proved. The principal condition is that there exists $\delta>0$ such that for any $(x, t)\in Q_T$, $|u|\leqslant M$

$$a^{ij}(u, x, t)\xi_i\xi_j-\delta\sum_{s,j=1}^{m}(a^{ij}_{x_s}(u, x, t)\xi_i)^2\geqslant 0.$$

§ 1. Introduction

In a recent paper[1], we have studied the global solutions of the first boundary value problem for the quasilinear equation of the form

$$Lu\equiv\frac{\partial u}{\partial t}-\frac{\partial}{\partial x_i}\left(a^{ij}(u, x, t)\frac{\partial u}{\partial x_j}\right)-\frac{\partial}{\partial x_j}f^i(u, x, t)=g(u, x, t) \tag{1.1}$$

with $a^{ij}=a^{ji}$ and

$$a^{ij}(u, x, t)\xi_i\xi_j\geqslant 0,\ \forall u\in R,\ (x, t)\in\bar{Q}_T,\ \xi=(\xi_1, \xi_2, \cdots, \xi_m)\in R^m,$$

where $Q_T=\Omega\times(0, T)$ and $\Omega\subset R^m$ is a bounded region with an appropriately smooth boundary Σ. The boundary value condition and the initial value condition are

$$u|_{\Sigma\times[0,T]}=0 \tag{1.2}$$

and

$$u|_{t=0}=u_0(x) \tag{1.3}$$

respectively. Under certain conditions, in[1], the solvability was established in $BV(Q_T)$, a class of all integrable functions whose generalized derivatives are measures with bounded variations.

In this paper, we shall show that the conditions described in[1] for the existence of solutions may be weakened. But with the weaker condition which will be stated in

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Theorem 1, we can only obtain the solutions in a more wide class of functions, denoted by $BV_{1,\frac{1}{2}}(Q_T)$ (see [2]), each element $v(x, t)$ of which is integrable on Q_T and satisfies

$$\iint_{Q_T} |v(x, t+\Delta t)-v(x, t)|\,dx\,dt \leqslant K|\Delta t|^{\frac{1}{2}},$$

$$\iint_{Q_T} |v(x+\Delta x, t)-v(x, t)|\,dx\,dt \leqslant K|\Delta x|$$

for some constant K. Here we set $v=0$ for $(x, t)\overline{\in} Q_T$. Clearly $BV(Q_T)\subset BV_{1,\frac{1}{2}}(Q_T)$. The generalized derivatives of any function in $BV_{1,\frac{1}{2}}(Q_T)$ with respect to $x_i(i=1, 2,\cdots, m)$ are measures with bounded variations, but in general, the generalized derivative with respect to t is not.

We shall assume that $a^{ij}(u, x, t)$, $f^i(u, x, t)$ and $g(u, x, t)$ are appropriately smooth for $u\in R$, $(x, t)\in\overline{Q}_T$ and g_u, f_{x_iu} are bounded, and $u_0(x)$ is appropriately smooth for $x\in\overline{\Omega}$. In addition, certain compatibility conditions will be assumed, namely, $u_0(x)$ itself and its first and second order derivatives vanish on Σ and

$$f^i_{x_i}(0, x, 0)+g(0, x, 0)=0.$$

Let

$$S_1=\{(x, t)\in\Sigma\times[0, T],\ a^{ij}(0, x, t)n_in_j=0\},$$
$$S_2=\{(x, t)\in\Sigma\times[0, T],\ a^{ij}(0, x, t)n_in_j>0,$$

where $n=(n_1, \cdots, n_m)$ denotes the inner unit normal on Σ. We shall assume that

$$S_1\cap\overline{S}_2=\emptyset;$$

this means that

$$S_1=\Sigma_1\times[0, T],\ S_2=\Sigma_2\times[0, T]$$

with

$$\Sigma_1\cup\Sigma_2=\Sigma,\ \Sigma_1\cap\overline{\Sigma}_2=\emptyset.$$

As in[1], the existence of global generalized solutions will be proved by means of the method of parabolic regularization. Thus we shall consider the regularized equations

$$\begin{aligned} L_\varepsilon u=&\frac{\partial u_\varepsilon}{\partial t}-\varepsilon\Delta u_\varepsilon-\frac{\partial}{\partial x_i}\left(a^{ij}(u_\varepsilon, x, t)\frac{\partial u_\varepsilon}{\partial x_j}\right)-\frac{\partial}{\partial x_i}f^i(u_\varepsilon, x, t)\\ =&g(u_\varepsilon, x, t),\quad(\varepsilon>0) \end{aligned}\tag{1.4}$$

with the conditions (1.2) and (1.3) and need to establish some estimates on the family $\{u_\varepsilon\}$ of solutions of these problems.It is well-known that under the conditions stated above, for any $\varepsilon>0$, the problem (1.4), (1.2), (1.3) has a unique appropriately smooth solution.

§ 2. Definition of generalized solutions

Definition. *A bounded function $u\in BV_{1,\frac{1}{2}}(Q_T)$ is said to be the generalized solution of the first boundary value problem* (1.1), (1.2), (1.3), *if the following conditions are fulfilled:*

1°) *There exist functions* $g^i \in L^2(Q_T)$ $(i=1, 2, \cdots, m)$ *such that for any* $\varphi \in C_0^2(Q_T)$

$$\iint_{Q_T} \varphi g^i dx\, dt = -\iint_{Q_T} \frac{\partial \varphi}{\partial x_j} R^{ij}(u, x, t) dx\, dt - \iint_{Q_T} \varphi R^{ij}_{x_j}(u, x, t) dx\, dt \ (i=1, 2, \cdots, m), \tag{2.1}$$

where
$$R^{ij}(u, x, t) = \int_0^u r^{ij}(S, x, t) dS.$$

2°) *There exists a subset* $E_0 \subset [0, T]$ *with zero measure such that for* $t \in [0, T] \setminus E_0$, *as a function of* x, u *is defined almost evevywhere on* Ω *and*

$$\lim_{\substack{t\to 0 \\ t\in[0,T]\setminus E_0}} \int_\Omega |u(x, t) - u_0(x)| dx = 0.$$

3°) u *satisfies*

$$\begin{aligned}
&\iint_{Q_T} \Big\{ |u-k| \frac{\partial \varphi_1}{\partial t} - \operatorname{sgn}(u-k) \Big[r^{ij}(u, x, t) g^j \frac{\partial \varphi_1}{\partial x_i} \\
&\quad + (f^i(u, x, t) - f^i(k, x, t)) \frac{\partial \varphi_1}{\partial x_i} - (f^i_{x_i}(k, x, t) + g)\varphi_1 \Big] \Big\} dx\, dt \\
&\quad + \operatorname{sgn} k \iint_{Q_T} \Big[u \frac{\partial \varphi_2}{\partial t} - r^{ij}(u, x, t) g^j \frac{\partial \varphi_2}{\partial x_i} - (f^i(u, x, t) - f^i(k, x, t)) \frac{\partial \varphi_2}{\partial x_i} \\
&\quad + (f^i_{x_i}(k, x, t) + g)\varphi_2 \Big] dx\, dt - \int_0^T \int_\Sigma \operatorname{sgn}(\gamma u - k) (A^{ij}(\gamma u, x, t) \\
&\quad - A^{ij}(k, x, t)) \frac{\partial \varphi_1}{\partial x_i} n_j d\sigma\, dt + \int_0^T \int_\Sigma \operatorname{sgn} k \Big[A^{ij}(k, x, t) \frac{\partial \varphi_1}{\partial x_i} \\
&\quad - A^{ij}(\gamma u, x, t) \frac{\partial \varphi_2}{\partial x_i} \Big] n_j d\sigma\, dt \geqslant 0,
\end{aligned} \tag{2.2}$$

where $A^{ij}(u, x, t) = \int_0^u a^{ij}(S, x, t) dS$, $\varphi_1, \varphi_2 \in C^2(\bar{Q}_T)$, $\varphi_1 \geqslant 0$, $\varphi_1|_{\Sigma\times[0,T]} = \varphi_2|_{\Sigma\times[0,T]}$, $\operatorname{supp} \varphi_1$, $\operatorname{supp} \varphi_2 \subset \bar{\Omega} \times (0, T)$.

By integrating by parts, (2.1) may be rewritten as

$$\iint_{Q_T} \varphi g^i dx\, dt = \int_0^T \Big(\int_\Omega \varphi \hat{r}^{ij}(u, x, t) \frac{\partial u}{\partial x_j} dx \Big) dt \quad (i=1, 2, \cdots, m), \tag{2.3}$$

where $\hat{r}^{ij}(u, x, t)$ denotes the composite mean value of $r^{ij}(u, x, t)$ and $u(x, t)$. (2.3) means that for almost all $t \in [0, T]$, $\hat{r}^{ij}(u, x, t) \frac{\partial u}{\partial x_j}$ is equivalent to g^i. Furthermore we can show that for almost all $t \in [0, T]$, $\hat{a}^{ij}(u, x, t) \frac{\partial u}{\partial x_j}$ is equivalent to $r^{ij}(u, x, t) g^j$

$$\hat{a}^{ij}(u, x, t) \frac{\partial u}{\partial x_j} = r^{ij}(u, x, t) g^j. \tag{2.4}$$

Obviously, a generalized solution in the sense of Definition 1 in[1] is also a generalized solution in the sense of the above definition.

In a similar way as in [1], we can prove that (2.2) is equivalent to the total of the following two conditions:

a) u satisfies

$$\iint_{Q_T}\Big\{|u-k|\frac{\partial\varphi_1}{\partial t}-\operatorname{sgn}(u-k)\Big[r^{ij}(u, x, t)g^j\frac{\partial\varphi_1}{\partial x_i}$$
$$+(f^i(u, x, t)-f^i(k, x, t))\frac{\partial\varphi_1}{\partial x_i}-(f^i_{x_i}(k, x, t)+g)\varphi_1\Big]dx\,dt$$
$$+\operatorname{sgn} k\iint_{Q_T}\Big[u\frac{\partial\varphi_2}{\partial t}-r^{ij}(u, x, t)g^j\frac{\partial\varphi_2}{\partial x_i}-(f^i(u, x, t)-f^i(k, x, t))\frac{\partial\varphi_2}{\partial x_i}$$
$$+(f^i_{x_i}(k, x, t)+g)\varphi_2\Big]dx\,dt\geqslant 0. \qquad (2.5)$$

b) For almost all $t\in[0, T]$

$$A^{ij}(\gamma u, x, t)n_in_j=0, \quad \text{a. e. on } \Sigma, \qquad (2.6)$$

this means that

$$\gamma u=0, \quad \text{a. e. on } S_2$$

and for almost all points of S_1 such that $\gamma u\neq 0$, one has

$$a^{ij}(S, x, t)n_j=0, \ \forall S\in I(0, \gamma u),$$

where $I(\alpha, \beta)$ denotes the closed interval with endpoints α and β.

§ 3. Estimates of solutions of regularized problems

Let u_ε be the solutions of regularized problems (1.4), (1.2), (1.3).

The first estimate we need follows from the maximum principle

$$|u_\varepsilon|\leqslant M \qquad (3.1)$$

for some constant M independent of ε.

Lemma 1. *The solutions u_ε of regularized problems* (1.4), (1.2), (1.3) *satisfy*

$$\varepsilon\int_0^t\int_\Sigma\left|\frac{\partial u_\varepsilon}{\partial n}\right|d\sigma\,dS+\int_0^t\int_\Sigma a^{ij}(0, x, t)n_in_j\left|\frac{\partial u_\varepsilon}{\partial n}\right|d\sigma\,dS\leqslant C_1+C_2\int_0^t|\operatorname{grad} u_\varepsilon|_{L^1(\Omega)}dS,$$

where constants C_1, C_2 are independent of ε and $\operatorname{grad} u=(u_{x_1}, u_{x_2}, \cdots, u_{x_m})$.

Proof Without loss of generality, we may assume that $u_0(x)\equiv 0$. Let v_1 be the solution of the prollem

$$\begin{cases}\dfrac{\partial v_1}{\partial t}-\dfrac{\partial}{\partial x_i}\Big(a^{ij}(u_\varepsilon, x, t)\dfrac{\partial v_1}{\partial x_j}\Big)-\varepsilon\Delta v_1=f^+,\\ v_1|_\Sigma=0,\\ v_1|_{t=0}=0,\end{cases}$$

where

$$f=f(x, t)=\frac{\partial}{\partial x_i}f^i(u_\varepsilon, x, t)+g(u_\varepsilon, x, t),$$

$$f^+=\begin{cases}f, \text{ if } f>0,\\ 0, \text{ if } f\leqslant 0,\end{cases} \quad f^-=\begin{cases}-f, \text{ if } f<0,\\ 0, \text{ if } f\geqslant 0.\end{cases}$$

Then by maximum principle, we have $v_1\geqslant 0$ in Q_T, and hence $\left.\dfrac{\partial v_1}{\partial n}\right|_\Sigma\geqslant 0$.

Similarly, for the solution v_2 of the problem

$$\begin{cases} \dfrac{\partial v_2}{\partial t}-\dfrac{\partial}{\partial x_i}\left(a^{ij}(u_\varepsilon,\ x,\ t)\dfrac{\partial v_2}{\partial x_j}\right)-\varepsilon\Delta v_2=f^-, \\ v_2|_\Sigma=0, \\ v_2|_{t=0}=0, \end{cases}$$

we have $\left.\dfrac{\partial v_2}{\partial n}\right|_\Sigma \geqslant 0$.

Since $u=v_1-v_2$ is a solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t}-\dfrac{\partial}{\partial x_i}\left(a^{ij}(u_\varepsilon,\ x,\ t)\dfrac{\partial u}{\partial x_j}\right)-\varepsilon\Delta u=f, \\ u|_\Sigma=0, \\ u|_{t=0}=0 \end{cases}$$

by uniqueness, $u_\varepsilon=v_1-v_2$.

Integrating the equation satisfied by v_1 we obtain

$$\varepsilon\int_0^t\int_\Sigma \frac{\partial v_1}{\partial n}\,d\sigma\,dS+\int_0^t\int_\Sigma a^{ij}n_in_j\frac{\partial v_1}{\partial n}\,d\sigma\,dS=\int_0^t\int_\Omega f^+\,dx\,dS-\int_\Omega v_1(x,\ t)\,dx.$$

Similarly

$$\varepsilon\int_0^t\int_\Sigma \frac{\partial v_2}{\partial n}\,d\sigma\,dS+\int_0^t\int_\Sigma a^{ij}n_in_j\frac{\partial v_2}{\partial n}\,d\sigma\,dS=\int_0^t\int_\Omega f^-\,dx\,dS-\int_\Omega v_2(x,\ t)\,dx.$$

Therefore

$$\varepsilon\int_0^t\int_\Sigma\left|\frac{\partial u_\varepsilon}{\partial n}\right|d\sigma\,dS+\int_0^t\int_\Sigma a^{ij}n_in_j\left|\frac{\partial u_\varepsilon}{\partial n}\right|d\sigma\,dS$$

$$\leqslant\varepsilon\int_0^t\int_\Sigma\frac{\partial v_1}{\partial n}\,d\sigma\,dS+\int_0^t\int_\Sigma a^{ij}n_in_j\frac{\partial v_1}{\partial n}\,d\sigma\,dS+\varepsilon\int_0^t\int_\Sigma\frac{\partial v_2}{\partial n}\,d\sigma\,dS$$

$$+\int_0^t\int_\Sigma a^{ij}n_in_j\frac{\partial v_2}{\partial n}\,d\sigma\,d\tau=\int_0^t\int_\Omega(f^++f^-)\,dx\,dS-\int_\Omega(v_1(x,\ t)+v_2(x,\ t))\,dx$$

$$\leqslant C_1+C_2\int_0^t|\mathrm{grad}\,u_\varepsilon|_{L^1(\Omega)}dS.$$

Theorem 1. *Suppose $S_1\cap\bar{S}_2=\emptyset$ and $a^{ij}(0,\ x,\ t)$ can be extended to a neighborhood of S_1 such that in this neighborhood*

$$a^{ij}(0,\ x,\ t)\xi_i\xi_j\geqslant0,\ \forall\xi\in R^m. \tag{3.2}$$

Suppose there exists a constant $\delta>0$ such that for $(x,\ t)\in\bar{Q}_T$ and $|u|\leqslant M$

$$a^{ij}\xi_i\xi_j-\delta\sum_{s,j=1}^m(a^{ij}_{x_s}\xi_i)^2\geqslant0,\ \ \forall\xi\in R^m. \tag{3.3}$$

Then the solutoins u_ε of regularized problems (1.4), (1.2), (1.3) *satisfy*

$$|\mathrm{grad}\,u_\varepsilon|_{L^1(\Omega)}\leqslant M_1. \tag{3.4}$$

$$\int_\Omega|u_\varepsilon(x,\ t+\Delta t)-u_\varepsilon(x,\ t)|\,dx\leqslant K|\Delta t|^{\frac12},\ \text{for } t,\ t+\Delta t\in[0,\ T]. \tag{3.5}$$

Here grad $u=(u_{x_1},\ u_{x_2},\ \cdots,\ u_{x_m})$ *and the constants M_1 and K are independent of ε.*

Remark. Notice that (3.3) does not involve the derivative of a^{ij} with respect to t. Hence it is always fulfilled, for example. if $a^{ij}(u,\ x,\ t)=a^{ij}(u,\ t)$.

Proof Differentiate (1.4) with respect to x_s and sum up for S from 1 up to m

after multiplying the resulting formula by $u_{x_s}\dfrac{\operatorname{sgn}_\eta|\operatorname{grad} u|}{|\operatorname{grad} u|}$. The integration over Q_t yields

$$\int_\Omega I_\eta(|\operatorname{grad} u|)dx-\int_\Omega I_\eta(|\operatorname{grad} u_0|)dx-\varepsilon\int_0^t\int_\Omega(\Delta u_{x_s})u_{x_s}\frac{\operatorname{sgn}_\eta|\operatorname{grad} u|}{|\operatorname{grad} u|}dxdS$$
$$-\int_0^t\int_\Omega\frac{\partial}{\partial x_i}(a_u^{ij}u_{x_s}u_{x_j}+a_{x_s}^{ij}u_{x_j}+a^{ij}u_{x_sx_j})u_{x_s}\frac{\operatorname{sgn}_\eta|\operatorname{grad} u|}{|\operatorname{grad} u|}dx\,dS$$
$$-\int_0^t\int_\Omega\frac{\partial}{\partial x_i}(f_u^iu_{x_s}+f_{x_s}^i)u_{x_s}\frac{\operatorname{sgn}_\eta|\operatorname{grad} u|}{|\operatorname{grad} u|}dx\,dS$$
$$=\int_0^t\int_\Omega\left(\frac{\partial}{\partial x_s}g\right)u_{x_s}\frac{\operatorname{sgn}_\eta|\operatorname{grad} u|}{|\operatorname{grad} u|}dx\,dS.$$

By integrating by parts, we obtain

$$\int_\Omega I_\eta(|\operatorname{grad} u|)dx-\int_\Omega I_\eta(|\operatorname{grad} u_0(x)|)dx+\int_0^t\int_\Omega\varepsilon\frac{\partial^2 I_\eta}{\partial\xi_s\partial\xi_p}u_{x_sx_i}u_{x_px_i}dx\,dS$$
$$+\int_0^t\int_\Omega a^{ij}\frac{\partial^2 I_\eta}{\partial\xi_s\partial\xi_p}u_{x_sx_j}u_{x_px_i}dx\,dS+\int_0^t\int_\Omega a_{x_s}^{ij}u_{x_j}\frac{\partial^2 I_\eta}{\partial\xi_s\partial\xi_p}u_{x_px_i}dx\,dS$$
$$-\int_0^t\int_\Omega\left(\frac{\partial}{\partial x_i}a_u^{ij}\right)u_{x_j}(|\operatorname{grad} u|\operatorname{sgn}_\eta|\operatorname{grad} u|-I_\eta)dx\,dS$$
$$-\int_0^t\int_\Omega a_u^{ij}u_{x_ix_j}(|\operatorname{grad} u|\operatorname{sgn}_\eta|\operatorname{grad} u|-I_\eta)dx\,dS$$
$$-\int_0^t\int_\Omega\left(\frac{\partial}{\partial x_i}f_u^i\right)(|\operatorname{grad} u|\operatorname{sgn}_\eta|\operatorname{grad} u|-I_\eta)dx\,dS$$
$$+\varepsilon\int_0^t\int_\Sigma\frac{\partial I_\eta}{\partial x_i}n_i\,d\sigma\,dS+\int_0^t\int_\Sigma a^{ij}\frac{\partial I_\eta}{\partial x_j}n_i\,d\sigma\,dS$$
$$+\int_0^t\int_\Sigma a_u^{ij}u_{x_j}I_\eta n_i\,d\sigma\,dS+\int_0^t\int_\Sigma f_u^iI_\eta n_i\,d\sigma\,dS$$
$$+\int_0^t\int_\Sigma a_{x_s}^{ij}u_{x_j}u_{x_s}\frac{\operatorname{sgn}_\eta|\operatorname{grad} u|}{|\operatorname{grad} u|}n_i\,d\sigma\,dS$$
$$=\int_0^t\int_\Omega\left[\left(\frac{\partial}{\partial x_i}f_{x_s}^i\right)+\frac{\partial}{\partial x_s}g\right]u_{x_s}\frac{\operatorname{sgn}_\eta|\operatorname{grad} u|}{|\operatorname{grad} u|}dx\,dS. \tag{3.6}$$

As in the proof of Theorem 1 in [1], from (3.3) we may deduce

$$\int_0^t\int_\Omega a^{ij}\frac{\partial^2 I_\eta}{\partial\xi_s\partial\xi_p}u_{x_sx_j}u_{x_px_i}dx\,dS+\int_0^t\int_\Omega a_{x_s}^{ij}u_{x_j}\frac{\partial^2 I_\eta}{\partial\xi_s\partial\xi_p}u_{x_px_i}dx\,dS$$
$$\geqslant-\beta\int_0^t\int_\Omega|\operatorname{grad} u|dx\,dS$$

for some constant β.

Using Lemma 1, in a similar way as in the proof of Theorem 1 in [1], we can estimate the five surface integrale in (3.6) by $\int_0^t|\operatorname{grad} u|_{L^1(\Omega)}dS$. Thus, letting $\eta\to0$, from (3.6) we can obtain

$$\int_\Omega|\operatorname{grad} u|dx\leqslant C_3+C_4\int_0^t\int_\Omega|\operatorname{grad} u|(x,S)|dx\,dS,$$

whence the estimate (3.4) follows.

It is remarkable that under the weaker condition (3.3), in general, we can not obtain the estimate of $\left|\frac{\partial u}{\partial t}\right|_{L^1(\Omega)}$ or $\left|\frac{\partial u}{\partial t}\right|_{L^1(Q_T)}$ what we can obtain is the estimate (3.5). To prove this, we shall apply the following lemma.

Lemma 2[3]. *Suppose $u(x, t)$ is measurable on Q_T, $|u(x, t)|\leqslant M$ and*

$$J(u, \Delta x)=\int_\Omega |u(x+\Delta x, t)-u(x, t)|dx\leqslant w(|\Delta x|) \tag{3.7}$$

(For $x+\Delta x\overline{\in}\Omega$, we set $u=0$). If for t, $t+\Delta t\in[0, T]$, and any $\psi(x)\in C_0^2(\Omega)$

$$\left|\int_\Omega \psi(x)(u(x, t+\Delta t)-u(x, t))dx\right|\leqslant C|\Delta t|\max\Big(|\psi(x)|+|\psi_x(x)|+\Big). \tag{3.8}$$

then for t, $t+\Delta t\in[0, T]$

$$\int_\Omega |u(x, t+\Delta t)-u(x, t)|dx\leqslant \min_{0<h\leqslant h_0}\left(h+\omega(h)+\frac{|\Delta t|}{h^2}\right). \tag{3.9}$$

Here $\omega(h)$ is a continuous increasing function which is defined for $0\leqslant h\leqslant h_0$ such that $\omega(0)=0$.

Now from the equation (1.4) which satisfied by u_ε, we have

$$\begin{aligned}\int_\Omega \psi(x)[u_\varepsilon(x, t+\Delta t)-u_\varepsilon(x, t)]dx&=\int_\Omega\int_t^{t+\Delta t}\psi(x)\frac{\partial u_\varepsilon}{\partial t}d\tau\, dx\\ &=\int_t^{t+\Delta t}\int_\Omega \psi(x)\Big[\frac{\partial}{\partial x_i}\Big(a^{ij}\frac{\partial u_\varepsilon}{\partial x_j}\Big)+\varepsilon\Delta u_\varepsilon+\frac{\partial f^i}{\partial x_i}+g\Big]dx\, d\tau\\ &=-\int_t^{t+\Delta t}\int_\Omega \psi_{x_i}(x)\Big(a^{ij}\frac{\partial u_\varepsilon}{\partial x_j}+\varepsilon\frac{\partial u_\varepsilon}{\partial x_i}\Big)dxd\tau\\ &\quad+\int_t^{t+\Delta t}\int_\Omega \psi(x)\Big(\frac{\partial f^i}{\partial x_i}+g\Big)dxd\tau.\end{aligned}$$

Using the estimate (3.4) we obtain

$$\left|\int_\Omega \psi(x)[u_\varepsilon(x, t+\Delta t)-u_\varepsilon(x, t)]dx\right|\leqslant C_2|\Delta t|\max_{x\in\Omega}\big(|\psi|+|\psi_x|\big).$$

Clearly (3.4) implies

$$\int_\Omega |u_\varepsilon(x+\Delta x, t)-u_\varepsilon(x, t)|dx\leqslant \widetilde{M}_1|\Delta x|,$$

where $\widetilde{M}_1=\max(M, M_1)$. Here we set $u_\varepsilon=0$ for $x+\Delta x\overline{\in}\Omega$. Thus u_ε satisfies (3.7) with $\omega(h)=\widetilde{M}_1 h$. According to Lemma 2, for t, $t+\Delta t\in[0, T]$,

$$\int_\Omega |u_\varepsilon(x, t+\Delta t)-u_\varepsilon(x, t)|dx\leqslant C_1\min_{0<h\leqslant h_0}\left[h+\widetilde{M}_1 h+\frac{|\Delta t|}{h}\right]=K|\Delta t|^{\frac{1}{2}}.$$

Thus the proof of Theorem 1 is completed.

Using Theorem 1, as in [1], we can obtain

$$\iint_{Q_T} a^{ij}(u_\varepsilon, x, t)\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_j}dx\,dt\leqslant M_2. \tag{3.10}$$

for some constant M_2 not depending on ε.

§ 4. Existence of generalized solutions

On the basis of estimates (3.1), (3.4), (3.5), (3.10), we can prove

Theorem 2. *Suppose the conditions of Theorem 1 are fulfilled. Then the first boundary value problem* (1.1), (1.2), (1.3) *has a generalized solution which is a limit point in* $L^1(Q_T)$, *of the family* $\{u_\varepsilon\}$ *of solutions of regularized problems* (1.4), (1.2), (1.3).

Proof By Kolmogoroff's theorem, there exists a subsequence $\{u_{\varepsilon_n}\}$ of $\{u_\varepsilon\}$, converging both in $L^1(Q_T)$ and in $L_\infty(Q_T)$ to some function u. Clearly $|u|\leqslant M$ and for almost all $t\in[0, T]$

$$\int_\Omega |u(x, t+\Delta t)-u(x, t)|dx\leqslant K|\Delta t|^{\frac{1}{2}},$$

$$\int_\Omega |u(x+\Delta x, t)-u(x, t)|dx\leqslant \widetilde{M}_1|\Delta x|,$$

in particular, $u\in BV_{1,\frac{1}{2}}(Q_T)$.

Estimat (3.10) implies the weak compactness of $\left\{r^{ij}(u_{\varepsilon_n}, x, t)\frac{\partial u_{\varepsilon_n}}{\partial x_j}\right\}$ in $L^2(Q_T)$. For simplicity of notation, we assume that $\left\{r^{ij}(u_{\varepsilon_n}, x, t)\frac{\partial u_{\varepsilon_n}}{\partial x_j}\right\}$ itself weakly converges in $L^2(Q_T)$ and denote the limit function by $g^i(x, t)$. Then for any $\varphi\in C_0^2(Q_T)$

$$\begin{aligned}\iint_{Q_T}\varphi g^i\,dx\,dt&=\lim_{\varepsilon_n\to0}\iint_{Q_T}\varphi r^{ij}(u_{\varepsilon_n}, x, t)\frac{\partial u_{\varepsilon_n}}{\partial x_j}\,dx\,dt\\&=\lim_{\varepsilon_n\to0}\iint_{Q_T}\varphi\frac{\partial}{\partial x_j}R^{ij}(u_{\varepsilon_n}, x, t)dx\,dt-\iint_{Q_T}\varphi R^{ij}_{x_j}(u, x, t)dx\,dt\\&=-\lim_{\varepsilon_n\to0}\iint_{Q_T}\frac{\partial\varphi}{\partial x_j}R^{ij}(u_{\varepsilon_n}, x, t)dx\,dt-\iint_{Q_T}\varphi R^{ij}_{x_j}(u, x, t)dx\,dt\\&=-\iint_{Q_T}\frac{\partial\varphi}{\partial x_j}R^{ij}(u, x, t)dx\,dt-\iint_{Q_T}\varphi R^{ij}_{x_j}(u, x, t)dx\,dt\ (i=1, 2, \cdots, m).\end{aligned}$$

Thus u satisfies the condition 1°) in the definition of genevalized solutions.

It is easy to prove that u satisfies the condition 2°) in the definition of generalized solutions.

In order to prove that u satisfies the condition 3°), let $\varphi_1\in C^2(\bar{Q}_T)$, $\varphi_1\geqslant0$, $\operatorname{supp}\varphi_1\subset\bar{\Omega}\times(0, T)$, multiply (1.4) by $\varphi_1\operatorname{sgn}_\eta(u_\varepsilon-k)$ and integrate over Q_T. By integrating by parts, we obtain

$$\begin{aligned}&-\iint_{Q_T}I_\eta(u_\varepsilon-k)\frac{\partial\varphi_1}{\partial t}\,dx\,dt+\iint_{Q_T}\operatorname{sgn}_\eta(u_\varepsilon-k)\Big[\varepsilon\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial\varphi_1}{\partial x_i}\\&+a^{ij}\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\varphi_1}{\partial x_i}+(f^i(u_\varepsilon, x, t)-f^i(k, x, t))\frac{\partial\varphi_1}{\partial x_i}\Big]dx\,dt\\&+\iint_{Q_T}\operatorname{sgn}'_\eta(u_\varepsilon-k)\Big(\varepsilon\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_i}+a^{ij}\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial u_\varepsilon}{\partial x_i}\Big)\varphi_1\,dx\,dt\end{aligned}$$

$$+\iint_{Q_T}(f^i(u_\varepsilon, x, t)-f^i(k, x, t))\operatorname{sgn}'_\eta(u_\varepsilon-k)\frac{\partial u_\varepsilon}{\partial x_i}\varphi_1\,dx\,dt$$
$$-\iint_{Q_T}\operatorname{sgn}_\eta(u_\varepsilon-k)(f^i_{x_i}(k, x, t)+g(u_\varepsilon, x, t))\varphi_1\,dx\,dt$$
$$-\operatorname{sgn}_\eta k\int_0^T\int_\Sigma\Big(\varepsilon\frac{\partial u_\varepsilon}{\partial x_i}+a^{ij}\frac{\partial u_\varepsilon}{\partial x_j}\Big)\varphi_1 n_i\,d\sigma\,dt$$
$$-\operatorname{sgn}_\eta k\int_0^T\int_\Sigma(f^i(0, x, t)-f^i(k, x, t))\varphi_1 n_i\,d\sigma\,dt=0.$$

Noticing that the third term is nonnegative and that the fourth term tends to zero as $\eta\to 0$, we deduce

$$\iint_{Q_T}\Big\{|u_\varepsilon-k|\frac{\partial\varphi_1}{\partial t}-\operatorname{sgn}(u_\varepsilon-k)\Big[\varepsilon\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial\varphi_1}{\partial x_i}+a^{ij}\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\varphi_1}{\partial x_i}$$
$$+(f^i(u_\varepsilon, x, t)-f^i(k, x, t))\frac{\partial\varphi_1}{\partial x_i}-(f^i_{x_i}(k, x, t)+g(u_\varepsilon, x, t))\varphi_1\Big]\Big\}dx\,dt$$
$$+\operatorname{sgn} k\int_0^T\int_\Sigma\Big(\varepsilon\frac{\partial u_\varepsilon}{\partial x_i}+a^{ij}\frac{\partial u_\varepsilon}{\partial x_j}\Big)\varphi_1 n_i\,d\sigma\,dt$$
$$+\operatorname{sgn} k\int_0^T\int_\Sigma(f^i(0, x, t)-f^i(k, x, t))\varphi_1 n_i\,d\sigma\,dt\geqslant 0. \tag{4.1}$$

Obviously

$$\iint_{Q_T}\operatorname{sgn}(u_\varepsilon-k)a^{ij}\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial\varphi_1}{\partial x_i}\,dx\,dt=\iint_{Q_T}\operatorname{sgn}(u_\varepsilon-k)\frac{\partial}{\partial x_j}(A^{ij}(u_\varepsilon, x, t)$$
$$-A^{ij}(k, x, t))\frac{\partial\varphi_1}{\partial x_i}-\iint_{Q_T}\operatorname{sgn}(u_\varepsilon-k)\Big(\int_k^{u_\varepsilon}a^{ij}_{x_j}(\tau, x, t)d\tau\Big)\frac{\partial\varphi_1}{\partial x_i}\,dx\,dt$$
$$=-\int_0^T\int_\Sigma\operatorname{sgn} kA^{ij}(k, x, t)\frac{\partial\varphi_1}{\partial x_i}n_j\,d\sigma\,dt$$
$$-\iint_{Q_D}\operatorname{sgn}(u_\varepsilon-k)(A^{ij}(u_\varepsilon, x, t)-A^{ij}(k, x, t))\frac{\partial^2\varphi_1}{\partial x_i\partial x_j}\,dx\,dt$$
$$-\iint_{Q_T}\operatorname{sgn}(u_\varepsilon-k)\Big(\int_k^{u_\varepsilon}a^{ij}_{x_j}(\tau, x, t)\,d\tau\Big)\frac{\partial\varphi_1}{\partial x_i}\,dx\,dt.$$

Hence

$$\lim_{\varepsilon_n\to 0}\iint_{Q_T}\operatorname{sgn}(u_{\varepsilon_n}-k)a^{ij}\frac{\partial u_{\varepsilon_n}}{\partial x_j}\frac{\partial\varphi_1}{\partial x_i}\,dx\,dt=-\int_0^T\int_\Sigma\operatorname{sgn} kA^{ij}(k, x, t)\frac{\partial\varphi_1}{\partial x_i}n_j\,d\sigma\,dt$$
$$-\iint_{Q_T}\operatorname{sgn}(u-k)(A^{ij}(u, x, t)-A^{ij}(k, x, t))\frac{\partial^2\varphi_1}{\partial x_i\partial x_j}\,dx\,dt$$
$$-\iint_{Q_T}\operatorname{sgn}(u-k)\Big(\int_k^u a^{ij}_{x_j}(\tau, x, t)d\tau\Big)\frac{\partial\varphi_1}{\partial x_i}\,dx\,dt. \tag{4.2}$$

Since for fixed t, $\operatorname{sgn}(u-k)(A^{ij}(u, x, t)-A^{ij}(k, x, t))\in BV(\Omega)$ and

$$\frac{\partial}{\partial x_j}[\operatorname{sgn}(u-k)(A^{ij}(u, x, t)-A^{ij}(k, x, t))]$$
$$=\operatorname{sgn}(u-k)\frac{\partial}{\partial x_j}[A^{ij}(u, x, t)-A^{ij}(k, x, t)],$$

we have

$$\lim_{\varepsilon_n\to 0}\iint_{Q_T}\operatorname{sgn}(u_{\varepsilon_n}-k)a^{ij}\frac{\partial u_{\varepsilon_n}}{\partial x_j}\frac{\partial\varphi_1}{\partial x_i}dx\,dt$$

$$=-\int_0^T\int_\Sigma \operatorname{sgn} k A^{ij}(k,\ x,\ t)\frac{\partial\varphi_1}{\partial x_i}n_j\,d\sigma\,dt+\int_0^T\int_\Sigma \operatorname{sgn}(\gamma u-k)(A^{ij}(\gamma u,\ x,\ t)$$

$$-A^{ij}(k,\ x,\ t))\frac{\partial\varphi_1}{\partial x_i}n_j\,d\sigma\,dt+\int_0^T\Big[\int_\Omega \operatorname{sgn}(u-k)\hat{a}^{ij}\frac{\partial u}{\partial x_j}\frac{\partial\varphi_1}{\partial x_i}dx\Big]dt. \qquad (4.3)$$

Let $\varphi_2\in C^2(\bar{Q}_T)$, $\varphi_2|_{\Sigma\times[0,T]}=\varphi_1|_{\Sigma\times[0,T]}$. Then

$$\int_0^T\int_\Sigma \varepsilon\frac{\partial u_\varepsilon}{\partial x_i}\varphi_1 n_i\,d\sigma\,dt=-\iint_{Q_T}\varepsilon\Big[\Delta u_\varepsilon\varphi_2+\frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial\varphi_2}{\partial x_i}\Big]dx\,dt.$$

Using the equation (1.4), we obtain

$$\lim_{\varepsilon_n\to 0}\int_0^T\int_\Sigma\Big(\varepsilon\frac{\partial u_{\varepsilon_n}}{\partial x_i}+a^{ij}\frac{\partial u_{\varepsilon_n}}{\partial x_j}\Big)\varphi_1 n_i\,d\sigma\,dt$$

$$=\lim_{\varepsilon_n\to 0}\iint_{Q_T}\Big[-a^{ij}\frac{\partial u_{\varepsilon_n}}{\partial x_j}\frac{\partial\varphi_2}{\partial x_i}-\frac{\partial u_{\varepsilon_n}}{\partial t}\varphi_2+\frac{\partial}{\partial x_i}f^i(u_{\varepsilon_n},\ x,\ t)\varphi_2+g\varphi_2\Big]dx\,dt.$$

It is easy to verify that

$$\lim_{\varepsilon_n\to 0}\iint_{Q_T}-a^{ij}\frac{\partial u_{\varepsilon_n}}{\partial x_j}\frac{\partial\varphi_2}{\partial x_i}dx\,dt=-\int_0^T\int_\Sigma A^{ij}(\gamma u,\ x,\ t)\frac{\partial\varphi_2}{\partial x_i}n_j\,d\sigma\,dt$$

$$-\int_0^T\Big[\int_\Omega\hat{a}^{ij}\frac{\partial u}{\partial x_j}\frac{\partial\varphi_2}{\partial x_i}dx\Big]dt.$$

$$\lim_{\varepsilon_n\to 0}\iint_{Q_T}\Big[-\frac{\partial u_{\varepsilon_n}}{\partial t}\varphi_2+\Big(\frac{\partial}{\partial x_i}f^i(u_{\varepsilon_n},\ x,\ t)\Big)\varphi_2+g\varphi_2\Big]dx\,dt$$

$$=-\int_0^T\int_\Sigma f^i(0,\ x,\ t)\varphi_1 n_i\,d\sigma\,dt+\iint_{Q_T}\Big[u\frac{\partial\varphi_2}{\partial t}-f^i(u,\ x,\ t)\frac{\partial\varphi_2}{\partial x_i}+g\varphi_2\Big]dx\,dt.$$

Therefore

$$\lim_{\varepsilon_n\to 0}\int_0^T\int_\Sigma\Big(\varepsilon_n\frac{\partial u_{\varepsilon_n}}{\partial x_i}+a^{ij}\frac{\partial u_{\varepsilon_n}}{\partial x_j}\Big)\varphi_1 n_i\,d\sigma\,dt$$

$$=-\int_0^T\int_\Sigma A^{ij}(\gamma u,\ x,\ t)\frac{\partial\varphi_2}{\partial x_i}n_j\,d\sigma\,dt-\int_0^T\int_\Sigma f^i(0,\ x,\ t)\varphi_1 n_i\,d\sigma\,dt$$

$$-\int_0^T\Big[\int_\Omega\hat{a}^{ij}\frac{\partial u}{\partial x_j}\frac{\partial\varphi_2}{\partial x_i}dx\Big]dt+\iint_{Q_T}\Big[u\frac{\partial\varphi_2}{\partial t}-f^i(u,\ x,\ t)\frac{\partial\varphi_2}{\partial x_i}+g\varphi_2\Big]dx\,dt. \qquad (4.4)$$

Now the inequality (2.2) follows from (4.1) by letting $\varepsilon=\varepsilon_n\to 0$ and usng (4.3), (4.4).

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