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COINDUCED REPRESENTATIONS AND INJECTIVE MODULES FOR HYPERALGEBRA $\boldsymbol{b}_r$

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Abstract

Let G be a simply connected semisimple linear algebraic group over an algebraically closed field of positive characteristic p, B its Borel subgroup, and $\boldsymbol{b}_r$ the r-th standard subalgebra of the hyperalgebra of B. Assume the roots in B to be negative.

Using the coinduced representations, in this paper the author proves:

(1) $J(r, \lambda)=\mathrm{St}_r\otimes((p^r-1)\delta+\lambda)$ is the $\boldsymbol{b}_r$-injective envelope of the one-dimensional $\boldsymbol{b}_r$-module λ, where St_r is the r-th Steinberg module of G, and δ half the sum of the positive roots.

(2) With respect to the natural homomorphism ρ_{rs}: $J(r, \lambda)\hookrightarrow J(s, \lambda)(r\leqslant s)$, $J(\infty, \lambda)=\varinjlim J(r, \lambda)$ is the B-injective envelope of B-module λ.

The above conclusions positively answer two questions posed by J. E. Humphreys at Shanghai in 1980. Moreover, this paper gives a complete description of injective $\boldsymbol{b}_r$-modules.

§ 1. Notation and introductory remarks

Let G be a simply connected semisimple linear algebraic group over an algebraically closed field k of characteristic $p>0$, T a fixed maximal torus of G. Choose an ordering of the root system $\Phi=\Phi(G, T)$, and let $B^-\supset T$ be the Borel subgroup corresponding to the negative roots. It is known that $B^-=U^-T$ (semidirect product), where U^- is the product of the root subgroups associated with the negative roots. The product of the root subgroups associated with the positive roots is denoted by U^+. We consider G as the Chevalley group constructed by reduction mod$p$ from the corresponding complex Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ has a corresponding decomposition $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$, where $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{n}^-$ (resp. $\mathfrak{n}^+$) the direct sum of the root subspaces associated with the negative (resp. positive) roots. Let $\mathscr{U}$ be the universal enveloping algebra of $\mathfrak{g}$. Owing to the Poincaré-Birkhoff-Witt theorem (cf. [8, § 17.3]), $\mathscr{U}=\mathscr{N}^-\otimes_{\mathbf{C}}\mathscr{H}\otimes_{\mathbf{C}}\mathscr{N}^+$, where $\mathscr{N}^-$ (resp. $\mathscr{H}$, $\mathscr{N}^+$) is the enveloping algebra of $\mathfrak{n}^-$(resp. $\mathfrak{h}$, $\mathfrak{n}^+$). Choose a Chevalley basis $\{Y_\alpha, H_i, X_\alpha(\alpha\in\Phi^+, 1\leqslant i\leqslant \mathrm{rank}\, \mathfrak{g})\}$ of $\mathfrak{g}$. The subring $\mathscr{U}_{\mathbf{Z}}$ generated by $X_\alpha^m/m!$ and $Y_\alpha^m/m!$ $(m\in\mathbf{Z}^+)$ is the Kostant $\mathbf{Z}$-form of $\mathscr{U}$ and has a corresponding decomposition $\mathscr{U}_{\mathbf{Z}}=\mathscr{N}^-_{\mathbf{Z}}\otimes_{\mathbf{Z}}\mathscr{H}_{\mathbf{Z}}\otimes\mathscr{N}^+_{\mathbf{Z}}$. Let $\mathscr{U}_k=\mathscr{U}_{\mathbf{Z}}\otimes_{\mathbf{Z}}k$. $\mathscr{N}^\pm_k$ and $\mathscr{H}_k$ are defined similarly; let $\mathscr{B}^-_k=\mathscr{N}^-_k\otimes_k\mathscr{H}_k$. Then $\mathscr{U}_k$(resp.

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$\mathcal{N}_k^{\pm}$, $\mathcal{H}_k$, $\mathcal{B}_k^-$) is the hyperalgebra of G (resp. $U^{\pm}$, T, B^-) and has a standard filtration by its finite-dimensional subalgebras $\boldsymbol{u}_r$ (resp. $\boldsymbol{n}_r^{\pm}$, $\boldsymbol{h}_r$, $\boldsymbol{b}_r^-$), $r\in\mathbf{Z}^+$. The subalgebra $\boldsymbol{n}_r^-$ (resp. $\boldsymbol{h}_r$, $\boldsymbol{n}_r^+$) is generated by $Y_{\alpha,m}$ (resp. $H_{i,m}$, $X_{\alpha,m}$) with $0\leqslant m<p^r$, and $\boldsymbol{b}_r^-=\boldsymbol{n}_r^-\otimes_k\boldsymbol{h}_r$, $\boldsymbol{u}_r=\boldsymbol{b}_r^-\otimes_k\boldsymbol{n}_r^+$, where

$$Y_{\alpha,m}=(Y_\alpha^m/m!)\otimes 1,\ X_{\alpha,m}=(X_\alpha^m/m!)\otimes 1,\ H_{i,m}=\binom{H_i}{m}\otimes 1.$$

For convenience we shall omit the symbol "$-$".

Let X be the weight lattice of Φ, which may be identified with the character group of T or B, or considered as the character group of $\mathcal{H}_k$ or $\mathcal{B}_k$. Therefore, for the above-mentioned groups or algebras (and its subalgebras), the one-dimensional module determined by $\lambda\in X$ is also denoted by λ. Let

$$X_r=\{\lambda\in X\,|\,0\leqslant\langle\lambda,\ \alpha\rangle<p^r,\ \forall\text{ simple root }\alpha\},$$

where $\langle\lambda,\ \alpha\rangle=2(\lambda,\ \alpha)/(\alpha,\ \alpha)$ and $(\ \ ,\ \ )$ is the Euclidean inner product. Let, in addition, δ be half the sum of the positive roots.

Denote by $k[U]$ the affine algebra of U. Using the fact that U is a normal subgroup of B, we define a rational B-module structure on $k[U]$ as follows:

$$(tu.\ f)(v)=f(t^{-1}\ vtu),\ \forall u,\ v\in U,\ t\in T,\ f\in k[U].$$

It is known that B-module $k[U]\otimes\lambda$ is the B-injective envelope of B-module λ (cf. [10, §6]). It was also proved by Cline, Parshall and Scott[3] that

$$k[U]\otimes\lambda=\varinjlim \mathrm{St}_r\otimes((p^r-1)\delta+\lambda),$$

where St_r is the r-th Steinberg module of G. But we do not know if the above direct limit is taken with respect to the natural linear ordering. When giving his lectures at Shanghai Normal University in the spring of 1980, J. E. Humphreys raised two further questions:

(1) Is $\mathrm{St}_r\otimes((p^r-1)\delta+\lambda)$ the $\boldsymbol{b}_r$-injective envelope of $\boldsymbol{b}_r$-module λ? (The case $r=1$ was settled by himself [6].)

(2) Is the above direct limit taken with respect to the natural ordering?

In this paper we shall give the affirmative answers to these questions. In Section 2 we introduce the tool we shall use, the coinduced representations. Section 3 is devoted to a discuss of the first question, and Section 4 to a discuss of the second question.

For convenience, let

$$J(r,\ \lambda)=\mathrm{St}_r\otimes((p^r-1)\delta+\lambda)\text{ and }J(\infty,\ \lambda)=k[U]\otimes\lambda.$$

§ 2. Coinduced representations

In this section, R and S are rings with 1 (not necessarily commutative), and φ: $S\to R$ a ring homomorphism preserving 1. Thus, a left R-module V may be considered as a left S-module viaφ. This S-module, denoted by $\mathrm{Res}_\varphi\ V$, is called the restriction of V to S. When no confusion arises, the symbol Res_φ is often omitted. The functor Res_φ from the category of left R-modules to the category of left S-modules has a left adjoint, which is called the induced functor. This functor is familiar to us: for a left S-module W, $\mathrm{Ind}_\varphi\ W = R \otimes_S W$, where R is considered as a right S-module via φ and the action of R on $\mathrm{Ind}_\varphi\ W$ is the multiplication on the left factor.

From the category of left S-modules to the category of left R-modules, we can define another functor, called the coinduced functor and denoted by Coind_φ, as follows:

$$\mathrm{Coind}_\varphi\ W = \mathrm{Hom}_S\ (R,\ W),$$

where R is considered as a left S-module via φ, and the action of R on $\mathrm{Coind}_\varphi\ W$ is

$$(x.\ f)(y) = f(yx),\ \forall x,\ y \in R,\ f \in \mathrm{Coind}_\varphi\ W.$$

Of course, we have to verify that $x.\ f \in \mathrm{Coind}_\varphi\ W$ and that Coind_φ indeed becomes a functor. This is straightforward.

We have the following

Proposition 2.1. Coind_φ *is the right adjoint of* Res_φ,

This is equivalent to the following

Proposistion 2.2 (reciprocity). *For any left R-module V and left S-module W, there exists a natural isomorphism*

$$\mathrm{Hom}_R\ (V,\ \mathrm{Coind}_\varphi W) \cong \mathrm{Hom}_S(\mathrm{Res}_\varphi\ V,\ W).$$

In order to prove the above propositions, it is enough to prove the following

Proposition 2.3. *The evaluation mapping Ev: $\mathrm{Coind}_\varphi W \to W$, which sends $f \in \mathrm{Coind}_\varphi\ W$ to $f(1)$, is a natural S-module homomorphism and has the following universal property: for any R-module V and any S-module homomorphism θ: $V \to W$, there exists a unique R-module homomorphism $\tilde{\theta}$: $V \to \mathrm{Coind}_\varphi\ W$, making the following diagram commutative*:

$$\begin{tikzcd} V \arrow[r, "\tilde{\theta}"] \arrow[dr, "\theta"'] & \mathrm{Coind}_\varphi\ W \arrow[d, "Ev"] \\ & W \end{tikzcd}$$

Proof (i) At first we prove that Ev is a natural S-module homomorphism. Let $f \in \mathrm{Coind}_\varphi W$, $s \in S$, then

$$Ev(\varphi(s) \cdot f) = (\varphi(s) \cdot f)(1) = f(\varphi(s)) = s \cdot f(1) = s \cdot Ev(f).$$

And, for an S-module homomorphism σ: $W \to W'$, the diagram

$$\begin{tikzcd} \mathrm{Coind}_\varphi\ W \arrow[r, "\sigma_*"] \arrow[d, "Ev"] & \mathrm{Coind}_\varphi\ W' \arrow[d, "Ev"] \\ W \arrow[r, "\sigma"] & W' \end{tikzcd}$$

is commutative, because for $f \in \mathrm{Coind}_\varphi\ W$ we have

$$\sigma \circ Ev(f) = \sigma(f(1)),$$

$$Ev\circ\sigma_*(f)=\sigma_*(f)(1)=(\sigma\circ f)(1)=\sigma(f(1)).$$

(ii) The uniqueness of $\tilde{\theta}$. Let $v\in V$, then, for any $x\in R$, we have

$$\tilde{\theta}(v)(x)=(x\cdot\tilde{\theta}(v))(1)=\tilde{\theta}(x\cdot v)(1)=(Ev\circ\tilde{\theta})(x\cdot v)=\theta(x\cdot v).$$

Hencт, $\tilde{\theta}(v)$ is completely determined by θ and v.

(iii) The existence of $\tilde{\theta}$. We Claim that the function $\tilde{\theta}(v)$ defined by $\tilde{\theta}(v)(x)=\theta(x\cdot v)$ belongs to $\mathrm{Coind}_\varphi W$, and that $\tilde{\theta}$ is the required R-module homomorphism.

First, for $s\in S$, $x\in R$, we have

$$\tilde{\theta}(v)(\varphi(s)x)=\theta(\varphi(s)x\cdot v)=s\cdot\theta(x\cdot v)=s\cdot\tilde{\theta}(v)(x),$$

so that $\tilde{\theta}(v)\in\mathrm{Coind}_\varphi$ W. Next, if x, $y\in R$, $v\in V$, then

$$\tilde{\theta}(y\cdot v)(x)=\theta(xy\cdot v)=\tilde{\theta}(v)(xy)=(y\cdot\tilde{\theta}(v))(x),$$

so that $\tilde{\theta}$ is an R-module homomorphism. Finally,

$$Ev\circ\tilde{\theta}(v)=\tilde{\theta}(v)(1)=\theta(v).$$

Corollary 2.4. *If I is an injective S-module, then* $\mathrm{Coind}_\varphi I$ *is an injective R-module.*

Corollary 2.5. *If R_1, R_2, R_3 are rings with* 1 *and ψ: $R_1\to R_2$ and φ: $R_2\to R_3$ are ring homomorphisms preserving* 1, *then* $\mathrm{Coind}_\varphi\mathrm{Coind}_\psi=\mathrm{Coind}_{\varphi\psi}$.

Proof Coind_φ Coind_ψ is the right adjoint of Res_ψ $\mathrm{Res}_\varphi=\mathrm{Res}_{\varphi\psi}$ (cf. [5, II, Prop. 7.1]).

From the definition we immediately deduce the following

Lemma 2.6. Coind_φ *is a left exact functor; it is exact iff R is projective as an S-module.*

Remark 2.7. If R and S are algebras over a field with R symmetric (and finite-dimensional), it is easy to see that Coind_φ and Ind_φ are equivalent. In particular, a homomorphism between group algebras of finite groups is the case. It is this fact that allows us to use two definitions of induced representations simultaneously in the representation theory of finite groups.

Remarks 2.8. The rationally induced representations of algebraic groups are indeed coinduced representations. They have the properties $(2.1)\sim(2.5)$ and (2.6) may be modified as follows: a ration-ally induced functor is left exact; it is exact iff the quotient variety is affine (see [2, 3] or [10, § 5] for details).

§ 3. Injective modules for hyperalgebra $\boldsymbol{b}_r$

Now we apply the results of Section 2 to the embedding homomorphism j_r: $\boldsymbol{h}_r\hookrightarrow\boldsymbol{b}_r$, and denote Coind_{j_r} simply by Coind_r. In this section, all modules under consideration is finite-dimensional left modules. Noting that $\boldsymbol{b}_r$ is a free $\boldsymbol{h}_r$-module and that a k basis of $\boldsymbol{n}_r$ is a free $\boldsymbol{h}_r$-basis of $\boldsymbol{b}_r$, we get the following

Lemma 3.1. Coind_r *is an exact functor. Let W be an $\boldsymbol{h}_r$-module. then* dim Coind_r W

$=p^{rN}\dim W$, *where* $N=\mathrm{Card}\,\Phi^+$.

Proof $\mathrm{Hom}_{h_r}(\boldsymbol{b}_r, W)\cong\mathrm{Hom}_k(\boldsymbol{n}_r, W)\cong\boldsymbol{n}_r^*\otimes_k W$, and $\dim \boldsymbol{n}_r=p^{rN}$.

Lemma 3.2. *$\mathrm{Coind}_r\,W$ is an injective $\boldsymbol{b}_r$-module. moreover, if $\lambda\in X$, then $\mathrm{Coind}_r\lambda$ is the $\boldsymbol{b}_r$-injective envelope of $\boldsymbol{b}_r$-module λ.*

Proof Since $\boldsymbol{h}_r$-modules are completely reducible, the first conclusion follows from (2.4). To prove the second conclusion, it is enough to note that for any $\mu\in X$

$$\dim \mathrm{Hom}_{b_r}(\mu, \mathrm{Coind}_r\,\lambda)=\dim \mathrm{Hom}_{h_r}(\mu, \lambda)$$

$$=\begin{cases}1, & \text{if } \mu\equiv\lambda(\mathrm{mod}\; p^rX),\\ 0, & \text{otherwise.}\end{cases}$$

Lemma 3.3. *As a $\boldsymbol{b}_r$-module,* St_r *has a unique irreducible submodule* $-(p^r-1)\delta$.

Proof (following [10, §4]) We prove a more general conclusion: if $\lambda\in X_r$, then, as a $\boldsymbol{b}_r$-module, the irreducible G-module $M(\lambda)$ with highest weight λ has a unique irreducible submodule $w_0\lambda$, where w_0 is the longest element of the Weyl group.

Let v^- be a minimal vector of G-module $M(\lambda)$, then v^- has weight $w_0\lambda$. The line kv^- is certainly an irreducible $\boldsymbol{b}_r$-module. Let N be the $\boldsymbol{b}_r$-socle of $M(\lambda)$, then $kv^-\subseteq N$. Since the adjoint action of T stabilizes $\boldsymbol{n}_r$ and $\boldsymbol{h}_r$, N is a T-submodule, hence a direct sum of its T-weight subspaces. If $N\neq kv^-$, we choose a T-weight vector v of N such that the weight μ of v is minimal among the T-weights of N distinct from $w_0\lambda$. The commutative formulas (cf. [10, §4]) tell us that all $Y_{\alpha,m}(m\in\mathbf{Z}^+)$ preserve N stable. In particular, $Y_{\alpha,m}\cdot v\in N$. But $Y_{\alpha,m}\cdot v$ is a T-weight vector of weight $\mu-m\alpha<\mu$. The choice of v forces $Y_{\alpha,m}\cdot v=0$ or $\mu-m\alpha=w_0\lambda$. For a simple root α, the second case leads $m<p^r$ (because $\mu=w_0\lambda+m\alpha$ is a weight of $M(\lambda)$, and all such weights of $M(\lambda)$ are in the α-string through $w_0\lambda$: $w_0\lambda$, $w_0\lambda+\alpha$, $\cdots$, $w_0\lambda-\langle w_0\lambda, \alpha\rangle\alpha$; $\lambda\in X_r$ forces $-\langle w_0\lambda, \alpha\rangle\leqslant p^r-1$). Therefore, $Y_{\alpha,m}\in\boldsymbol{n}_r$, and we get $Y_{\alpha,m}\cdot v=0$ again. Hence, for a simple root α, we always have $Y_{\alpha,m}\cdot v=0$. It follows that v is a minimal vector of G-module $M(\lambda)$. This is a contradiction.

Theorem 3.4. *As $\boldsymbol{b}_r$-modules, $J(r, \lambda)$ is the injective envelope of λ.*

Proof Since λ appears as a T-weight of $J(r, \lambda)$ exactly once, we can (essentially uniquely) define a T-module homomorphism θ: $J(r, \lambda)\to\lambda$. θ is injective when restricted to the $\boldsymbol{b}_r$-socle of $J(r, \lambda)$. Now the commutative diagram

$$\begin{tikzcd}
J(r, \lambda) \arrow[r, "\tilde{\theta}"] \arrow[dr, "\theta"'] & \mathrm{Coind}_r\,\lambda \arrow[d, "Ev"] \\
 & \lambda
\end{tikzcd}$$

gives a $\boldsymbol{b}_r$-module homomorphism $\tilde{\theta}$: $J(r, \lambda)\to\mathrm{Coind}_r\,\lambda$. The restriction of $\tilde{\theta}$ to the $\boldsymbol{b}_r$-socle of $J(r, \lambda)$ must be injective, hence it is injective itself. Now the dimension comparison

$$\dim J(r, \lambda) = \dim \mathrm{St}_r = p^{rN},$$
$$\dim \mathrm{Coind}_r \lambda = p^{rN} \text{ (by (3.1))}$$

ensures that $\tilde{\theta}$ is an isomorphism.

Note that if we identify $J(r, \lambda)$ with $\mathrm{Coind}_r \lambda$ via $\tilde{\theta}$, the evaluation mapping is just θ, which is a T-module homomorphism.

Corollary 3.5. *Every injective $\boldsymbol{b}_r$-module is a direct sum of certain $J(r, \lambda)$'s. Hence, it has a natural B-module structure and a natural $\boldsymbol{b}_s$-module structure for all $s>r$. $\{J(r, \lambda), \lambda\in X_r\}$ is a complete representative system of isomorphic classes of indecomposable injective $\boldsymbol{b}_r$-modules. Moreover, if $\lambda=\mu_0+p\mu_1+\cdots (\mu_i\in X_1)$ is the p-adic expression of $\lambda\in X$, and $\lambda_r=\mu_0+p\mu_1+\cdots+p^{r-1}\mu_{r-1}$, then there are $\boldsymbol{b}_r$-module isomorphisms*

$$J(r, \lambda)\cong J(r, \lambda_r)\cong J(1, \mu_0)\otimes J(1, \mu_1)^{(p)}\otimes\cdots\otimes J(1, \mu_{r-1})^{(p^{r-1})}.$$

Proof The only non-trivial fact is the last isomorphism. However, the definition of $J(r, \lambda)$ and the Steinberg's tensor product theorem (cf. [10, §3]) give the following B-module isomorphisms

$$\begin{aligned} J(r, \lambda_r) &= \mathrm{St}_r\otimes((p^r-1)\delta+\lambda_r) \\ &\cong \mathrm{St}_1\otimes \mathrm{St}_1^{(p)}\otimes\cdots\otimes \mathrm{St}_1^{(p^{r-1})}\otimes((p^r-1)\delta+\lambda_r) \\ &\cong (\mathrm{St}_1\otimes((p-1)\delta+\mu_0))\otimes(\mathrm{St}_1\otimes((p-1)\delta+\mu_1))^{(p)}\otimes\cdots \\ &\quad\cdots\otimes(\mathrm{St}_1\otimes((p-1)\delta+\mu_{r-1}))^{(p^{r-1})} \\ &\cong J(1, \mu_0)\otimes J(1, \mu_1)^{(p)}\otimes\cdots\otimes J(1, \mu_{r-1})^{(p^{r-1})}. \end{aligned}$$

Remark 3.6. Although $\boldsymbol{b}_r$ is a Frobenius algebra (because it is a finite-dimensional Hopf algebra), and its injective modules are just projective modules, but we think that it is more convenient and natural for our questions to discuss injective modules direc-tly applying the tool introduced in Section 2. And we believe that the tool is also usefull for the discussion of injective modules over a ring.

§ 4. Rationally injective B-modules

It is known that $J(\infty, \lambda)$, which is the B-injective envelope of B-module λ, is (co) induced from T-module λ. That is

$$J(\infty, \lambda) = \{\text{regular function } f\colon B\to k \mid f(tx)=\lambda(t)f(x), \; \forall t\in T,\ x\in B\}.$$

Theorem 4.1. *Let $s, r\in\mathbf{Z}^+\cup\{\infty\}$ with $s\leqslant r$, then*

$$\dim \mathrm{Hom}_B(J(s, \lambda), J(r, \lambda))=1,$$

and the non-zero homomorphism $\rho_{sr}\colon J(s, \lambda)\to J(r, \lambda)$ is injective.

Proof (i) The existence of an injective B-module homomorphism. Assume for a moment that $s<r<\infty$, $r-s=t$. Owing to the Steinberg's tensor product theorem, we have a G-module isomorphism $\mathrm{St}_r \cong \mathrm{St}_s \otimes \mathrm{St}_t^{(p^s)}$, and hence we have B-module isomorphisms

$$\begin{aligned} J(r, \lambda) &= \mathrm{St}_r \otimes ((p^r-1)\delta+\lambda) \\ &\cong \mathrm{St}_s \otimes ((p^s-1)\delta+\lambda) \otimes \mathrm{St}_t^{(p^s)} \otimes (p^t-1)p^s\delta \\ &\cong J(s, \lambda) \otimes J(t, 0)^{(p^s)}. \end{aligned}$$

Noting that $J(t, 0)$ has a trivial one-dimensional B-submodule, we see that $J(s, \lambda)$ is isomorphic with a B-submodule of $J(r, \lambda)$.

Now let $s<r=\infty$. Since $J(\infty, \lambda)$ is the B-injective envelope of B-module λ, while $J(s, \lambda)$ has B-socle λ, we get an injective B-module homomorphism $\rho_s=\rho_{s\infty}$: $J(s, \lambda) \hookrightarrow J(\infty, \lambda)$.

(ii) The uniqueness of non-zero B-module homomorphism. Let $\tilde{\tau}$: $J(s, \lambda) \to J(r, \lambda)$ be a non-zero B-module homomorphism. Since each of these modules has B-socle λ and λ appears as its weights only once, the restriction of λ to the socle must be injective. Therefore, T-module homomorphism $\mathrm{Ev}\circ\tilde{\tau}=\tau$ is non-zero. However, up to scalar factors, there exists a unique non-zero T-module homomorphism from $J(s, \lambda)$ to λ. Now $\tilde{\tau}$ is the b_r-(or B-, for $r=\infty$) homomorphism making the following diagram commutative:

$$\begin{tikzcd} J(s, \lambda) \arrow[r, "\tilde{\tau}"] \arrow[dr, "\tau"'] & J(r, \lambda) \arrow[d, "Ev"] \\ & \lambda \end{tikzcd}$$

When τ is given, $\tilde{\tau}$ is unique.

Remark 4.2. For the uniqueness, it is enough to give a proof for $r=\infty$. However, this does not simplify the argument.

Corollary 4.3. *For the linearly ordered system* $\{J(r, \lambda), \rho_{sr}(s, r\in \mathbf{Z}^+$ *and* $s\leqslant r)\}$, *we have*

$$J(\infty, \lambda)=\varinjlim J(r, \lambda).$$

Proof It remains to prove only that $J(\infty, \lambda)=\bigcup_r \mathrm{Im}\,\rho_r$. This can be done as follows: for any $\mu\in X$, there exists a large r such that the weight appears in $J(r, \lambda)$ as many times as in $J(\infty, \lambda)$. For the details see, for example, [3]or [10, §6].

Remark 4.4. (3.5) and (4.3) has the same form as the correspon-ding result about $\mathfrak{u}_r$ and G. The latter, of course, is much more complicated and difficult, and so far it has been settled onld for $p\geqslant 2h-2$, h being the Coxeter number. This is known as the Humphreys-Ballard-Jantzen's theorem(cf. [1, 11] or[12]).

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