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A FUNDAMENTAL INEQUALITY AND ITS APPLICATION

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Abstract

Let $f(z)$ be meromorphic in $|z|<R(0<R\leqslant\infty)$ and k, τ be two positive integers such that $\tau>k+4+\left[\frac{2}{k}\right]$. In this note, a fundamental inequality is established such that the characreristic function $T(r, f)$ can be limited by $N\left(r, \frac{1}{f}\right)$ and $\overline{N}_{\tau-1)}\left(r, \frac{1}{f^{(k)}-1}\right)$. As an application, the following criterion for normality is also proved: Let $\mathscr{F}$ be a family of meromorphic functions in a region D. If for every $f(z)\in\mathscr{F}$, $f(z)\neq 0$ and all the zeros of $f^{(k)}(z)-1$ are of multiplicity $>k+4+\left[\frac{2}{k}\right]$ in D, then $\mathscr{F}$ is normal there.

§ 1. Introduction

In the fundamental inequalities[4], limiting the characteristic function usually needs three counting functions, in which one or two can be concerned with their derivatives. From these inequalities, one can eliminate so called "initial values" and establish the corresponding criterion[1] for normality.

In 1959, W. K. Hayman[2,3] obtained an interesting result. He used only two counting functions for limiting the characteristic function.

(**A**) If $f(z)$ is meromorphic and transcendent al in $|z|<R(0<R\leqslant\infty)$ and k is a positive integer, then

$$T(r, f)<\left(2+\frac{1}{k}\right)N\left(r, \frac{1}{f}\right)+\left(2+\frac{2}{k}\right)\overline{N}\left(r, \frac{1}{f^{(k)}-1}\right)+S(r, f). \tag{1.1}$$

Recently Ku Yunghsing[5] successed to prove a criterion for normality which corresponds to the proposition (A).

(**B**) Let $\mathscr{F}$ be a family of meromorphic functions in the region D and let k be a positive integer. If, for every $f(z)$ of $\mathscr{F}$, $f(z)\neq 0$ and $f^{(k)}(z)\neq 1$ in D, then $\mathscr{F}$ is normal.

In this note, we will establish a fundamental inequality (Theorem 1) and the corresponding criterion (Theorem 2). Both theorems improve the propositions (A) and (B).

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§ 2. Preliminary Lemmas

Lemma 1. *Suppose that $f(z)$ is meromorphic in $|z|<R(0<R\leqslant\infty)$. If $f(0)\neq 0$, ∞; $f^{(k)}(0)\neq 1$ and $f^{(k+1)}(0)\neq 0$, then we have*

$$T(r, f)<\overline{N}(r, f)+N\left(r, \frac{1}{f}\right)+\overline{N}\left(r, \frac{1}{f^{(k)}-1}\right)$$
$$-N_0\left(r, \frac{1}{f^{(k+1)}}\right)+S(r, f) \tag{2.1}$$

for $0<r<R$, where

$$S(r, f)=m\left(r, \frac{f^{(k)}}{f}\right)+m\left(r, \frac{f^{(k+1)}}{f}\right)+m\left(r, \frac{f^{(k+1)}}{f^{(k)}-1}\right)$$
$$+\log\left|\frac{f(0)(f^{(k)}(0)-1)}{f^{(k+1)}(0)}\right|+\log 2 \tag{2.2}$$

and in $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ only zeros of $f^{(k+1)}(z)$, which are not zeros of $f^{(k)}(z)-1$, are to be considered.

Lemma 1[8] is substantially due to H. Milloux(See e. g. [2; 3 p.57]), but there is some difference in the terms $S(r, f)$.

Before proving Lemma 2, we recall some notations[6, 7].

Suppose that $F(z)$ is meromorphic in $|z|<R$ $(0<R\leqslant\infty)$ and that τ is an integer greater than 1. Let

$N_{\tau-1)}(r, F)$ be the counting function of polesof multiplicity $\leqslant\tau-1$ of $F(z)$ in $|z|\leqslant r$, each pole counted with its multiplicity;

$\overline{N}_{\tau-1)}(r, F)$ be the counting function of distinct poles of multiplicity $\leqslant\tau-1$ of $F(z)$ in $|z|\leqslant r$.

Setting

$$N_{(\tau}(r, F)=N(r, F)-N_{\tau-1)}(r, F),$$
$$\overline{N}_{(\tau}(r, F)=\overline{N}(r, F)-\overline{N}_{\tau-1)}(r, F),$$

it is clear that

$$\overline{N}_{1)}(r, F)=N_{1)}(r, F)$$

and

$$\overline{N}_{(\tau}(r, F)\leqslant\frac{1}{\tau}N_{\tau)}(r, F). \tag{2.3}$$

Lemma 2. *We have with the hypotheses of Lemma 1*

$$\overline{N}(r, f)\leqslant\frac{k+1}{2k+1}T(r, f)+\frac{1}{2k+1}\left\{\overline{N}\left(r, \frac{1}{f^{(k)}-1}\right)+N_0\left(r, \frac{1}{f^{(k+1)}}\right)\right.$$
$$\left.+m\left(r, \frac{g'}{g}\right)+\log\left|\frac{g(0)}{g'(0)}\right|\right\}, \tag{2.4}$$

where

$$g(z)=\frac{(f^{(k+1)}(z))^{k+1}}{(1-f^{(k)}(z))^{k+2}}. \tag{2.5}$$

Proof We divide $\overline{N}(r,f)$ as

$$\overline{N}(r,f)=\frac{k+1}{2k+1}N_{1)}(r,f)+\frac{k}{2k+1}N_{1)}(r,f)+\overline{N}_{(2}(r,f).$$

According to a lemma of Hayman[2; 3, p.60]

$$kN_{1)}(r,f)\leqslant\overline{N}_{(2}(r,f)+\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)+N_0\left(r,\frac{1}{f^{(k-1)}}\right)$$
$$+m\left(r,\frac{g'}{g}\right)+\log\left|\frac{g(0)}{g'(0)}\right|.$$

Thus

$$\overline{N}(r,f)=\frac{k+1}{2k+1}N_{1)}(r,f)+\left(1+\frac{1}{2k+1}\right)N_{(2}(r,f)$$
$$+\frac{1}{2k+1}\left\{\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)+N_0\left(r,\frac{1}{f^{(k+1)}}\right)+m\left(r,\frac{g'}{g}\right)+\log\left|\frac{g(0)}{g'(0)}\right|\right\}.$$

Since

$$\frac{k+1}{2k+1}N_{1)}(r,f)+\left(1+\frac{1}{2k+1}\right)\overline{N}_{(2}(r,f)$$
$$\leqslant\frac{k+1}{2k+1}N_{1)}(r,f)+\frac{k+1}{2k+1}N_{(2}(r,f)\leqslant\frac{k+1}{2k+1}T(r,f),$$

(2.4) follows immediately.

Lemma 3. *Suppose that $f(z)$ satisfies the assumptions of Lemma 1 and that τ is an integer greater than 1. Then we have*

$$\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)\leqslant\frac{k^2+3k+1}{(2k+1)\tau-k}T(r,f)+\frac{(2k+1)(\tau-1)}{(2k+1)\tau-k}\overline{N}_{\tau-1)}\left(r,\frac{1}{f^{(k)}-1}\right)$$
$$+\frac{k}{(2k+1)\tau-k}\left\{N_0\left(r,\frac{1}{f^{(k+1)}}\right)+m\left(r,\frac{g'}{g}\right)+\log\left|\frac{g(0)}{g'(0)}\right|\right\}$$
$$+\frac{2k+1}{(2k+1)\tau-k}\left\{m\left(r,\frac{f^{(k)}}{f}\right)+\log 2+\log\frac{1}{|f^{(k)}(0)-1|}\right\}, \tag{2.6}$$

where $g(z)$ is given by (2.4).

Proof We divide $\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)$ into two parts.

$$\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)=\overline{N}_{\tau-1)}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_{(\tau}\left(r,\frac{1}{f^{(k)}-1}\right)$$
$$\leqslant\left(1-\frac{1}{\tau}\right)\overline{N}_{\tau-1)}\left(r,\frac{1}{f^{(k)}-1}\right)+\frac{1}{\tau}\overline{N}_{\tau-1)}\left(r,\frac{1}{f^{(k)}-1}\right)+\frac{1}{\tau}N_{(\tau}\left(r,\frac{1}{f^{(k)}-1}\right)$$
$$\leqslant\left(1-\frac{1}{\tau}\right)\overline{N}_{\tau-1)}\left(r,\frac{1}{f^{(k)}-1}\right)+\frac{1}{\tau}N\left(r,\frac{1}{f^{(k)}-1}\right)$$
$$\leqslant\left(1-\frac{1}{\tau}\right)\overline{N}_{\tau-1)}\left(r,\frac{1}{f^{(k)}-1}\right)+\frac{1}{\tau}\left\{T(r,f^{(k)})+\log 2+\log\frac{1}{|f^{(k)}(0)-1|}\right\}. \tag{2.7}$$

For the term $T(r,f^{(k)})$, we have

$$T(r,f^{(k)})=m(r,f^{(k)})+N(r,f^{(k)})$$
$$\leqslant\left\{m(r,f)+m\left(r,\frac{f^{(k)}}{f}\right)\right\}+\{N(r,f)+k\overline{N}(r,f)\}$$
$$=T(r,f)+k\overline{N}(r,f)+m\left(r,\frac{f^{(k)}}{f}\right).$$

By (2.4)

$$T(r,f^{(k)})\leqslant\left\{1+\frac{k(k+1)}{2k+1}\right\}T(r,f)+\frac{k}{2k+1}\left\{\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)\right.$$
$$\left.+N_0\left(r,\frac{1}{f^{(k+1)}}\right)+m\left(r,\frac{g'}{g}\right)+\log\left|\frac{g(0)}{g'(0)}\right|\right\}+m\left(r,\frac{f^{(k)}}{f}\right). \quad (2.8)$$

Substituting (2.8) into (2.7), then (2.6) can be derived.

Lemma 4. *We have with the hypotheses of Lemma 3*

$$\overline{N}(r,f)\leqslant\frac{(k+1)\tau+1}{(2k+1)\tau-k}T(r,f)+\frac{\tau-1}{(2k+1)\tau-k}\overline{N}_{\tau-1)}\left(r,\frac{1}{f^{(k)}-1}\right)$$
$$+\frac{\tau}{(2k+1)\tau-k}\left\{N_0\left(r,\frac{1}{f^{(k+1)}}\right)+m\left(r,\frac{g'}{g}\right)+\log\left|\frac{g(0)}{g'(0)}\right|\right\}$$
$$+\frac{1}{(2k+1)\tau-k}\left\{m\left(r,\frac{f^{(k)}}{f}\right)+\log 2+\log\frac{1}{|f^{(k)}(0)-1|}\right\}. \quad (2.9)$$

(2.9) follows by substituing (2.6) into (2.4).

§ 3. A fundamental inequality.

We proceed to prove our main theorem.

Theorem 1. *Let $f(z)$ be meromorphic in $|z|<R$ $(0<R\leqslant\infty)$ and k, τ be two positive integer such that*

$$\tau>k+4+\left[\frac{2}{k}\right]. \quad (3.1)$$

If $f(0)\neq 0,\infty$, $f^{(k)}(0)\neq 1$, $f^{(k+1)}(0)\neq 0$ and

$$(k+1)f^{(k+2)}(0)\,(f^{(k)}(0)-1)-(k+2)f^{(k+1)}(0)^2\neq 0,$$

then

$$\left\{1-\frac{(k+1)(\tau+k+2)}{(2k+1)\tau-k}\right\}T(r,f)<N\left(r,\frac{1}{f}\right)$$
$$+\frac{2(k+1)(\tau-1)}{(2k+1)\tau-k}\overline{N}_{\tau-1)}\left(r,\frac{1}{f^{(k)}-1}\right)+S_1(r,f), \quad (3.2)$$

where

$$S_1(r,f)\leqslant\frac{5}{4}m\left(r,\frac{f^{(k)}}{f}\right)+m\left(r,\frac{f^{(k+1)}}{f}\right)+\frac{1}{2}m\left(r,\frac{f^{(k+2)}}{f^{(k+1)}}\right)+\frac{3}{2}m\left(r,\frac{f^{(k+1)}}{f^{(k)}-1}\right)$$
$$+\log|f(0)|+\frac{2(k+1)(\tau-1)}{(2k+1)\tau-k}\log|f^{(k)}(0)-1|+\frac{2k(\tau-1)}{(2k+1)\tau-k}\log\frac{1}{|f^{(k+1)}(0)|}$$
$$+\frac{1}{2}\log^+\frac{1}{|(k+1)f^{(k+2)}(0)\,(f^{(k)}(0)-1)-(k+2)f^{(k+1)}(0)^2|}$$
$$+2\log 2+\frac{1}{2}\log(k+2). \quad (3.3)$$

Proof Substituting (2.6) and (2.9) in (2.1) we obtain

$$\left\{1-\frac{(k+1)(\tau+k+2)}{(2k+1)\tau-k}\right\}T(r,f)<N\left(r,\frac{1}{f}\right)+\frac{2(k+1)(\tau-1)}{(2k+1)\tau-k}\overline{N}_{\tau-1)}\left(r,\frac{1}{f^{(k)}-1}\right)$$
$$-\left\{1-\frac{k+\tau}{(2k+1)\tau-k}\right\}N_0\left(r,\frac{1}{f^{(k+1)}}\right)+S_2(r,f), \tag{3.4}$$

where

$$S_2(r,f)=m\left(r,\frac{f^{(k)}}{f}\right)+m\left(r,\frac{f^{(k+1)}}{f}\right)+m\left(r,\frac{f^{(k+1)}}{f^{(k)}-1}\right)$$
$$+\log\left|\frac{f(0)(f^{(k)}(0)-1)}{f^{(k+1)}(0)}\right|+\log 2+\frac{k+\tau}{(2k+1)\tau-k}\left\{m\left(r,\frac{g'}{g}\right)+\log\left|\frac{g(0)}{g'(0)}\right|\right\}$$
$$+\frac{2(k+1)}{(2k+1)\tau-k}\left\{m\left(r,\frac{f^{(k)}}{f}\right)+\log 2+\log\frac{1}{|f^{(k)}(0)-1|}\right\}. \tag{3.5}$$

Since

$$\frac{k+\tau}{(2k+1)\tau-k}<\frac{1}{2} \tag{3.6}$$

under (3.1), the term $N_0\left(r,\frac{1}{f^{(k+1)}}\right)$ in (3.4) can be omitted.

By (2.5)

$$\frac{g'}{g}=(\log g)'=(k+2)\left\{\frac{k+1}{k+2}\frac{f^{(k+2)}(z)}{f^{(k+1)}(z)}-\frac{f^{(k+1)}(z)}{f^{(k)}(z)-1}\right\},$$

so that

$$m\left(r,\frac{g'}{g}\right)\leqslant m\left(r,\frac{f^{(k+2)}}{f^{(k+1)}}\right)+m\left(r,\frac{f^{(k+1)}}{f^{(k)}-1}\right)+\log(k+2)+\log 2.$$

Also

$$\log\left|\frac{g(0)}{g'(0)}\right|=\log\left|\frac{f^{(k+1)}(0)\,(f^{(k)}(0)-1)}{(k+1)f^{(k+2)}(0)\,(f^{(k)}(0)-1)-(k+2)f^{(k+1)}(0)^2}\right|.$$

In view of (3.6) and

$$\frac{2(k+1)}{2(k+1)\tau-k}\leqslant\frac{1}{4}$$

under (3.1), we obtain

$$S_2(r,f)\leqslant\frac{5}{4}m\left(r,\frac{f^{(k)}}{f}\right)+m\left(r,\frac{f^{(k+1)}}{f}\right)+\frac{1}{2}m\left(r,\frac{f^{(k+2)}}{f^{(k+1)}}\right)$$
$$+\frac{3}{2}m\left(r,\frac{f^{(k+1)}}{f^{(k)}-1}\right)+\log|f(0)|+\left\{1+\frac{k+\tau}{(2k+1)\tau-k}-\frac{2(k+1)}{(2k+1)\tau-k}\right\}$$
$$\times\log|f^{(k)}(0)-1|+\left\{1-\frac{k+\tau}{(2k+1)\tau-k}\right\}\log\frac{1}{|f^{(k+1)}(0)|}$$
$$+\frac{1}{2}\log^+\left|\frac{1}{(k+1)f^{(k+2)}(0)\,(f^{(k)}(0)-1)-(k+2)f^{(k+1)}(0)^2}\right|$$
$$+\frac{7}{4}\log 2+\frac{1}{2}\log(k+2).$$

Thus the estimate (3.3) can be deduced immediately.

Corollary. *Suppose that $f(z)$ is meromorphic and transcendental in the plane. Then either $f(z)$ assumes every finite value infinitely often or $f^{(k)}(z)$ assumes every finite* *non-zero value with multiplicity* $\leqslant k+4+\left[\frac{2}{k}\right]$ *infinitely often.*

Remark. *If r tends to infinity, then the fundamental inequality* (3.2) *reduces to the Hayman's inequality* (1.1).

§ 4. Application.

Lemma 5. *Suppose that $f(z)$ satisfies the assumptions of Theorem* 1 *and that in addition $f(z)\neq 0$ and all the roots of $f^{(k)}(z)=1$ are of multiplicity $>k+4+\left[\frac{2}{k}\right]$ n $|z|<R$. Then we have*

$$\log M\left(r, \frac{1}{f}\right)<C\frac{R}{R-r}\left(1+B+\log\frac{R}{R-r}\right)$$

for $0<r<R$, *where*

$$B=\log^+ R+\log^+\frac{1}{R}+\log^+|f(0)|+\log^+|f^{(k)}(0)|+\log^+\frac{1}{|f^{(k+1)}(0)|}$$
$$+\log^+\frac{1}{|(k+1)f^{(k+2)}(0)(f^{(k)}(0)-1)-(k+2)f^{(k+1)}(0)^2|}.$$

In fact, we start from Theorem 1 and estimate the terms of $S_1(r, f)$ in (3.3), then the proof of Lemma 5 can be completed as [8].

Lemma 6. *Let k be a positive integer and $f(z)$ be meromorphic in $|z|<1$ If $f(z)\neq 0$ and all the roots of $f^{(k)}(z)=1$ are of multiplicity $>k+4+\left[\frac{2}{k}\right]$ in $|z|<1$, then either $|f(z)|<1$ or $|f(z)|>C$ uniformly in $|z|<\frac{1}{32}$, where C is a positive constant depending only on k.*

Proof The conclusion will hold with $C=1$ unless there is a point z_1 such that

$$|f(z_1)|=1,\ |z_1|<\frac{1}{32}.$$

There are two mutually exclusive cases.

(A) One has

$$\sum_{j=0}^{k+1}|f^{(j)}(z)|\geqslant\frac{1}{4}\ \text{uniformly in}\ |z|<\frac{1}{8}.$$

It follows that

$$m\left(r, z_1, \frac{1}{f}\right)\leqslant\sum_{j=0}^{k+1}m\left(r, z_1, \frac{f^{(j)}}{f}\right)+\log 4(k+2)$$

for $0<r<\frac{3}{32}$. Since[4, p.4]

$$m\left(r, z_1, \frac{f^{(j)}}{f}\right)<C\left\{1+\log\frac{1}{\rho-r}+\log^+T(\rho, z_1, f)\right\}$$
$$=C\left\{1+\log\frac{1}{\rho-r}+\log^+T\left(\rho, z_1, \frac{1}{f}\right)\right\}$$

for $j=1, 2, \cdots, k+1$ and $\frac{1}{32}<r<\rho<\frac{3}{32}$, then

$$T\left(r, z_1, \frac{1}{f}\right) < C\left\{1 + \log\frac{1}{\rho - r} + \log^+ T\left(\rho, z_1, \frac{1}{f}\right)\right\}. \quad \left(\frac{1}{32} < r < \rho < \frac{3}{32}\right).$$

Applying Bureau's lemma[1, 237], it yields

$$\log M\left(\frac{1}{32}, \frac{1}{f}\right) \leqslant \log M\left(\frac{1}{16}, z_1, \frac{1}{f}\right) \leqslant 9T\left(\frac{5}{64}, z_1, \frac{1}{f}\right) < C.$$

(B) There is a point z_2 such that

$$\sum_{j=0}^{k+1} |f^{(j)}(z_2)| < \frac{1}{4}, \quad |z_2| < \frac{1}{8}.$$

According to [8], we can find a point z_0 on the segment $\overline{z_2 z_1}$ such that

$$|f^{(k+2)}(z_0)| \geqslant 1, \ \frac{1}{12} < |f^{(k+1)}(z_0)| < \frac{1}{2}, \quad |f^{(k)}(z_0)| < \frac{1}{2}, \quad |f(z_0)| < \frac{1}{2},$$

so that

$$|(k+1)f^{(k+2)}(z_0)(f^{(k)}(z_0) - 1) - (k+2)f^{(k+1)}(z_0)^2| > \frac{1}{4}.$$

Thus we have

$$\log M\left(\frac{1}{32}, \frac{1}{f}\right) < \log M\left(\frac{1}{2}, z_0, \frac{1}{f}\right) < C$$

from Lemma 5.

Theorem 2. *Let $\mathscr{F}$ be a family of meromorphic functions in the region D and let k be a positive integer. If for every $f(z) \in \mathscr{F}$, $f(z) \neq 0$ and all the roots of $f^{(k)}(z) = 1$ are of multiplicity $> k + 4 + \left[\frac{2}{k}\right]$ in D, then $\mathscr{F}$ is normal.*

Theorem 2 follows at once from Lemma 6. As another application of Lemma 6, we derive the following result on the existence of a singular direction.

Theorem 3. *Let $f(z)$ be meromorphic in the plane. If*

$$\varlimsup_{r\to\infty} \frac{T(r, f)}{(\log r)^3} = \infty,$$

then there exists a number θ_0 such that $0 \leqslant \theta_0 < 2\pi$ and for every positive number ε and every positive integer k, either $f(z)$ assumes every finite value infinitely often or $f^{(k)}(z)$ assumes every finite non-zero value with multiplicity $\leqslant k + 4 + \left[\frac{k}{2}\right]$ infinitely often in the angle $|\arg z - \theta_0| < \varepsilon$. The proof can be completed as [8].

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