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AN EXAMPLE OF THE NON-UNIQUENESS OF THE $L_5(10)$ ASSOCIATION SCHEME

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Abstract

It is known from Bose and Shimamoto [1] that the existence of a Latin square type $L_5(10)$ association scheme with parameters $v=100$, $n_1=45$, $p_{11}^1=20$, and $p_{11}^2=20$ presupposes the existence of a set of three mutually orthogonal Latin squares of order 10. But as yet we have not known whether such a set of orthogonal Latin squares exists.

In this note the author gives an association scheme with the above parameters which is not an $L_5(10)$ association scheme.

Consider $v=s^2$ treatments which may be set forth in a $s\times s$ square scheme. Superpose a set of $i-2$ mutually orthogonal Latin squares of order s (if such a set exists) onto the scheme. We then define two treatments to be first associates if they occur together in the same row or same column of the square scheme, or if they correspond to the same symbol of one of the Latin squares and second associates otherwise. The square scheme so defined may be called an $L_i(s)$ association scheme. By the notation of Bose and Shimamoto[1], the parameters of the $L_i(s)$ association scheme are given by

$$v=s^2,\ n_1=i(s-1),\ p_{11}^1=(s-2)+(i-1)(i-2),\ p_{11}^2=i(i-1). \tag{1}$$

One may ask the question whether the parameters (1) characterize the $L_i(s)$ association scheme. Bruck[2] showed that the answer is in the affirmative if

$$s>\frac{1}{2}(i-1)(i^3-i^2+i+2). \tag{2}$$

an association scheme with parameters (1) may be called a pseudo-$L_i(s)$ association scheme. We can then say that a pseudo-$L_i(s)$ association scheme is an $L_i(s)$ association scheme provided that Bruck's condition (2) holds.

No general results are known for the case when (2) is not satisfied, but some progress has been made for small values of i and s before Bruck. For the case $i=2$, Shrikhande[5] showed that the answer is in the affirmative except when $s=4$. For the case $s=4$, there is exactly one pseudo-$L_2(4)$ association scheme which is not the $L_2(4)$ association scheme. For the case $i=3$, Liu[4] showed also that the answer is in the affirmative except when $5\leqslant s<24$ and gave a pseudo-$L_3(5)$ association scheme

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which is not the $L_3(5)$ association scheme. Xu (P. L. Hsu)[6] then obtained a pseudo-$L_3(6)$ association scheme which is not the $L_3(6)$ association scheme.

When $i=5$ and $s=10$, Bruck's condition (2) is not satisfied. For this case we give a pseudo-$L_5(10)$ association scheme which is not the $L_5(10)$ association scheme.

The association matrix A of the pseudo-$L_5(10)$ association scheme is as follows:

$$\begin{pmatrix}
0 & j' & -j' & j' & -j' & j' & -j' & j' & -j' & j' & -j' & j' \\
j & J-I & -C+I & C-I & -C+I & C-I & -C+I & -C-I & C+I & -C-I & C+I & -C-I \\
-j & -C+I & C & -C-I & -C+I & C-I & C+I & J & -D+I & -D-I & -D+I & -D-I \\
j & C-I & -C-I & C & -C-I & -C+I & C-I & D+I & J & -D+I & -D-I & -D+I \\
-j & -C+I & -C+I & -C-I & C & -C-I & -C+I & D-I & D+I & J & -D+I & -D-I \\
j & C-I & C-I & -C+I & -C-I & C & -C-I & D+I & D-I & D+I & J & -D+I \\
-j & -C+I & C+I & C-I & -C+I & -C-I & C & D-I & D+I & D-I & D+I & J \\
j & -C-I & J & D+I & D-I & D+I & D-I & -C & C-I & C+I & -C-I & -C+I \\
-j & C+I & -D+I & J & D+I & D-I & D+I & C-I & -C & C-I & C+I & -C-I \\
j & -C-I & -D-I & -D+I & J & D+I & D-I & C+I & C-I & -C & C-I & C+I \\
-j & C+I & -D+I & -D-I & -D+I & J & D+I & -C-I & C+I & C-I & -C & C-I \\
j & -C-I & -D-I & -D+I & -D-I & -D+I & J & -C+I & -C-I & C+I & C-I & -C
\end{pmatrix},$$

where j is the all-one vector, J the all-one square matrix, I the unit matrix, of order 9, and

$$C=\begin{pmatrix}
0 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \\
-1 & 0 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
-1 & -1 & 0 & 1 & 1 & -1 & 1 & 1 & -1 \\
-1 & 1 & 1 & 0 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 0 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & 0 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 & 1 & 1 & 0 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & 0 & -1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 0
\end{pmatrix}, \quad D=\begin{pmatrix}
0 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\
-1 & 0 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & -1 & 0 & -1 & 1 & 1 & 1 & -1 & 1 \\
1 & 1 & -1 & 0 & -1 & -1 & 1 & -1 & 1 \\
-1 & 1 & 1 & -1 & 0 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & 0 & -1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & 0 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & 0 & -1 \\
-1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 0
\end{pmatrix}.$$

The association matrix A is obtained from the symmetric Hadamard matrix with constant diagonal of order 100, which has been constructed by Goethals and Seidel [3], by taking complementation with respect to some rows and columns and by replacing O's on the diagonal. Further discussion shall be given in a separate paper.

References

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