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NEW SIMPLE LIE ALGEBRAS OF CHARACTERISTIC p

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Abstract

Over an algebraically closed field F of characteristic $p>3$, classes of Lie algebras $\bar{\Sigma}=\bar{\Sigma}(n, m, \underset{\sim}{r}, G)(m>0)$, $\Sigma^*=\Sigma^*(n,m,r,G)(n+2\not\equiv 0(\mathrm{mod}p))$and $\tilde{\Sigma}=\tilde{\Sigma}(n,m,\underset{\sim}{r}, G)((n+2\equiv 0(\mathrm{mod}p))$are constructed, where n, m are non-negative numbers, $\underset{\sim}{r}=(s_0+1, s_1+1, \cdots, s_n+1, t_1+1, \cdots, t_n+1)$is a$(2n+1)$-tuple of positive numbers and G is a subgroup of the additive group of F. $\bar{\Sigma}$, Σ^* and $\tilde{\Sigma}$ are shown to be all simple Lie algebras with dimensions p^N-2, p^N and p^N-1 respectively, where

$$N=\sum_{i=0}^{n}(s_i+1)+\sum_{i=1}^{n}(t_i+1)+m.$$

Their derivation algebras are determined. It is shown that they are of generalized Cartan type K when $m=0$ and of generalized Cartan type H when $m>0$. It is then determined that Σ^* and Σ are new simple Lie algebras if $n>0$. Conditions of isomorphism are obtained. And a special graded algebra structure, the K-like gradation, is discussed.

§ 0. Introduction

In this paper, a class of Lie algebras $\Sigma(n, m, \underset{\sim}{r}, G)$ is constructed over an algebraically closed field F of characteristic $p>2$, where n, m are non-negative integers, $\underset{\sim}{r}$ isa$(2n+1)$-tuple of positive integers and G is a subgroup of the additive group of F. When $1\in G$, Σ has a 1-dimensional center $\mathfrak{C}$ and $\bar{\Sigma}=(\Sigma/\mathfrak{C})'$is simple, dim $\bar{\Sigma}=p^N-2$, where N is the sum of the components of $\underset{\sim}{r}$ and m. When $1\bar{\in}G$, if $n+2\not\equiv 0(\mathrm{mod}p)$, $\Sigma^*=\Sigma$ is simple, dim$\Sigma^*=p^N$,if $n+2\equiv 0(\mathrm{mod}p)$, then $\tilde{\Sigma}=\Sigma'$is simple and dim $\tilde{\Sigma}=p^N-1$. The derivation algebras of $\bar{\Sigma}$, Σ^* and $\tilde{\Sigma}$ are determined and the dimensions of the outer derivation algebras are $N+2$, $N-(2n+1)$ and $N-2n$ respectively. When $m=0$(i.e.,$G=\{0\}$), Σ^* and$\tilde{\Sigma}$ are of generdlized cartan type K. If $m>0$, $\bar{\Sigma}$, Σ^* and $\tilde{\Sigma}$ are all of generaliled Cartan type H, and Σ^* and $\tilde{\Sigma}$ are new ones when $n>0$. Σ^* and $\tilde{\Sigma}$ also furnish examples of simple graded Lie algebras which are neither classidal nor the"graded Lie algebras of Cartan type"of Kostrikin–Šafarevič.

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§ 1. Construction

Suppose R is a commutative ring of characteristic $p>0$. Let $\mathfrak{A}=R[x_1, \cdots, x_n]$, $x_i^p=\eta_i\in R$, be the truncated polynomial algebra over R. Now let $R=F$ be a field of characteristic $p>2$ (we may assume that F is algebraically closed). Let $D_1, \cdots, D_m$, $D'_1, \cdots, D'_m$ be $2m$ mutually commuative derivations of $\mathfrak{A}$. In $\mathfrak{A}$, we define a commutator operation

$$[f, g]=\sum_{i=1}^{m}((D_i f)(D'_i g)-(D_i g)(D'_i f)),\ f, g\in\mathfrak{A}. \tag{1.1}$$

It can be directly verified that Jacobi identity holds, and $\mathfrak{A}$ thus becomes a Lie algebra (over F) which is denoted by $V=V(\mathfrak{A}, D_i, D'_i)$. Let μ_i, ν_i, $i=0, 1, \cdots, n$, be $2(n+1)$ elements of F such that

$$\mu_i+\nu_i=1,\ i=0, 1, \cdots, n, \tag{1.2}$$

and especially, let

$$\mu_0=0,\ \nu_0=1. \tag{1.3}$$

Let A be the additive subgroup of F generated by $\{\mu_i, \nu_i | i=0, 1, \cdots, n\}$ and let $\{\xi_0=1, \xi_1, \cdots, \xi_{t_0}\}$ be a basis of A over the prime field Π of F. Let

$$\overline{\mathfrak{A}}=F[x_{00}, \cdots, x_{0s_0}, \cdots, x_{n0}, \cdots, x_{ns_n};\ y_{00}, \cdots, y_{0t_0}, \cdots, y_{n0}, \cdots, y_{nt_n}], \tag{1.4}$$

where

$$y_{0i}^p=1,\ i=0, 1, \cdots, t_0, \tag{1.5}$$

and

$$x_{ij}^p=0,\ i=0, 1, \cdots, n,\ j=0, \cdots, s_i-1,\ x_{is_i}^p=\varepsilon_i(=0 \text{ or } 1), \tag{1.6}$$

$$y_{ij}^p=0,\ i=1, \cdots, n,\ j=0, \cdots, t_i-1,\ y_{it_i}^p=\varepsilon'_i\ (=0 \text{ or } 1). \tag{1.6$'$}$$

Set

$$D'_0=\frac{\partial}{\partial y_{00}}+\sum_{i=1}^{t_0}\xi_i y_{00}^{-1} y_{0i}\frac{\partial}{\partial y_{0i}}. \tag{1.7}$$

Let

$$y^{l_0}_0=y_{00}^{l_{00}}\cdots y_{0t_0}^{l_{0t_0}},\ l_0=\sum_{i=0}^{t_0} l_{0i}\xi_i\in A,\ 0\leqslant l_{0i}<p,\ i=0, 1, \cdots, t_0. \tag{1.8}$$

It is easy to see that

$$D'_0 y_0^{l_0}=l_0 y_0^{l_0-1},\ l_0\in A. \tag{1.9}$$

Let

$$D_i=\frac{\partial}{\partial x_{i0}}+\sum_{v=1}^{s_i}\left(\prod_{u=0}^{v-1}x_{iu}^{p-1}\right)\frac{\partial}{\partial x_v},\ i=0, 1, \cdots, n, \tag{1.10}$$

$$D'_i=\frac{\partial}{\partial y_{i0}}+\sum_{v=1}^{t_i}\left(\prod_{u=0}^{v-1}y_{iu}^{p-1}\right)\frac{\partial}{\partial y_v},\ i=1, \cdots, n. \tag{1.10$'$}$$

If k_i and l_i are integers such that $0\leqslant k_i<p^{s_i+1}$, $0\leqslant l_i<^{t_i+1}$, they can be uniquely expressed in p-adic form

$$k_i=\sum_{v=0}^{s_i}k_{iv}p^v,\ 0\leqslant k_{iv}<p;\ l_i=\sum_{v=0}^{t_i}l_{iv}p^v,\ 0\leqslant l_{iv}<p. \tag{1.11}$$

We put

$$x_i^{k_i}=x_{i0}^{k_{i0}}\cdots x_{is_i}^{k_{is_i}} \text{ and } y_i^{l_i}=y_{i0}^{l_{i0}}\cdots y_{it_i}^{k_{it_i}}. \tag{1.12}$$

It can be easily shown that, if $k_i, l_i\neq 0$, then

$$D_ix_i^{k_i}=k_i^*x_i^{k_i-1},\ D_i'y_i^{l_i}=l_i^*y_i^{l_i-1}, \tag{1.13}$$

where k_i^* and l_i^* are the first nonzero numbers of $(k_{i0}, \cdots, k_{is_i})$ and $(l_{i0}, \cdots, l_{it_i})$ respectively (we assnme $x_i^{k_i}=0$ if $k_i<0$ and $y_i^{l_i}=0$ if $l_i<0$). It follows that

$$D_ix_i^{k_i}=0 \text{ iff } k_i=0;\ D_i'y_i^{l_i}=0 \text{ iff } l_i=0. \tag{1.14}$$

Let $\bar{k}=(k_0, \cdots, k_n)$, $0\leqslant k_i<p^{s_i+1}$, $i=0, 1,\cdots, n$, $\bar{l}=(l_0, \cdots, l_n)$, $l_0\in A$, $0\leqslant l_i<p^{t_i+1}$, $i=1, \cdots, n$. put

$$x^{\bar{k}}y^{l}=x_0^{k_0}\cdots x_n^{k_n}y_0^{l_0}\cdots y_n^{l_n}. \tag{1.15}$$

We have, for $i=0, 1, \cdots, n$

$$x_i^{k_i}x_i^{k_i'}=\begin{cases} x_i^{k_i+k_i'}, \text{ if } k_{iv}+k_{iv}'<p,\ v=0, 1, \cdots, s_i, \\ 0, \text{ if } k_{iv}+k_{iv}'\geqslant p \text{ for some } v<s_i \text{ or } k_{is_i}+k_{is_i}'\geqslant p \text{ and } \varepsilon_i=0, \\ x_i^{k_i+k_i'-p^{s_i+1}} \text{if } k_{iv}+k_{iv}'<p,\ v<s_i,\ k_{is_i}+k_{is_i}'\geqslant p \text{ and } \varepsilon_i=1 \end{cases} \tag{1.16}$$

Similar formulas hold for $y_i^{l_i}y_i^{l_i'}, i=1, \cdots, n$, and, by (1.5), (1.8)

$$y_0^{l_0}y_0^{l_0'}=y^{l_0+l_0'},\ l_0, l_0'\in A. \tag{1.16$'$}$$

Consider $V=V(\mathfrak{A}, D_0, \cdots, D_n; D_0', \cdots, D_n')$. Let

$$\bar{e}_i=(\delta_{0i}, \delta_{1i}, \cdots, \delta_{ni}),\ i=0, 1, \cdots, n. \tag{1.17}$$

By (1.19), (1.13), (1.16) and (1.16)$'$, we have

$$[x^{\bar{k}}y^{l}, x^{\bar{k}'}y^{l'}]=(k_0^*l_0'-k_0'^*l_0)\bar{X}_0\bar{Y}_0+\sum_{i=1}^{n}(k_i^*l_i'^*-k_i'^*l_i^*)\bar{X}_i\bar{Y}_i, \tag{1.18}$$

where

$$\bar{X}_i=x^{\bar{k}-\bar{e}_i}x^{\bar{k}_i'}=x^{\bar{k}}x^{\bar{k}'-\bar{e}_i},\ \bar{Y}_i=y^{l-\bar{e}_i}y^{l'}=y^{l}y^{l'-\bar{e}_i},\ i=0, 1\cdots, n. \tag{1.18$'$}$$

By (1.16) and (1.16)$'$, we see that

$$\text{either } \bar{X}_i=0 \text{ or } \bar{X}_i=x^{\bar{a}} \text{ with } \bar{a}\equiv\bar{k}_i+\bar{k}_k'-\bar{e}_i(\text{mod}p),\ i=0, 1\cdots, \cdots n. \tag{1.19}$$

$$\text{either } \bar{Y}_i=0 \text{ or } \bar{Y}_i=y^{\bar{b}} \text{ with } \bar{b}\equiv\bar{l}_i+\bar{l}_i'-\bar{e}_i(\text{mod}p),\ i=0, 1, \cdots, n. \tag{1.19$'$}$$

Let D be the linear transformation of V defined by

$$D(x^{\bar{k}}y^{l})=\Big(\ l_0+\sum_{i=1}^{n}(k_i\mu_i+l_i\nu_i)-1\ \Big)x^{\bar{k}}y^{l}. \tag{1.20}$$

By (1.18), (1.19) and (1.19)$'$, it is easy to check that D is a derivation of V. Then $\mathfrak{C}=\{f\in V\mid Df=0\}$ is a subalgebra of V which is spanned by all $x^{\bar{k}}y^{l}$ satisfying

$$l_0+\sum_{i=1}^{n}(k_i\mu_i+l_i\nu_i)=1. \tag{1.21}$$

For any vector $(k_0, k_1, \cdots, k_n, l_1, \cdots, l_n)$, there is one and only one $l_0\in A$ satisfying (1.21). Corresponding to $\bar{k}$, $\bar{l}$, put

$$k=(k_1, \cdots, k_n),\ l=(l_1, \cdots, l_n). \tag{1.22}$$

Let $\mathfrak{A}=F[x_{00}, \cdots, x_{0s_0}, \cdots, x_{n0}, \cdots, x_{ns_n}; y_{10}, \cdots, y_{1t_1}, \cdots, y_{n0}, \cdots, y_{nt_n}]$. Define linear map σ: $\bar{\mathfrak{A}}\to\mathfrak{A}$ by

$$\sigma(x^{\bar{k}}y^{l})=x^{k}y^{l}. \tag{1.23}$$

Then the restriction of σ on $\mathfrak{C}$ is bijective. By (1.5), we have

$$\sigma(fg)=\sigma(f)\sigma(g),\ f,\ g\in\bar{\mathfrak{A}}. \tag{1.24}$$

And evidently

$$D_i(\sigma f)=\sigma(D_i f),\ i=0,\ 1,\ \cdots,\ n,\ D_i'\sigma(f)=\sigma(D_i' f),\ i=1,\ \cdots,\ n. \tag{1.24}$$

We shall identify $f\in\mathfrak{C}$. with $\sigma(f)\in\mathfrak{A}$ and express the elements of $\mathfrak{C}$ in the form of elements of $\mathfrak{A}$. Let

$$\varphi(k,\ l)=\sum_{i=1}^{n}(k_i\mu_i+l_i\nu_i). \tag{1.25}$$

Then the commutator operation of $\mathfrak{C}$ is

$$[x^{k}y^{l},\ x^{k'}y^{l'}]=(k_0^*(1-\varphi(k',\ l'))-k_0'^*(1-\varphi(k,\ l)\bar{X}_0Y_0+\sum_{i=1}^{n}(k_i^*l_i'^*-k_i'^*l_i^*)\bar{X}_iY_i, \tag{1.26}$$

where $\bar{X}_i$ are the same as in (1.18)' and

$$Y_0=y^l y^{l'},\ Y_i=y^{l-e_i}y^{l'}=y^l y^{l'-e_i},\ i=1,\ \cdots,\ n. \tag{1.27}$$

where

$$e_i=(\delta_{1i},\ \cdots,\ \delta_{ni}),\ i=1,\ \cdots,\ n. \tag{1.17)'$$

Let

$$\partial_0=I-\sum_{i=1}^{n}\left(\mu_i x_{i0}\frac{\partial}{\partial x_{i0}}+\nu_i y_{i0}\frac{\partial}{\partial y_{i0}}\right), \tag{1.28}$$

where I is the identity mapping. From (1.26) and (1.27), we have

$$[f,\ g]=(D_0f)(\partial_0 g)-(D_0 g)(\partial_0 f)+\sum_{i=1}^{n}((D_i f)(D_i' g)-(D_i g)(D_i' f)),\ f,\ g\in\mathfrak{C}_0 \tag{1.26)'$$

Let $\mathfrak{C}=\mathfrak{C}(n,\ \underline{\tau},\ \mu_i,\ \varepsilon_i,\ \varepsilon_i')$, where

$$\underline{\tau}=(s_0+1,\ \cdots,\ s_n+1,\ t_1+1,\ \cdots,\ t_n+1)\in(\mathbf{Z}^+)^{(2n+1)}, \tag{1.29}$$

Now consider $\mathfrak{C}(n+m,\ \underline{\tau},\ \mu_i,\ \varepsilon_i,\ \varepsilon_i')$ with $s_{n+1}=0$, $\varepsilon_{n+1}=1, i=1,\ \cdots,\ m$. Write z_i for x_{n+i}, w_i for y_{n+1} and γ_i for μ_{n+i}, $i=1,\ \cdots,\ m$. We have

$$z_i^p=1,\ i=1,\ \cdots,\ m. \tag{1.30}$$

Moreover, we assume that

$$\gamma_1,\ \cdots,\ \gamma_m \text{ are linearly independent over } \Pi. \tag{1.31}$$

Let G be the additive subgroup of F generated by $\gamma_1,\ \cdots,\ \gamma_m$. Every element of G can be uniquely expressed as

$$u=\sum_{i=1}^{m}u_i\gamma_i,\ 0\leqslant u_i<p. \tag{1.32}$$

We write

$$z^u=z_1^{u_1}\cdots z_m^{u_m},\ u\in G. \tag{1.33}$$

By (1.30), we have

$$z^u\cdot z^v=z^{u+v},\ u,\ v\in G. \tag{1.34}$$

It is evident that the elements in $\mathfrak{C}$ which do not contain w_i, $i=1,\ \cdots,\ m$, form a Lie subalgebra of $\mathfrak{C}$, which will be denoted by $\Sigma=\Sigma(n, m, \underset{\sim}{r}, G, \mu_i, s_i, s'_i)$. It follows from (1.29) that the commutator operation of Σ can be expressed as

$$[f, g]=(D_0f)(\partial'_0g)-(D_0g)(\partial'_0f)+\sum_{i=1}^{n}((D_if)(D'_ig)-(D_ig)(D'_if)), f, g\in\Sigma, \tag{1.35}$$

where

$$\partial'_0=I-\sum_{i=1}^{n}\left(\mu_ix_{i0}\frac{\partial}{\partial x_{i0}}+\nu_iy_{i0}\frac{\partial}{\partial y_{i0}}\right)-\sum_{i=1}^{m}\gamma_iz_i\frac{\partial}{\partial z_i}. \tag{1.36}$$

Define

$$\deg x^{\bar{k}}y^lz^r=2k_0+\sum_{i=1}^{n}(k_i+l_i). \tag{1.37}$$

An element $a\in\Sigma$ is said to be homogeneous of degree t (denoted by $\deg a=t$) if a is a linear combination of $x^{\bar{k}}y^lz^r$ with all $\deg x^{\bar{k}}y^lz^r=t$. If a, b are homogeneous, then

$$\deg[a, b]=\deg a+\deg b-2. \tag{1.38}$$

It is evident that $\dim\Sigma=p^N$, where

$$N=\sum_{i=0}^{n}(s_i+1)+\sum_{i=1}^{n}(t_i+1)+m. \tag{1.39}$$

Note1.1. If s'_i, t'_i are non-negative integers such that $s'_i\leqslant s_i$, $t'_i\leqslant t_i$, then the subspace $\Sigma'=\langle x^{\bar{k}}y^lz^r\,|\,k_i<p^{s'_i+1}, l_i<p^{t'_i+1}\rangle$ is a subalgebra of Σ and $\Sigma'=\Sigma(n, m, \underset{\sim}{r}', G, \mu_i, \eta_i, \eta'_i)$, where $\underset{\sim}{r}=(s'_0+1, \cdots, s'_n+1, t'_1+1, \cdots, t'_n+1)$ and $\eta_i=s_i$ if $s'_i=s_i$, $s'_i=0$, if $s'_i<s_i$; $\eta'_i=s'_i$ if $t'_i=t_i$, $\eta'_i=0$ if $t'_i<t_i$.

Σ is said to be of type I if $1\in G$, and is of type II if $1\bar{\in} G$. We shall write

$$x_i=x_i^1=x_{i0}, i=0, 1, \cdots, n, y_i=y_i^1=y_{i0}, i=1, \cdots, n. \tag{1.40}$$

If Σ is of type I, we shall also write

$$z=z^1. \tag{1.40}$$

Let

$$\pi_i=p^{s_i+1}-1, i=0, 1, \cdots, n, \pi'_i=p^{t_i+1}-1, i=1, \cdots, n, \tag{1.41}$$

and

$$\bar{\pi}=(\pi_0, \pi_1, \cdots, \pi_n), \pi'=(\pi'_1, \cdots, \pi'_n). \tag{1.42}$$

Lemma 1.1 *If Σ is of type I, then $\mathfrak{C}=\langle z\rangle$ is the center and the commutator subalgebra $\Sigma'=\langle x^{\bar{k}}y^lz^r\,|\,(\bar{k}, l, r)\neq(\bar{\pi}, \pi', n+2)\rangle$.*

Proof We have

$$\partial'_0z^u=(1-u)z^u, u\in G. \tag{1.43}$$

Therefore $\partial'_0z=0$. It follows from (1.35) that $[f, z]=0$, $\forall f\in\Sigma$. On the other hand, if $[g, f]=0$, $\forall f\in\Sigma$, then $D_0g=[g, 1]=0$. It follows that $D_ig=[g, y_i]=0$, $D'_ig=-[g, x_i]=0$, $i=1, \cdots, n$. Thus by (1.14), $g=\sum_{u\in G}\alpha_uz^u$. Then $[g, x_0]=\sum\alpha_u(u-1)z^u=0$, so $\alpha_u=0$ if $u\neq1$, that is, $g\in\langle z\rangle$.

Let $\Sigma_1=\langle x^{\bar{k}}y^lz^r\,|\,(\bar{k}, l, r)\neq(\bar{\pi}, \pi, n+2)\rangle$. If $k_0<\pi_0$, then

$$x^{\bar{k}}y^{l}z^{r}=\frac{1}{(k_0+1)^*}[x^{\bar{k}+\bar{e}_0}y^{l}z^{r},\ 1]\in\Sigma'.$$

If $k_i<\pi_i$ for some $i>0$, we have

$$[x^{\bar{k}+\bar{e}_i}y^{l}z^{u},\ y_i]=(1-\nu_i)y_iD_0(x^{\bar{k}+\bar{e}_i}y^{l}z^{u})+(k_i+1)^*x^{\bar{k}}y^{l}z^{u}\in\Sigma'.$$

Since the first term of the right side is in Σ', $x^{\bar{k}}y^{l}z^{u}\in\Sigma'$. The situation is the same if $l_i<\pi'_i$ for some i. If $(\bar{k},\ l)=(\bar{\pi},\ \pi)$ and $u\neq n+2$, then by (1.35)

$$[x_0,\ x^{\bar{k}}y^{l}z^{u}]=(1-\varphi(\pi,\ \pi')-u-\pi_0^*)x^{\bar{k}}y^{l}z^{r}=(2+n-u)x^{\bar{k}}y^{l}z^{r}\in\Sigma'.$$

So $x^{\bar{k}}y^{l}z^{u}\in\Sigma'$ and $\Sigma_1\subseteq\Sigma'$. On the other hand, by (1.35), we have

$$[x^{\bar{k}}y^{l}z^{u},\ x^{\bar{k}'}y^{l'}z^{u'}]=(k_0^*(1-\varphi(k',\ l')-u')-k_0'^*(1-\varphi(k,\ l)-u)x_0^{k_0-\bar{e}_0}x^{\bar{k}'}y^{l}y^{l'}z^{u+u'}$$
$$+\sum_{i=1}^{n}(k_i^*l_i'^*-k_i'^*l_i^*)x^{\bar{k}-\bar{e}_i}x^{\bar{k}'}y^{l-e_i}y^{l'}z^{u+u'}.\qquad(1.35)'$$

If $x^{\bar{\pi}}y^{\pi}z^{n+2}$ occurs in the right side, then either (i) $\bar{k}+\bar{k}'-\bar{e}_0=\bar{\pi}$, $l+l'=\pi$ and $u+u'=n+2$, In this case, $k_0+k_0'-1=p^{s_i+1}-1$ and $k_0^*+k_0'^*=p$, so the coefficient $k^*(1-\varphi(k',\ l')-u')-k'^*(1-\varphi(k,\ l)-u)=k_0^*(2-\varphi(\pi,\ \pi')-(u+u'))=0$, or (ii) $\bar{k}+\bar{k}'-\bar{e}_i=\pi$, $l+l'-e_i=\pi'$ and $u+u'=n+2$ for some $i>0$. Then $k_i^*+k_i'^*=p$, $l_i^*+l_i'^*=p$ and the coefficient $k_i^*l_i'^*-k_i'^*l_i^*=0$. Therefore $x^{\bar{\pi}}y^{\pi'}z^{n+2}$ does not occur in Σ' and $\Sigma'\subseteq\Sigma_1$.

Lemma 1.2. *If Σ is of type II then $\Sigma'=\Sigma$ when $n+2\not\equiv0\ (\mathrm{mod}\,p)$ and $\Sigma'=\langle x^{\bar{k}}y^{l}z^{u}\mid(\bar{k},\ l,\ u)\neq(\bar{\pi},\ \pi,\ 0)\rangle$ when $n+2\equiv0\ (\mathrm{mod}\,p)$*

Proof Similar to Lemma 1.1.

Definition 1.1. *If $1\in G$, set $\bar{\Sigma}=\bar{\Sigma}(n,\ m,\ \underline{r},\ G,\ \mu_i,\ s_i,\ s_i')=\Sigma'/\mathfrak{C}$.*

Definition 1.2. *If $1\bar{\in}G$ and $n+2\not\equiv0\ (\mathrm{mod}\,p)$,* set

$$\Sigma^*=\Sigma^*(n,\ m,\ \underline{r},\ G,\ \mu_i,\ s_i,\ s_i')=\Sigma.$$

Definition 1.3. *If $1\bar{\in}G$ and $n+2\equiv0\ (\mathrm{mod}\,p)$ set*

$$\tilde{\Sigma}=\tilde{\Sigma}(n,\ m,\ \underline{r}G,\ \mu_i,\ s_i,\ s_i')=\Sigma'.$$

§ 2. Simplicity

Theorem 2.1. *Σ^* is simple,* $\dim\Sigma^*=p^N$.

Proof Let $\mathfrak{B}$ be a nonzero ideal of Σ^*, $0\neq f\in\mathfrak{B}$. Let $f=f_0x_0^k+f_1x_0^{k-1}+\cdots+f_k$, where f_i does not contain x_i, $i=0,\ 1,\ \cdots,\ k$, and $f_0\neq0$. By (1.35) we have

$$[g,\ 1]=D_0g,\ \forall g\in\Sigma.\qquad(2.1)$$

If $k\neq0$, $\mathfrak{B}\ni[f,\ 1]=k^*f_0x_0^{k-1}+\cdots\neq0$. So we may assume $k=0$, i.e., f does not contain x_0. Again by (1.35)

$$[g,\ x_i]=(1-\mu_i)x_i(D_0g)-D_i'g,\ \forall g\in\Sigma,\ i=1,\ \cdots,\ n,\qquad(2.2)$$
$$[g,\ y_i]=(1-\nu_i)y_i(D_0g)+D_ig,\ \forall g\in\Sigma,\ i=1,\ \cdots,\ n.\qquad(2.2)'$$

If for some $i>0$, $f=a_0x_i^k+a_1x_i^{k-1}+\cdots+a_k$, where a_j does not contain x_i, $a_0\neq0$ and $k>0$, then $\mathfrak{B}\ni[f,\ y_i]=k^*a_0x_i^{k-1}+\cdots\neq0$. Thus we may assume $f=f(z)$. Let $f=\sum_{u\in G}c_uz^u$, where $c_u\in F$. choose f such that the number of nonzero $c_u's$ is minimal. If $f=c_1z^{u_1}+$

$c_2 z^{u_2}+\cdots$ with $c_1\neq 0$, $c_2\neq 0$, $u_1\neq u_2$, then $f'=[f, x_0]=c_1(u_1-1)z^{u_1}+c_2(u_2-1)z^{u_2}+\cdots$ $\neq 0$ and $\mathfrak{B}\ni f'-(u_1-1)f=c_2(u_2-u_1)z^{u_2}+\cdots$ which is nonzero with less terms than f. This contradicts our choice of f, so we may assume $f=z^u(u\neq 1$, since $1\bar{\in} G)$. Now $1=(1-u)^{-1}[x_0 z^{-u}, f]\in\mathfrak{B}$. If $k_0<\pi_0$, then

$$x^{\bar{k}}y^l z^r=\frac{1}{(k_0+1)^*}[x^{\bar{k}+\bar{e}_0}y^l z^r, 1]\in\mathfrak{B}.$$

Especially, $x_0\in\mathfrak{B}$, $x_i, y_i\in\mathfrak{B}, i=1, 2, \cdots, n$. If $k_i<\pi_i$ for some $i>0$, then

$$x^{\bar{k}}y^l z^r=\frac{1}{(k_i+1)^*}\{[x^{\bar{k}+\bar{e}_i}y^l z^r; y_i]-\mu_i y_i D_0(x^{\bar{k}+\bar{e}_i}y^l z^r)\}\in\mathfrak{B}.$$

Similarly, if $l_i<\pi'_i$ for some i, then $x^{\bar{k}}y^l z^r\in\mathfrak{B}$. We have $[x^{\bar{\pi}}y^{\pi'}z^r, x_0]=(r-(n+2))$ $x^{\bar{\pi}}y^{\pi}z^r$. Since $1\bar{\in} G$, $G\cap\Pi=\{0\}$. Since $n+2\not\equiv 0 \pmod p$, $r-(n+2)\neq 0$. Therefore $x^{\bar{\pi}}y^{\pi}z^r\in\mathfrak{B}$, $\forall r\in G$ and $\mathfrak{B}=\Sigma^*$.

Similarly, we have

Theorem 2.2. *$\tilde{\Sigma}$ is simple,* $\dim\tilde{\Sigma}=p^N-1$.

Theorem 2.3. *$\bar{\Sigma}$ is simple,* $\dim\bar{\Sigma}=p^N-2$.

§ 3. Generators

In this section we assume $p>3$.

For simplicity, in what follows we denote by f any element $f+\langle z\rangle$ of $\bar{\Sigma}(f\in\Sigma)$. By this convention we have

$$\alpha z=0 \text{ in } \bar{\Sigma},\ \forall\alpha\in F. \tag{3.1}$$

Theorem 3.1. *Assume $p>3$. Let $X=\Sigma^*$, $\tilde{\Sigma}$ or $\bar{\Sigma}$ and*

$\Delta=\{x_i^k,\ 0\leqslant k<p;\ x_i^{p^s},\ 1\leqslant s\leqslant s_i,\ i=0, 1, \cdots, n;\ y_i^l,\ 0\leqslant l<p,\ y_i^{p^t},\ 1\leqslant t\leqslant t_i,$
$i=1, \cdots, n;\ z^r,\ r\in G\}$.

Then X is generated by Δ.

We shall prove Theorem3.1 in detail for $X=\Sigma^*$ only.

Lemma 3.1. *Let $i>0$ and $H^{(i)}$be the subalgebra $\langle x_i^k y_i^l\,|\,0\leqslant k\leqslant\pi_i,\ 0\leqslant l\leqslant\pi'_i,\ (k, l)$ $\neq(\pi_i, \pi'_i)\rangle$ of X. Then$H^{(i)}$ is generated by$\{x_i^k,\ y_i^l\,|\,0\leqslant k\leqslant\pi_i,\ 0\leqslant l\leqslant\pi'_i\}$.*

Proof If $k<\pi_i$, $l<\pi'_i$; then

$$x_i^k y_i^l=\frac{1}{(k+1)^*(l+1)^*}[x_i^{k+1}, y_i^{l+1}].$$

Let $l<\pi'_i$. If $l^*\neq p-1$, then

$$x_i^{\pi_i}y_i^l=\frac{-1}{2(l+1)^*}[x_i^{\pi_i-1}y_i^l, x_i^2 y_i].$$

If $l^*=p-1$, then $l+1<\pi'_i$, so

$$x_i^{\pi_i}y_i^{l_i}=\frac{1}{(l+1)^*}[x_i, x_i^{\pi_i}y_i^{l+1}].$$

Lemma 3.2. *Let $X=\Sigma^*$ and Y be the subalgebra generated by Δ. Then*
$\bar{\Delta}=\{x_i^k,\ 0\leqslant k\leqslant\pi_i,\ i=0, 1, \cdots, n;\ y_i^l,\ 0\leqslant l\leqslant\pi'_i,\ i=1, \cdots, n;\ z^r,\ r\in G\}\subseteq Y$.

Proof (i) $x_0x_i\in Y$, $x_0y_i\in Y$, $i=1, \cdots, n$.

Since $\mu_i+\nu_i=1$, we may assume $\nu_i\neq 1$. Then $[x_0^2, y_i]=2(1-\nu_i)x_0y_i$ and $x_0y_i\in Y$. $[x_i^2, x_0y_i]=(2\mu_i-1)x_i^2y_i+2x_0x_i$. By lemma3.1(applying to the case $\pi_i=\pi_i'=p-1$), the first term is in Y, so $x_0x_i\in Y$.

(ii) $x_i^{k_i}\in Y$, $y_i^{l_i}\in Y$, $\forall 0\leqslant k_i\leqslant \pi_i$, $0\leqslant l_i\leqslant \pi_i'$, $i=1, \cdots, n$.

Induction on k_i. Let $k_i=\sum_{v=0}^{n} c_vp^v$, $k_i^*=c_j$. ⓐ $j=0$. If $c_0\neq 1$, $[x_i^{k_i-1}, x_i^2y_i]=(c_0-1)\ x_i^{k_i}$ and $x_i^{k_i}\in Y$. If $c_0=1$, then $k_i-1\equiv 0 (\mathrm{mod}\, p)$. We have $[x_0x_i, x_i^{k_i-1}]=(1-(k_i-1)\mu_i)x_i^{k_i}=x_i^{k_i}\in Y$. ⓑ $j>0$. We have $[x_0^2, x_i^{p^j}]=2x_0x_i^{p^j}\in Y$ and $x_i^{k_i}=[x_0x_i^{p^j}, x_i^{k_i-p^j}]\in Y$.

(iii) $x_i^{k_0}\in Y$, $\forall 0\leqslant k_0\leqslant \pi_0$.

Induction on k_0. We may assume $k_0>p-1$ and $k_0\neq p^s$, $s=1, \cdots, s_0$. Let

$$k_0=\sum_{v=0}^{u} c_vp^v,$$

$c_u\neq 0$, $k_0^*=c_j$, then, $j<u$ or $j=u$, $c_j>1$. ⓐ $j=0$. We have $[x_0^2, x_0^{k_0-1}]=(3-c_0)x_0^{k_0}$. If $c_0\neq 3$, $x_0^{k_0}\in Y$. If $c_0=3$, then $[x_0^3, x_0^{k_0-2}]=2x_0^{k_0}$ and $x_0^{k_0}\in Y$. ⓑ $j>0$. Then p^j+1, $p^j+2<k_0$. If $c_j\neq 2$, $Y\ni[x_0^{p^j+1}, x_0^{k_0-p^j}]=(2-c_j)x_0^{k_0}$. If $c_j=2$, then $Y\ni[x_0^{p^j+2}, x_0^{k_0-p^j}]=x_0^{k_0+1}$ and $x_0^{k_0}=[x_0^{k_0+1}, 1]\in Y$.

Proof of Theorem 3.1 Let $X=\Sigma^*$. Let Y be the subalgebra generated by Δ. We have $\bar{\Delta}\subseteq Y$.

(i) $x_0^kz^r\in Y$, $\forall 0\leqslant k\leqslant \pi_0$, $r\in G$.

If $k_0<\pi_0$, $[x_0^{k+1}, z^r]=(k+1)^*(1-r)x_0^kz^r\neq 0$ and $x_0^kz^r\in Y$. We have $[x_0^2z^r, x_0^{\pi_0-1}]=2(2-r)x_0^{\pi_0}z^r\neq 0$ and also $x_0^{\pi_0}z^r\in Y$.

(ii) $x_0^{k_0}x_i^{\pi_i}y_i^{\pi_i'}z^r\in Y$, $\forall 0\leqslant k_0\leqslant \pi_0$, $i=1, \cdots, n$, $r\in G$

$[x_0^{k_0+1}z^r, x_i^ky_i^l]\ (k_0+1)^*(1-k\mu_i-l\nu_i)\ x_0^{k_0}x_i^ky_i^lz^r$. By Lemma 3.1, if $k_0<\pi_0$, $(k, l)\neq(\pi_i, \pi_i')$, $1-k\mu_i-l\nu_i\neq 0$, then $x_0^{k_0}x_k^iy_i^lz^r\in Y$. Take integer t such that $0<t<p$, $1-t\mu_i\neq 0$, $2+(t-1)\mu_i\neq 0$. Then $x_0^{k_0}x_i^tz^r\in Y$. If $k_0<\pi_0-1$, $[x_0^{k_0+1}x_i^tz^r, x_i^{\pi_i-t}y_i^{\pi_i'}]=(k_0+1)^*(2+(t-1)\mu_i)\ x_0^{k_0}x_i^{\pi_i}y_i^{\pi_i'}z^r-tx_0^{k_0+1}x_i^{\pi_i-1}y_i^{\pi_i'-1}z^r\in Y$. Since $1-(\pi_i-1)\mu_i-(\pi_i'-1)\nu_i=3\neq 0$, $x_0^{k_0+1}x_i^{\pi_i-1}y_i^{\pi_i'-1}z^r\in Y$, so $x_0^{k_0}x_i^{\pi_i}y_i^{\pi_i'}z^r\in Y$. Take integer s such that $2<s<p$. Then $[x_0^sz^r, x_0^{\pi_0-s+1}x_i^{\pi_i}y_i^{\pi_i'}]=(3s-rs)\ x_0^{\pi_0}x_i^{\pi_i}y_i^{\pi_i'}z^r\in Y$ and $x_0^{\pi_0}x_i^{\pi_i}y_i^{\pi_i'}z^r\in Y$. Since $[x_0^{k_0}x_i^{\pi_i}y_i^{\pi_i'}z^r, 1]=k_0^*x_0^{k_0-1}x_i^{\pi_i}y_i^{\pi_i'}z^r$, applying 1 successively. to $x_0^{\pi_0}x_i^{\pi_i}y_i^{\pi_i'}z^r$, we have $x_0^{k_0}x_i^{\pi_i}y_i^{\pi_i'}z^r\in Y$ for all $0\leqslant k_0\leqslant \pi_0$.

(iii) $x_0^{k_0}x_i^ky_i^lz^r\in Y$, $\forall 0\leqslant k_0\leqslant \pi_0$, $0\leqslant k\leqslant \pi_i$, $0\leqslant l\leqslant \pi_i'$, $r\in G$, $i=1, \cdots, n$.

Induction on $d=\pi_i+\pi_i'-k-l$. If $d=0$, the conlusion follows from (ii). Assune $k>0$ (or $l>0$). $[x_0^{k_0}x_i^ky_i^lz^r, y_i]=k_0^*(1-\nu_i)x_0^{k_0-1}x_i^k(y_i^ly_i)z^r+k^*x_0^{k_0}x_i^{k-1}y_i^lz^r\in Y$. The left side and the first term of the right side is in Y by the assumption of induction, so $x_0^{k_0}x_i^{k-1}y_i^lz^r\in Y$.

(iv) $x_0^{k_0}x^{\pi}y^{\pi'}z^r\in Y$, $\forall 0\leqslant k_0\leqslant \pi_0$ (here $x^{\pi}=x_1^{\pi_1}\cdots x_n^{\pi_n}$), $r\in G$.

Let $a_h(k_0)=x_0^{k_0}x_1^{\pi_1}y_1^{\pi_1'}\cdots x_h^{\pi_h}y_h^{\pi_k'}z^r$, $h=1, \cdots, n$. We shall show $a_h(k_0)\in Y$ by induction on h. ⓐ If $k_0<\pi_0$, then $[a_h(k_0+1),\ x_{h+1}^{\pi_{h+1}}y_{h+1}^{\pi_{h+1}}]=2(k_0+1)^* a_{h+1}(k_0)\in Y$. ⓑ If $h+3\not\equiv 0$ (mod$p$), then $[x_0^2,\ a_{h+1}(\pi_0-1)]=2(h+3-r)a_{h+1}(\pi_0)\in Y$ and $a_{h+1}(\pi_0)\in Y$. ⓒ If $h+3\equiv 0$(mod$p$), then $h+1<n$. From ⓐ, $[x_0^2 x_{h+2}^{\pi_{h+2}}y_{h+2}^{\pi'_{h+2}},\ a_{h+1}(\pi_0-1)]=2(h+4-r)a_{h+2}(\pi_0)\in Y$. By an argument similar to (iii), we have $a_{h+1}(\pi_0)x_{h+2}^k y_{h+2}^l\in Y$, $\forall 0\leqslant k\leqslant\pi_{h+2}$, $0\leqslant l\leqslant\pi'_{h+2}$, and especially $a_{h+1}(\pi_0)\in Y$.

(v) $x^{\bar{k}}y^l z^r\in Y$, $\forall\bar{k}$, l, r.

We have $a_n(k_0)\in Y$, $\forall 0\leqslant k_0\leqslant\pi_0$. By an argument similar to (iii), we can show $x^{\bar{k}}y^l z^r\in Y$ by induction on

$$d=\sum_{i=1}^{n}(\pi_i+\pi'_i)-\sum_{i=1}^{n}(k_i+l_i).$$

If $X=\tilde{\Sigma}$, the proof is essentially the same. If $X=\bar{\Sigma}$, the situation is somewhat more complicated. We omit the details.

Note 3.1. If $m>0$, i.e., $G\neq\{0\}$, Theorem3.1 is valid for $p=3$.

Note 3.2. The number of generators can be reduced. If $m>0$, X is generated by $\Delta_1=\{z^r,\ r\in G;\ x_0^2;\ x_i^{p^s},\ y_i^{p^t},\ i=1,\cdots,n,\ s=0,1,\cdots,s_i,\ t=0,1\cdots,t_i\}$. If $m=0$, X is generated by $\Delta_2=\{1,\ x_0^3;\ x_i^{p^s},\ y_i^{p^s},\ i=1,\cdots,n,\ s=0,1,\cdots,s_i,\ t=0,\cdots,t_i\}$

§ 4. Derivation algebra

In this section we still assume $p>3$.

If L is a Lie algebra, $\mathscr{D}(L)$ will denote the derivatian algebra of L.

In $X=\Sigma^*$, $\tilde{\Sigma}$ or $\bar{\Sigma}$, it is easy to see that $\operatorname{ad}1=D_0$, $\operatorname{ad}x_i=D'_i-\nu_i x_i D_0$, $\operatorname{ad}y_i=-D_i-\mu_i y_i D_0$. Therefore

$$D_0^p=(\operatorname{ad}1)^p;\ D_i^p=-(\operatorname{ad}y_i)^p,\ D_i'^p=(\operatorname{ad}x_i)^p,\ i=1,\cdots,n. \tag{4.1}$$

Thus, D_0^p, D_i^p, $D_i'^p$, $i=1,\cdots,n$, are all derivations of X. Let

$$\mathscr{D}_1=\langle D_i^{p^s},\ i=0,1,\cdots,n,\ s=1,\cdots,s_i;\ D_i'^{p^t},\ i=1,\cdots,n,\ t=1,\cdots,t_i\rangle. \tag{4.2}$$

Then $\mathscr{D}_1$ is a subspace of $\mathscr{D}(X)$, dim $\mathscr{D}_1=N-(2n+m+1)$. Let Θ be the set of all additive mappings of G into F. Then Θ is an m-dimensional vector space over F. For $\theta\in\Theta$, define a linear transformation of X

$$D_\theta:\ D_\theta(x^{\bar{k}}y^l z^r)=\theta(r)x^{\bar{k}}y^l z^r. \tag{4.3}$$

It can be easily verified that D_θ is a derivaton. Let

$$\mathscr{D}_2=\{D_\theta\,|\,\theta\in\Theta\}. \tag{4.4}$$

Then $\theta\mapsto D_\theta$ is a one-one linear mapping of Θ onto $\mathscr{D}_2$. We have dim$\mathscr{D}_2=m$. We shall prove

Theorem 4.1. $\mathscr{D}(\Sigma^*)=\operatorname{ad}\Sigma^*\oplus\mathscr{D}_1\oplus\mathscr{D}_2$. *The dimensionality of the outer derivation algebra of* Σ^* *is* $N-(2n+1)$.

Let R be a commutative ring of characteristic p and $R(n)=R[x_1,\cdots,x_n]=[\sum\alpha_k x^k,\ \alpha_k\in R]$ where $k=(k_1,\cdots,k_n)$, $0\leqslant k_i<p^{s_i+1}$, $x^k=x_1^{k_1}\cdots k_n^{k_n}$ and $x_i^{k_i}$ is defined as

in (1.12). Let D_i, $i=1, \cdots, n$, be defined as in (1.10).

Lemma 4.1. *Let $A_i \in R(n)$, $i=1, \cdots, n$. There exists $f \in R(n)$ such that $A_i = D_i f$, $i=1, \cdots, n$, if and only if $D_i A_j = D_j A_i$, $i, j=1, \cdots, n$, and A_i is x_i-truncated,* i.e., *A_i does not contain any term x^k with $k_i = \pi_i$.*

Proof Necessity is clear. We prove the sufficiency part by induction on n. The case $n=1$ is obvious. Now $R(n) = R[x_n][x_1, \cdots, x_{n-1}]$. By the assumption of induction there exists $f_0 \in R(n)$ such that $D_i f_0 = A_i$, $i=1, \cdots, n-1$. Then $D_n A_i = D_n D_i f_0 = D_i(D_n f_0)$. Since $D_n A_i = D_i A_n$, we have $D_i(A_n - D_n f_0) = 0$, $i=1, \cdots, n-1$. It follows that $D_n f_0 - A_n = g(x_n)$ which contains x_n only. Both terms of the left side are x_n-truncated, so we have $g(x_n) = D_n h(x_n)$. Let $f = f_0 - h(x_n)$, then $D_i f = D_i f_0 = A_i$, $i=1, \cdots, n$, and $D_n f = D_n f_0 - D_n h(x_n) = A_n$.

Note 4.1. Under conditions $D_i A_j = D_j A_i$, $i, j=1, \cdots, n$, $\alpha x_i^{\pi_i}$, $\alpha \in R$ is the only term containing $x_i^{\pi_i}$ that is possible to arise in A_i is In fact, write $A_i = a x_i^{\pi_i} + A'$, where a does not contain x_i and A_i' does not contain $x_i^{\pi_i}$. Then for $j \neq i$, $D_i A_j = D_j A_i = (D_j a) x_i^{\pi_i} + D_j A_i'$. But $D_i A_j$ and $D_j A_i'$ are both x_i-truncated. This forces $D_j a$ to be zero, $j=1, \cdots, n$, and $a = \alpha \in R$. Thus the second condition in Lemma 4.1 can be restated as follows A_i does not contain scalar multiple of $x_i^{\pi_i}$.

Lemma 4.2. *Let $a \in \Sigma^*$. If $[a, 1]=0$, $[a, x_i] = [a, y_i] = 0$, $i=1, \cdots, n$, then $a = h(z)$ which contains only z.*

Proof $D_0 a = [a, 1] = 0$, $D_i a = [a, y_i] - \mu_i(D_0 a) y_i = 0$ and $D_i' a = -[a, x_i] - \nu_i(D_0 a) x_i = 0$, $i=1, \cdots, n$.

Lemma 4.3. *Let D be a derivation of Σ^* satisfying*

$$D1 = Dx_0 = 0; \ Dx_i = Dy_i = 0, \ i=1, \cdots, n. \tag{4.5}$$

Then $Dz^r = \theta(r) z^r$, where $\theta \in \Theta$, i. e., *$\theta(r) \in F$ and*

$$\theta(r+u) = \theta(r) + \theta(u), \ r, u \in G. \tag{4.6}$$

Proof We have $[z^r, 1]=0$. Applying D we obtain $D_0(Dz^r) = [Dz^r, 1] = 0$. Similarly, $D_i(Dz^r) = D_i'(Dz^r) = 0$, $i=1, \cdots, n$. Therefore, $Dz^r = h(z)$. Applying D to the identity $[z^r, x_0] = (r-1) z^r$, we have $[h(z), x_0] = (r-1) h(z)$. Let $h(z) = \sum_{u \in G} \alpha_u z^u$. Then $\sum_u (u-1) \alpha_u z^u = (r-1) \sum_u \alpha_u z^u$. Comparing coefficients, we have $\alpha_u = 0$, $\forall\ u \neq r$, that is $Dz^r = \theta(r) z^r$, where $\theta(r) = \alpha_r$. From $[x_0 z^r, 1] = z^r$, we have $D_0(Dx_0 z^r) = \theta(r) z^r$. It follows that $Dx_0 z^r = \theta(r) x_0 z^r + a_r$, where a_r does not contain x_0. Applying D to the identity $[z^r, x_0 z^u] = (r-1) z^{r+u}$, we have

$$[\theta(r) z^r, x_0 z^u] + [z^r, \theta(u) x_0 z^u + a_u] = (r-1) \theta(r+u) z^{r+u},$$

that is $(r-1)(\theta(r) z^{r+u} + \theta(u) z^{r+u}) = (r-1) \theta(r+u) z^{r+u}$. Since $1 \overline{\in} G$, $r-1 \neq 0$, and we have (4.5).

Lemma 4.4. *Let D be a derivation of Σ^* satisfying*

$$Dz^r = 0, \ r \in G; \ Dx_0 = 0; \ Dx_i = Dy_i = Dx_i y_i = 0, \ i=1, \cdots, n. \tag{4.5$'$}$$

Then $Dx_0^{k_0}=0$, $Dx_i^{k_i}=Dy_i^{l_i}=0$, $k_0,\ k_i,\ l_i=1, \cdots, p-1$, $i=1, \cdots, n$.

Proof (i) $Dx_i^{k_i}=0$, $Dy_i^{l_i}=0$, $i=1, \cdots, n$, $k_i,\ l_i=1, 2, \cdots, p-1$.

Induction on k_i, $[x_i^{k_i}, 1]=[x_i^{k_i}, x_j]=0$, $[x_i^{k_i}, y_j]=\delta_{ij}k_ix_i^{k_i-1}$, $i, j=1, \cdots, n$. Applying D, by Lemma 4.2, we have $Dx_i^{k_i}=h(z)$. $[x_i^{k_i}, x_iy_i]=k_ix_i^{k_i}$. Appling D, we obtain $k_ih(z)=[h(z), x_iy_i]=0$, so $h(z)=0$. Similarly, $Dy^{l_i}=0$.

(ii) $Dx_0^{k_0}=0$, $k_0=1, \cdots, p-1$.

Let $a=x_0^{s_0}x_i^{k_i}y_i^{l_i}z^r$, where $i>0$, $0\leqslant k_i,\ l_i<p$. We show $Da=0$ if deg $a<2k_0$, by induction on deg$a$. The case deg$a=0$ is clear. $[a, 1]=D_0a$, $[a, x_i]=\nu_i(D_0a)x_i-D_i'a$, $[a, y_i]=\mu_i(D_0a)y_i+D_ia$. Applying D, by the assumption of induction and Lemma 4.2, we have $Da=h(z)$. We shall show $h(z)=0$ by a second induction on s_0. When $s_0=0$, our assertion follows from Lemma 3.1 and (i) (note that since deg$a<2k_0\leqslant 2(p-1)$, $(k_i, l_i)\neq(p-1, p-1)$). ① If $k_i\neq l_i$, then $[a, x_iy_i]=(k_i-l_i)a$. Applying D, we have $(k_i-l_i)h(z)=[h(z), x_iy_i]=0$, so $h(z)=0$. ② If $k_i=l_i\neq 0$, then $k_i\neq p-1$, otherwise deg $a=s_0+2(p-1)$, contrary to our assumption deg$a<2k_0\leqslant 2(p-1)$. Now $[x_0^{s_0}x_i^{k_i+1}y_i^{k_i-1}, y_1^2]=s_0(1-k_1-\mu_1+\nu_1)\,x_0^{s_0-1}x_i^{k_i+1}y_i^{l_i+1}+2(k_i+1)a$. Applying D, by induction assumption and ① we have $Da=0$. ③ $k_i=l_i=0$, i.e., $a=x_0^{s_0}$. We may assume $s_0>0$. $[x_0, a]=(1-s_0)a$. Applying D, we have $[x_0, h(z)]=(1-s_0)h(z)$. Let $h(z)=\sum_{u\in G}\alpha_uz^u$, then $\sum(1-u)\alpha_uz^u=(1-s_0)\sum\alpha_uz^u$. Comparing coefficients, we have $(u-s_0)\alpha_u=0$. Since $1\overline{\in} G$, $G\cap\Pi=\{0\}$. It follows that $u-s_0\neq 0$, $\forall u\in G$. We have $\alpha_u=0$, $\forall u\in G$, that is, $h(z)=0$. Now $[x_0^{k_0}, 1]=k_0x_0^{k_0-1}$, $[x_0^{k_0}, x_i]=\nu_ix_0^{k_0-1}x_i$ and $[x_0^{k_0}, y_i]=\mu_ix_i^{k_0-1}y_{i0}$ Applying D, by Lemma 3.2, $Dx_0^{k_0}=h(z)$. Repeating the argument of ③, we have $Dx_0^{k_0}=0$.

Lemma 4.5. *Let D be a derivation of Σ^* satisfying* (4.5). *Let i, s be integers such that $0\leqslant i\leqslant n$, $1\leqslant s\leqslant s_i$. If $Dx_i^{p^u}=0$ for $u=1, \cdots, s-1$, then $Dx_i^{p^s}=\alpha\cdot 1$ with $\alpha\in F$. For $y_i^{p^t}$, $1\leqslant i\leqslant n$, $1\leqslant t\leqslant t_i$, similar result holds.*

Proof (i) $i>0$. By Theorem 3.1 and Lemma 4.4, $D\Sigma^*(n, m, \underset{\sim}{r}')=0$, where $\underset{\sim}{r}'$ is obtained by changing the i-th component of $\underset{\sim}{r}$ to s and all other components to 1 (see Note 1.1). We have $[x_i^{p^s}, 1]=0$, $[x_i^{p^s}, x_j]=0$ and $[r_i^{p^s}y_j]=\delta_{ij}x_i^{p^s-1}\in\Sigma^*(n, m, \underset{\sim}{r}')$. Applying D, by Lemma 4.2, we have $Dx_i^{p^s}=h(z)$. $[x_0, x_i^{p^s}]=x_i^{p^s}$. Applying D, $[x_0, h(z)]=h(z)$. If $h(z)=\sum_{r\in G}\alpha_rz^r$, then $\sum(1-r)\alpha_rz^r=\sum\alpha_rz^r$. It follows that $\alpha_r=0$, $\forall r\neq 0$. (ii) $i=0$. We have $[x_0^{p^s}, 1]=x_0^{p^s-1}$, $[x_0^{p^s}, x_j]=\nu_ix_0^{p^s-1}x_j$ and $[x_0^{p^s}, y_j]=\mu_ix_0^{p^s-1}y_j\,(j>0)$. All the right sides are in $\Sigma^*(n, m, \underset{\sim}{r}')$. Applying D, we have our conclusion by an argument similar to (i).

Lemma 4.6. (1) $\mathscr{D}_1\cap\mathscr{D}_2=\{0\}$; (2) $(\mathscr{D}_1\oplus\mathscr{D}_2)\cap\operatorname{ad}\Sigma^*=\{0\}$.

Proof (1) is obvious. (2) Let

$$\operatorname{ad}a=\sum_{i=0}^{n}\sum_{j=1}^{s_i}\alpha_{ij}D_i^{p^j}+\sum_{i=1}^{n}\sum_{j=1}^{t_i}\beta_{ij}D_i'^{p^j}+D_\theta.$$

Applying this to 1, we have $[a, 1]=0$. Similarly $[a, x_i]=[a, y_i]=0$, $i=1, \cdots, n$. By

Lemma 3.2, $a=h(z)$. If $a=h(z)=\sum\alpha_r z^r$, then $[h(z), x_0^g]=-x_0^{g-1}h^*(z)$, where $h^*(z)=\sum\alpha_r(1-r)z^r$. Applying ad a to x_0^g, we have $-x_0^{g-1}h^*(z)=\alpha_{00}\cdot 1$ which implies $h^*(z)=0$, that is $h(z)=0$.

Proof of Theorem 4.1 Let D be a derivation of $\Sigma^*(n, m, \underline{r})$

(i) Let $a_0=D1$, $a_i=Dy_i-\mu_i a_0 y_i$, $b_i=-Dx_i+\nu_i a_0 x_i$, $i=1, \cdots, n$. Since $[1, y_i]=0$, we have $[D1, y_i]+[1, Dy_i]=0$. Hence, we get $[a_0, y_i]=[1, a_i+\mu_i a_0 y_i]=0$. That is, $\mu_i(D_0a_0)y_i+D_ia_0-D_0a_i-\mu_iD_0(a_0y_i)=0$. It follows that

$$D_0a_i=D_ia_0,\ i=1, \cdots, n. \tag{4.7}$$

Similarly

$$D_0b_i=D_i'a_0,\ i=1, \cdots, n. \tag{4.8}$$

Since $[x_i, x_j]=0$, $i, j=1, \cdots, n$, we have $[Dx_i, x_j]+[x_i, Dx_j]=0$, That is

$$[-b_i+\nu_i a_0 x_i, x_j]+[x_i, -b_j+\nu_j a_0 x_j]=0.$$

We have

$$-\nu_j(D_0b_i)x_j+D_j'b_i+\nu_i\nu_j(D_0a_0)x_ix_j-\nu_i(D_j'a_0)x_i+\nu_i(D_0b_j)x_i-D_i'b_j$$
$$-\nu_i\nu_j(D_0a_0)x_jx_i+\nu_j(D_i'a_0)x_j=0.$$

Simplifying it and using (4.8), we get

$$D_j'b_i=D_i'b_j,\ i, j=1, \cdots, n. \tag{4.9}$$

Similarly

$$D_ja_i=D_ia_j,\ i, j=1, \cdots, n. \tag{4.10}$$

From $[x_i, y_j]=\delta_{ij}\cdot 1$, similar computation shows

$$D_i'a_j=D_jb_i,\ i, j=1, \cdots, n. \tag{4.11}$$

We shall show that a_i is x_i-truncated, $i=0, 1, \cdots, n$. By Note 4.1, t suffices to show that a_i does not contain $F[z]$-mnltiple of $x_i^{\pi_i}$. From $[x_0, 1]=1$, we have $[Dx_0, 1]+[x_0, a_0]=a_0$. It follows that

$$a_0-(\partial_0'a_0-x_0D_0a_0)=D_0(Dx_0). \tag{4.12}$$

Let $a_0=\alpha(z)x_0^{\pi_0}+a_0'$, where $\alpha(z)\in F[z]$ and a_0' is x_0-truncated. We have

$$\partial_0'a_0=(\partial_0'\alpha(z))x_0^{\pi_0}+\partial_0'a_0' \quad \text{and} \quad x_0(D_0a_0)=-\alpha(z)x_0^{\pi_0}+a_0'',$$

where $\partial_0'a_0'$ and a_0'' are also x_0-truncated. Substituting these into (4.12), we have

$$-(\partial_0'\alpha(z))x_0^{\pi_0}+(a_0'-\partial_0'a_0'+a_0'')=D_0(Dx_0).$$

Since the right side is x_0-truncated, we must have $\partial_0'\alpha(z)=0$, that is, $\alpha(z)=0$. For $i>0$, let $a_i=\alpha_i(z)x_i^{\pi_i}+a_i'$. From $[x_0, y_i]=\mu_i y_i$, by a similar argument, we can show that $\alpha_i(z)=0$, $i=1, \cdots, n$. Thus a_i is x_i-truncated. Also, b_i is y_i-truncated. By Lemma 4.1, there exists $a\in\Sigma^*$ such that $D_0a=a_0$, $D_ia=a_i$, $D_i'a=b_i$, $i=1, \cdots, n$. Let $D^{(1)}=D-\text{ad }a$, then $D^{(1)}1=D1-[a, 1]=0$, $D^{(1)}x_i=Dx_i-[a, x_i]=(\nu_ia_0x_i-b_i)-(\nu_i(D_0a)x_i-D_i'a)=0$ and also $D^{(1)}y_i=0$, $i=1, \cdots, n$.

(ii) By Lemma4.3, $D^{(1)}z^r=\theta(r)z^r$, $\theta\in\Theta$. Let $D^{(2)}=D^{(1)}-D_\theta$. We have $D^{(2)}z^r=0$, $\forall r\in G$ and still $D^{(2)}1=D^{(2)}x_i=D^{(2)}y_i=0$, $i=1, \cdots, n$.

(iii) $[x_0, 1]=1$, $[x_0, x_i]=\nu_ix_i$, $[x_0, y_i]=\mu_iy_i$ Applying $D^{(2)}$, by Lemma 4.2, we have $D^{(2)}x_0=h(z)$. $[x_iy_i,\ 1]=0$, $[x_iy_i,\ x_j]=-\delta_{ij}x_i$, $[x_iy_i,\ y_j]=\delta_{ij}y_i$, $i,\ j=1,\ \cdots,\ n$. Applying$D^{(2)}$, by Lemma 4.2, we have $D^{(2)}(x_iy_i)=h_i(z)$; $i=1,\ \cdots,\ n$. Appling $D^{(2)}$to the identity $[x_0,\ x_iy_i]=0$, we have$[x_0\ ,\ h_i(z)]=0$. That is $\partial_0'h_i(z)=0$, so $h_i(z)=0$, $i=1,\ \cdots,\ n$.Let $h(z)=\sum_{r\in G}\alpha_rz^r$ and $f(z)=\sum_{r\in G}(r-1)^{-1}\alpha_rz^r$, than $\partial_0f(z)=h(z)$. Let $D^{(3)}=D^{(2)}-\operatorname{ad}f(z)$, we have $D^{(3)}\ x_0=h(z)-[f(z),\ x_0]=h(z)-\partial_0'f(z)=0$. Now $D^{(3)}$ satisfies (4.5)$'$.

(iv) By Lemma 4.5, $D^{(3)}x_i^p=\alpha_i\cdot 1$, $i=0,\ 1,\cdots,\ n$, $D^{(3)}\cdot y_i^p=\beta_i 1$, $i=1,\ \cdots,\ n$, α_i, $\beta_i\in F$. Let $D^{(4)}=D^{(3)}+\sum_{i=0}^{n}\alpha_iD_i^p+\sum_{i=0}^{n}\beta_iD_i'^p$, then $D^{(4)}x_0^p=D^{(4)}x_i^p=D^{(4)}y_i^p=0$, $i=1,\cdots,n$. $D^{(4)}$ still satisfies(4.5)$'$. By Lemma 4.5 $D^{(4)}x_i^{p^2}=\alpha_i'\cdot 1$, $D^{(4)}y_i^{p^2}=\beta_i'\cdot 1$. Proceeding as above, we can finally obtain $\bar{D}=D^{(3)}+\sum_{i=0}^{n}\sum_{j=1}^{s_i}\alpha_{ij}D_i^{p^j}+\sum_{i=1}^{n}\sum_{j=1}^{t_i}\alpha_{ij}D_i'^{p^j}$ such that $\bar{D}$ satisfies(4.5)$'$ and $\bar{D}x_i^{p^j}=0$, $i=0,\ 1\cdots,\ n$, $j=1,\ \cdots,\ s_i$, $\bar{D}y_i^{p^j}=0$, $i=1,\ \cdots,\ n$, $j=1,\ \cdots,\ t_i$. By Lemma 4.4 and Theorem 3.1, we have $\bar{D}=0$ and $D\in\mathscr{D}_1\oplus\mathscr{D}_2\oplus\operatorname{ad}\Sigma^*$.

Corollary 4.1. *$\mathscr{D}(\Sigma^*)$ is the restricted closure of* $\operatorname{ad}\Sigma^*$.

Proof It suffices to show $\mathscr{D}_2\subseteq\operatorname{ad}\Sigma^*+(\operatorname{ad}\Sigma^*)^p+\cdots$. Let

$$b=x_0+\sum_{i=1}^{n}\nu_ix_iy_i.$$

Then $\operatorname{ad}b\cdot z^u=(1-u)z^u$, $\operatorname{ad}b\cdot x_0=\operatorname{ad}b\cdot x_i=0$ and $\operatorname{ad}b\cdot y_i=y_i$, $i=1,\ \cdots,\ n$. Let $A=(\operatorname{ad}b)^p-\operatorname{ad}b$. We have $Ax_0=Ax_i=Ay_i=Ax_iy_i=0$, $i=1,\ \cdots,\ n$, $Az^u=\theta_0(u)z^u$, where $\theta_0(u)=u^p-u$ and $\theta_0\in\Theta$. Since $G\cap\Pi=\{0\}$, $\theta_0(u)\neq 0$ if $u\neq 0$. It follows that θ_0, θ_0^p, $\cdots$, $\theta_0^{p^{m-1}}$ are linearly independent and form a basis of Θ. Let

$$\theta=\sum_{i=0}^{m-1}\alpha_i\theta_0^{p^i}$$

be any element in Θ and

$$\bar{A}=\sum_{i=0}^{m-1}\alpha_iA^{p^i}.$$

set $D=\bar{A}-D_\theta$. Then $Dz^u=0$ and D satisfies(4.5)$'$. By Lemma 4.4 and Lemma 4.5, $D=\sum\alpha_{ij}D_i^{p^j}+\sum\beta_{ij}D_i'^{p^j}$, then $D_\theta=\bar{A}-D\in\operatorname{ad}\Sigma^*+(\operatorname{ad}\Sigma^*)^p+\cdots$.

Theorem 4.2. *If $n+2\equiv 0(\operatorname{mod}p)$, let d_π be the restriction of* $\operatorname{ad}x^{\bar{\pi}}y^{\pi'}$ *on $\tilde{\Sigma}$, then $\mathscr{D}(\tilde{\Sigma})=\operatorname{ad}\tilde{\Sigma}\oplus\langle d_\pi\rangle\oplus\mathscr{D}_1\oplus\mathscr{D}_2$. The dimensionality of the outer derivation algebra of $\tilde{\Sigma}$ is $N-2n$.*

Proof Same as the proof of Theorem 4.1.

In $\bar{\Sigma}$, let $\bar{d}_\pi$ be the restriction of $\operatorname{ad}x^{\bar{\pi}}y^{\pi'}z^{n+2}$ on $\bar{\Sigma}$ and define linear transformations

$$\bar{d}_i:\ \bar{d}_if=x_i^{\pi_i}z(\partial_0'f),\ i=0,\ 1,\ \cdots,\ n;\ \bar{d}_i':\ \bar{d}_i'=y_i^{\pi_i'}z(\partial_0'f),\ i=1,\ \cdots,\ n \tag{4.13}$$

$$\bar{d}_z:\ \bar{d}_zf=zD_0f. \tag{4.14}$$

It can be directly verified that $\bar{d}_i$, $\bar{d}_i'$ and $\bar{d}_z$ are derivations of $\bar{\Sigma}$. Let

$$\mathscr{D}_3=\langle\bar{d}_i,\ i=0,\ 1,\ \cdots,\ n;\ \bar{d}_i',\ i=1,\ \cdots,\ n;\ \bar{d}_z\rangle. \tag{4.15}$$

Theorem 4.3. $\mathscr{D}(\bar{\Sigma})=\operatorname{ad}\ \bar{\Sigma}+\langle\bar{d}_\pi\rangle\oplus\mathscr{D}_1\oplus\mathscr{D}_2\oplus\mathscr{D}_3$. *The dimensionality of the* *outer derivation algebra of* $\bar{\Sigma}$ *is* $N+2$.

Proof The argument is similar to that of Theorem 4.1 but is more complicated. Because of (3.1), Lemma 4.2 is no longer true and must be refined. This causes the existence of the outer derivations $\bar{d}_i$, $\bar{d}'_i$ and $\bar{d}_z$. We omit the details.

§ 5. Filtration

In this section we assume $p>3$ and

$$n>0;\ m>0;\ \varepsilon_0=0,\ \varepsilon_i=\varepsilon'_i=0,\ i=1,\ \cdots,\ n. \tag{5.1}$$

If L is a Lie algebra, $d\in\mathscr{D}(L)$, let $I(d)=\dim(\operatorname{Im} d)$. Let $\mathscr{D}$ be a subalgebra of $\mathscr{D}(L)$, let $I(\mathscr{D})=\min\limits_{0\neq d\in\mathscr{D}} I(d)$.

Let $X=\Sigma^*$, $\tilde{\Sigma}$ or $\bar{\Sigma}$. Set

$$s(z)=\sum_{r\in G} z^r, \tag{5.2}$$

$$b=x^{\bar{\pi}}y^{\pi'}s(z),\ B=\operatorname{ad} b\,|_X, \tag{5.3}$$

Lemma 5.1. (1) $I(B)=2(n+1)$; (2) *Let* $\mathfrak{C}=\ker B$, *then* $\mathfrak{C}=\langle x=x^k y^l z^r \mid \deg x \geqslant 2,\ x\neq x_0\rangle\oplus\langle(1-r)-z^r,\ r\in G;\ x_0(1-z^r),\ r\in G;\ x_i(1-z^r),\ y_i(1-z^r),\ r\in G,\ i=1,\ \cdots,\ n\rangle$.

Proof Let $x=x^k y^l z^r$. By (1.38), it is evident that $Bx=0$ if $\deg x>2$. It is also easy to verify that if $\deg x=2$ and $x\neq x_0$, then $Bx=0$. Note that $s(z)z^r=s(z)$, $\forall r\in G$. We have $Bx_0z^r=-(n+2)b+x^{\bar{\pi}}y^{\pi'}(\sum_{u\in G} uz^u)$, which is independent of r, so $B(x_0-x_0z^r)=0$. For $i>0$, $Bx_iz^r=x^{\bar{\pi}}y^{\pi'-\varepsilon_i}s(z)$ and $Bx_i(1-z^r)=0$. Similarly, $By_i(1-z^r)=0$. We have $Bz^r=(r-1)x^{\bar{\pi}-\bar{\varepsilon}_0}y^{\pi'}s(z)$ and $B((1-r)-z^r)=0$. Let $\mathfrak{B}=\langle 1,\ x_0,\ x_i,\ y_i,\ i=1,\cdots,n\rangle$, then $X=\mathfrak{C}\oplus\mathfrak{B}$. We see that $B1$, Bx_0, Bx_i, By_i, $i=1,\ \cdots,\ n$, are linerly independent. Now $\dim\mathfrak{B}=2(n+1)$ and our conclusions follow.

Teroem 5.1. *Let* $p>3$ *and* $X=\Sigma^*$, $\tilde{\Sigma}$ or $\bar{\Sigma}$ *satisfying* (5.1). *Then* (1) $I(\mathscr{D}(X))=2(n+1)$; (2) $D\in\mathscr{D}(X)$, $I(D)=2(n+1)$ *if and only if* $D\in\langle B\rangle$.

Proof By an argument similar to [14, Theorem 2.1, 2.2], $I(D)\geqslant 2(n+1)$, $\forall D\in\mathscr{D}(X)$ and if $I(D)=2(n+1)$, then $D=\operatorname{ad} x^{\bar{\pi}}y^{\pi'}f(z)\,|_X$, where $f(X)\in F[z]$. If $f(z)\bar{\in}\langle s(z)\rangle$, since $\langle s(z)\rangle$ is the only one-dimensional ideal of $F[z]$ (cf., e.g., [15, Lemma 80]), there is z^t, $t\in G$, such that $f(z)$ and $f(z)z^t$ are linearly independent. Now $Dz^t=(t-1)x^{\bar{\pi}-\bar{\varepsilon}_0}y^{\bar{\pi}'}f(z)z^t$. The images of the $2n+3$ elements 1, z^t, x_0, x_i, y_i, $i=1,\ \cdots,\ n$, are linearly independent, so $I(D)>2(n+1)$. Therefore $f(z)\in\langle s(z)\rangle$ and $D\in\langle B\rangle$.

Let Γ be the induced representation of $\mathfrak{C}$ on $X/\mathfrak{C}$

$$\Gamma(x):\ y+\mathfrak{C}\mapsto[x,\ y]+\mathfrak{C},\ x\in\mathfrak{C},\ y\in X. \tag{5.4}$$

Lemma 5.2. (1) Γ *is irreducicle*, (2) $\mathfrak{C}$ *is an invariant maximal subalgebra of* X.

Proof (1) For any $a\in X$, the element $a+\mathfrak{C}\in X/\mathfrak{C}$ will be denoted by $\bar{a}$. Let M

be a nonzero submodule of $X/\mathfrak{C}$ and $0\neq\bar{a}=\gamma\cdot\bar{1}+\alpha\bar{x}_0+\sum_{i=1}^{n}(\alpha_i\bar{x}_i+\beta_i\bar{y}_i)\in M$. If $\alpha_i\neq 0$ (or $\beta_i\neq 0$) for some $i=1, \cdots, n$, then $-\Gamma(x_iy_i)\bar{a}=\alpha_i\bar{x}_i\in M$ (or $\beta_i\bar{y}_i\in M$). If $\alpha_i=\beta_i=0$, $i=1, \cdots, n$, when $\gamma\neq 0$, $\Gamma(x_0x_i)\bar{a}=\gamma\bar{x}_i\in M$ (similarly $\gamma\bar{y}_i\in M$). When $\gamma=0$, $\bar{a}=\alpha_0\bar{x}_0$ and $\Gamma(x_i-x_iz^r)\bar{a}=-\alpha_0 r\overline{x_iz^r}\in M$, that is $\bar{x}_i=\overline{x_iz^r}\in M$ (similarly, $\bar{y}_i\in M$). In all cases we have $\bar{x}_i\in M$ (or $\bar{y}_i\in M$) for some $i>0$. So

$$\bar{1}=-\Gamma\left(\frac{1}{r}y_i(1-z^r)\right)\cdot\bar{x}_i=+\Gamma\left(\frac{1}{r}x_i(1-z^r)\right)\cdot\bar{y}_i\in M.$$

Then $\bar{x}_0=\Gamma(x_0^2)\cdot\bar{1}\in M$, $\bar{x}_j=\Gamma(x_0x_j)\cdot 1\in M$ and $\bar{y}_j=\Gamma(x_0y_j)\cdot\bar{1}\in M$, $j=1, \cdots, n$, and $M=X/\mathfrak{C}$.

(2) $\mathfrak{C}$ is invariant by Theorem 5.1 and Lemma 5.1. Let L be any subalgebra containing $\mathfrak{C}$, then $L/\mathfrak{C}$ is a submodule of $X/\mathfrak{C}$. By (1), $L=\mathfrak{C}$ or X, and $\mathfrak{C}$ is maximal.

Theorem 5.2. *Let $p>3$ and $X=\Sigma^*$, $\tilde{\Sigma}$ or $\bar{\Sigma}$ satisfying (5.1). Then X is of generalized Cartan type $H(2n+2)$.*

Proof Let $a=1-r-z^r$, $r\neq 0$ and $r\neq 1$ in case $X=\bar{\Sigma}$. We have rank $\Gamma(a)=1$ and $(\mathrm{ad}\Gamma(a))^2\neq 0$ since $(\mathrm{ad}\Gamma(a))^2\cdot\Gamma(x_0^2)=4(1-r)^2\Gamma(2(1-r-z^r)-(1-2r-z^{2r}))\neq 0$. Thus the pair $(X, \mathfrak{C})$ satisfies condition (0.1) of wilson[18]. Moreover, $(\mathrm{ad}x_1^{\pi_1})^2=0$, so X is strongly degenerate. By [18, corollary], X is of generalized Cartan type. By [18, Lemma 3.6], $\Gamma(\mathfrak{C})$ must be one of gl$(X/\mathfrak{C})$, sl$(X/\mathfrak{C})$, sp$(X/\mathfrak{C})$ or csp $(X/\mathfrak{C})$. It is easy to see that $\Gamma(\mathfrak{C})$ has a basis $\{\Gamma(1-r-z^r), \Gamma(x_0-x_0z^r), \Gamma(x_i-x_iz^r), \Gamma(y_i-y_iz^r), i=1, \cdots, n, \Gamma(x_ix_j), \Gamma(x_iy_j), \Gamma(y_iy_j), i, j=1, \cdots, n, \Gamma(x_0x_i), \Gamma(x_0y_j) i=1, \cdots, n, \Gamma(x_0^2)\}$, where $r\neq 0$, 1 is a fixed element of G. We have dim $\Gamma(\mathfrak{C})=(2n+3)(n+1)$ and dim $X/\mathfrak{C}=2(n+1)$, so dim$\Gamma(\mathfrak{C})=$dim sp$(X/\mathfrak{C})$. Therefore $\Gamma(\mathfrak{C})=$sp$(X/\mathfrak{C})$. Let Gr $X=L_{-1}\oplus L_0\oplus L_1\oplus\cdots$ be the graded Lie algebra associated with X. Since $\sum_{i\leq 0}L_i\cong\sum_{i\leq 0}H(2n+2)_{[i],\underline{1}}$ and $L_2\neq 0$, by the argument of [18, § 7] or [7, proposition 3.2], we conclude that X is of generalized Cartan type $H(2n+2)$.

Remark 5.1. All known simple Lie algebras of Cartan type $H(2k)$ with $k>1$ are included in [10, 11, 16] (cf [18, § 7]). None of them are of dimensionality p^N. Therefore $\Sigma^*(n, m, \underline{r}, G)$ is a new simple Lie algebra if $n>0$ (under condition (5.1)). The outer derivation algebras of all known (p^N-1)-dimensional simple Lie algebras of generalized Cartan type H have dimensionality $\geqslant N$. We also conclude that $\tilde{\Sigma}(n, m, \underline{r}, G)$ is new under condition (5.1).

Let $\mathscr{L}_0=\mathfrak{C}$, $\mathscr{L}_{-1}=X$ and define $\mathscr{L}_i=\{x\in\mathscr{L}_{i-1}\,|\,[x, \mathscr{L}_{-1}]\subseteq\mathscr{L}_{i-1}\}$, $i>0$. We obtain an invariant filtration of X: $X=\mathscr{L}_{-1}\supseteq\mathscr{L}_0\supseteq\mathscr{L}_1\supseteq\cdots$. If $\alpha=\{\alpha_r\,|\,r\in G\}$ is a subset of F, define $p_t(\alpha)=\sum_{r\in G}(1-r)^t\alpha_r$, $t=0, 1, \cdots$. By direct computation we have

$$\mathscr{L}_i=\langle\sum_r\alpha_rz^r\,|\,p_t(\alpha)=0,\ t=1, \cdots, i+1\rangle\oplus\sum_{j=1}^{i}\langle x^{\bar{k}}y^{\bar{l}}(\sum_r\alpha_rz^r)\,|\,\deg x^{\bar{k}}y^{\bar{l}}=j,$$

$$p_t(\alpha)=0,\ t=0,\ 1,\ \cdots,\ i,\ j+1\rangle\oplus\langle x^k y^l z^r \mid \deg x^k y^l z^r > i+1\rangle.$$

By an argument similar to [14, Theorem 3.2] based on a discussion of dim $\mathscr{L}_i$, we can prove.

Theorem 5.3. *Let $p>3$ and $X=\Sigma^*$, $\tilde{\Sigma}$ or $\bar{\Sigma}$ satisfying* (5.1)*. If $X(n, m, \underline{r}, G)\cong X(n', m', \underline{r}', G')$ then $n=n'$ and $\{s_0+1, s_1+1, \cdots, s_n+1, t_1+1, \cdots, t_n+1, m\}=\{s'_0+1, s'_1+1, \cdots, s'_n+1, t'_1+1, \cdots, t'_n+1, m'\}$.*

Corollary 5.1. *If $X=\Sigma^*$ or $\tilde{\Sigma}$ satisfying* (5.1)*, then $X(n, m, \underline{r}, G)\cong X(n', m', \underline{r}', G')$ implies $m=m'$.*

Proof Let $S=\{s_0+1, s_1+1, \cdots, s_n+1, t_1+1, \cdots, t_n+1\}$, $M=\{m\}$. Let $V_t=(\mathrm{ad}X+(\mathrm{ad}X)^p+\cdots+(\mathrm{ad}X)^{p^t})/\mathrm{ad}X$, $t=1, 2, \cdots$. Set $d_t=\dim V_t$ which is an invariant of X. By Theorems 4.1, 4.2 and the proof of corollary 4.1

$$d_t=(\mathrm{card}\,S_t)t+\sum_{x\in S\setminus S_t}(x-1)+(\mathrm{card}M_t)t+\sum_{x\in M\setminus M_t}x,$$

where

$$S_t=\{x\in S \mid x-1>t\},\ M_t=\{x\in M \mid x>t\}.$$

(Let $\bar{V}_t=\mathrm{ad}\,X+(\mathrm{ad}\,X)^p+\cdots+(\mathrm{ad}X)^{p^t}$. A close investigation shows $\bar{V}_t\cap(\mathscr{D}_1\oplus\mathscr{D}_2)=\langle D_0^{p^t}, D_i^{d^k}, D_i'^{p^k}, i=1, \cdots, n, 0<k\leqslant t, \theta_0^{p^{k-1}}, 0<k\leqslant t\rangle)$. By (1.39), we have

$$d_t=N-(2n+1)+\mathrm{card}\,S_t-\sum_{x\in S_t}(x-t)-\sum_{x\in M_t}(x-t). \tag{5.4}$$

Let $t=m-1$. We have

$$d_{m-1}=N-(2n+1)+\mathrm{card}\,S_{m-1}-\sum_{x\in S_{m-1}}(x-t)-1.$$

Suppose $m'<m$. For $X(n', m', \underline{r}', G')$ we define S', M', S'_t, M'_t and d'_t analogously. If $X(n', m', \underline{r}, G')\cong X(n, m, \underline{r}, G)$, by Theorem 5.3, $S'=(S\setminus\{m'\})\cup\{m\}$ and $S'_{m-1}=S_{m-1}$. Now $M'_{m-1}=\phi$ but $M_{m-1}=\{m\}$. By (5.4), $d'_{m-1}=d_{m-1}+1$. This is impossible, since d_{m-1} is an invariant and we must have $m'=m$.

Note 5.1. If $m=0$, $X>0$, $X=\Sigma^*$ or $\tilde{\Sigma}$, then $X=\sum_{i\geqslant -2}L_i$ is a graded Lie algebra, where $L_i=\langle x=x^k y^l \mid \deg\ x=i+2\rangle$. It is easy to prove

$$\sum_{i\leqslant 0}L_i\cong\sum_{i\leqslant 0}K(2n+1,\ \underline{v})_{[i],\underline{1}+\varepsilon_{2n+1}},$$

where $\underline{v}=(s_1+1, \cdots, s_n+1, l_1+1, \cdots, l_n+1, s_0+1)$. By [18, proposition 4.1] or [7, proposition 3.2], $X\cong K(2n+1, \underline{v})$.

Note 5.2. If $n=0$, $m>0$, Theorem 5.2 is still valid, i.e., X is the Albert-Zassenhaus algebra of generalized Cartan type $H(2)$. If $n=m=0$, then it is easily seen that X is a Zassenhaus algebra, i. e., it is of generalized Cartan type $W(1)$.

§ 6. *K*-like gradation

In this section (5.1) is assumed. Let $X=\Sigma^*$, $\tilde{\Sigma}$ or $\bar{\Sigma}$. Define

$$K_{[i]}=\langle x=x^k y^l z^r\in X \mid \deg x=i+2\rangle,\ i=-2, -1, 0, 1, \cdots. \tag{6.1}$$

Then $X=\sum_{i\geq-2} K_{[i]}$ is a graded Lie algebra, called the K-like gradation of X. Let $K_0=\sum_{i\geq 0} K_{[i]}$.

Proposition 6.1. (1) *K_0 is a maximal subalgebra of X*; (2) *The representation of $K_{[0]}$ on $K_{[-1]}$ is irreducible.*

Proof (2) Let M be a nonzero submodule of $L_{[-1]}$ and

$$0\neq a=\sum_{j=1}^{n}(f_j(z)x_j+g_j(z)y_j)\in M.$$

Suppose $f_i(z)\neq 0$ for some i. Then $b=[a, x_iy_i]=f_i(z)x_i-g_i(z)y_i\in M$ and $c=[b, x_iy_i]=f_i(z)x_i+g_i(z)y_i\in M$, so $f_i(z)x_i=\frac{1}{2}(b+c)\in M$. Choose $x_ih(z)\in M$ such that $h(z)\neq 0$ and $h(z)=\sum_r \alpha_r z^r$ has minimal number of terms. If $h(z)$ has more than one terms, set $h(z)=\alpha_1 z^{u_1}+\alpha_2 z^{u_2}+\cdots$, $u_1\neq u_2$, α_1, $\alpha_2\neq 0$. We have

$$d=[x_0, x_ih(z)]=\alpha_1(\nu_i-u_1)z^{u_1}+(\nu_i-u_2)z^{u_2}+\cdots\in M.$$

Then $0\neq(\nu-u_1)x_ih(z)-d=\alpha_2(u_2-u_1)x_iz^{u_2}+\cdots\in M$ which has less terms than $x_ih(z)$, it is a contradiction. Thus M contains an element. x_iz^u. and $x_iz^v=[x_iz^u, x_iy_iz^{v-u}]\in M$, $\forall v\in G$. We have $[x_iz^v, x_jy_i]=x_jz^v\in M$ and $[x_iz^v, y_iy_j]=(1+\delta_{ij})y_jz^v\in M$, $j=1,\cdots, n$, $v\in G$, and $M=L_{-1}$.

(1) Let $\overline{K}$ be any subalgebra of X that properly contains K_0. Then

$$\overline{K}\ni b=f_0(z)+\sum_{i=1}^{n}(f_i(z)x_i+g_i(z)y_i)\neq 0.$$

If $f_0(z)=0$, then $b\in L_{[-1]}$. Let $f_0(z)\neq 0$, (i) If $f_i(z)\neq 0$(or $g_i(z)\neq 0$) for some $i>0$, then (ii) If $f_i(z)=g_i(z)=0$, $i=1,\cdots,n$, then $b=f_0(z)$ and $\overline{K}\ni[[b, x_iy_i]y_i']=f_i(z)\in\overline{K}$. $[x_0x_i, b]=x_i(\partial_0'f_0(z))\neq 0$(since $b\neq 0$, $f_0(z)\neq\alpha z$). In all cases, $\overline{K}\cap L_{[-1]}\neq\{0\}$. By the irreducibility of $L_{[-1]}$, we have $L_{[-1]}\subseteq\overline{K}$. Then $L_{[-2]}=[L_{[-1]}, L_{[-1]}]\subseteq\overline{K}$ and $\overline{K}=X$.

Remark 6.1. Σ^* and $\tilde{\Sigma}$ furnishes examples of simple graded Lie algebras which are neither classical nor the "graded Lie algebras of cartan type" of Kostrikin and Šafarevič[9].

Note 6.1. In the construction of Σ. other choices of D_i and D_i' are possible. For instance, we can take D_i and D_i' to be of the same type as D_0'(see(1.7)). Simple Lie algebras are then obtained, and results similar to that of § 2—§ 4 also hold. It is not clear if they are isomorphic to Σ^*, $\tilde{\Sigma}$ and $\overline{\Sigma}$ discussed above.

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