Chin. Ann. of Math.
4B (3) 1983

THE COMPARISON OF ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF EQUIVALUED BOUNDARY VALUE PROBLEMS FOR NONLINEAR AND LINEAR ELLIPTIC EQUATIONS

CHEN SHUXING (陈恕行)
(*Institute of Mathematics, Fudan University*)

Abstract

This paper shows the different asymptotic behavior of the solution of equivalued boundary value problems for nonlinear equation from the solution to linear one, while the boundary, on which the equivalued boundary value is carried, shrinks to afixed point.

§ 1. Introduction

In this paper we study the behavior, as Γ_2 shrinks to a point, of solutions of the following problems:

$$\begin{cases} \Delta u = |u|^{p-1}u, \ x \in \Omega, \\ u|_{\Gamma_1} = 0, \\ u|_{\Gamma_2} = \text{const.}, \\ \displaystyle\int_{\Gamma_2} \frac{\partial u}{\partial n} ds = 1, \end{cases} \tag{1}$$

$$\begin{cases} \Delta u = 0, \ x \in \Omega, \\ u|_{\Gamma_1} = 0, \\ u|_{\Gamma_2} = \text{const.}, \\ \displaystyle\int_{\Gamma_2} \frac{\partial u}{\partial n} ds = 1, \end{cases} \tag{2}$$

where $\Omega = \Omega_1 \backslash \Omega_2$, Ω_1 and Ω_2 are open bounded sets in $\mathbf{R}^n$ with smooth boundaries Γ_1 and $\bar{\Omega}_2$, $\Omega_2 \subset \Omega_1$, and $\frac{\partial}{\partial n}$ denotes the differential operator in the direction of the exterior normal to the boundary of Ω, $p \geqslant 1$. Since the (constant) value of u on Γ_2 is unknown, it will be determined together with the solution u. The value of $\int_{\Gamma_2} \frac{\partial u}{\partial n} ds$ can be any other positive number, but it will not cause more trouble.

Our main results show that, as Ω_2 shrinks to a point $x_0 \in \Omega_1$, (i) the solution of (1) tends to zero if $p \geqslant \frac{n}{n-2}$, but this solntion tends to a non-zero limit if $\frac{n+1}{n-1} \leqslant p$

Manuscript received Nevemher 17, 1981. Revised Appil 30, 1982.

$<\frac{n}{n-2}$, (ii) the solution of (2) approaches the solution of

$$\begin{aligned} \Delta u &= \delta(x_0) \quad \text{in } \Omega_1, \\ u &= 0 \quad \text{on } \Gamma_1. \end{aligned} \tag{3}$$

The section 2 is devoted to prove the existence and uniqueness of solution to (1) and so-called compact-uniformly boundedness of solutions to (1) or (2), while Γ_2 shrinks to a point. In the section 3 we state the main results and their proofs.

I would like to thank H. Brézis who introduced me to this subject, and I would like to thank also L. Veron and R. Rostamian for valuable discussions and suggestions.

§ 2. Existence and compact-uniformly boundedness

The existence and uniqueness of the solution of (2) are well known (see e. g. [1]), now we prove the corresponding fact for (1). We consider an auxiliary Dirichlet problem as follows.

$$\begin{cases} \Delta u = |u|^{p-1}u, \ x \in \Omega, \\ u|_{\Gamma_1} = 0, \\ u|_{\Gamma_2} = g. \end{cases} \tag{4}$$

The solution of (4) uniquely exists (see e. g. [2]), and following monotonicity properties hold.

Lemma 1. *If $g \geqslant 0$, then the solution of (4) is non-negative, furthermore, if $g_2 \geqslant g_1 \geqslant 0$, then for the corresponding solutions, we have $u_2 \geqslant u_1 \geqslant 0$ and*

$$\int_{\Gamma_2} \frac{\partial u_2}{\partial n} ds \geqslant \int_{\Gamma_2} \frac{\partial u_1}{\partial n} ds \geqslant 0.$$

Proof If $u<0$ at any point in Ω, then u must attain a negative minimum at some point in Ω, so at this point $\Delta u \geqslant 0$, $|u|^{p-1}u<0$. This is absurd.

In the case $g_2 \geqslant g_1 \geqslant 0$, we have $u_1 \geqslant 0$, $u_2 \geqslant 0$. Obviously, $u_2 - u_1|_{\Gamma_1} = 0$, $u_2 - u_1|_{\Gamma_2} \geqslant 0$, so if $u_2 - u_1 < 0$ at any point in Ω, $u_2 - u_1$ must attain a negative minimum at some point in Ω, so that $\Delta(u_2 - u_1) \geqslant 0$ and $|u_2|^{p-1}u_2 - |u_1|^{p-1}u_1 = u_2^p - u_1^p < 0$. This is absurd again.

Furthermore, when $g_2 \geqslant g_1 \geqslant 0$, we know $\frac{\partial u_2}{\partial n} \leqslant \frac{\partial u_1}{\partial n} < 0$ on Γ_1, since $u_2 \geqslant u_1 \geqslant 0$. Then from

$$\int_{\Gamma_2} \frac{\partial u_i}{\partial n} ds + \int_{\Gamma_1} \frac{\partial u_i}{\partial n} ds = \int_{\Omega} u_i^p dx \quad (i=1,\ 2),$$

we obtain

$$\int_{\Gamma_2} \frac{\partial u_2}{\partial n} ds \geqslant \int_{\Gamma_2} \frac{\partial u_1}{\partial n} ds \geqslant 0.$$

Lemma 2. *If u is the solution of (4) with $g \geqslant 0$, v is a harmonic function with* $v|_{\Gamma_1} \geqslant 0$ and $v|_{\Gamma_2} = h$, *then $h \geqslant g$ implies $v \geqslant u$.*

Proof $w = v - u$ satisfies

$$\begin{cases} \Delta w = -|u|^{p-1}u, \ x \in \Omega, \\ w|_{\Gamma_1} \geqslant 0, \\ w|_{\Gamma_2} = h - g \geqslant 0. \end{cases} \tag{5}$$

Therefore, if $w<0$ holds at a point in Ω, then w would attain a negative minimum in Ω. But this will lead to contradiction: $\Delta w \geqslant 0$, $-|u|^{p-1}u<0$.

Proposition 1. *There exists a unique solution of problem* (1).

Proof We set $u|_{\Gamma_2}=c$ and consider c as a parameter, and let c vary from 0 to infinity. Denote the solution of (4) by u^c, then $\int_{\Gamma_2} \frac{\partial u^c}{\partial n} ds$ varies monotonously. Obviously, when $c=0$, we have $u^c=0$, then $\int_{\Gamma_2} \frac{\partial u^c}{\partial n} ds=0$. Now we will show $\int_{\Gamma_2} \frac{\partial u^c}{\partial n} ds$ tends to infinity, when $c \to \infty$.

Since Γ_2 is smooth, we can introduce the distance d to Γ_2 as new variable in the neighborhood of Γ_2. Denote the isometric surface with distance d by Γ_{2d} and the domain between Γ_2 and Γ_{2d} by Ω_{2d}, then we have

$$\int_{\Gamma_2} u^c\, ds - \int_{\Gamma_{2d}} u^c\, ds = \int_0^d \int_{\Gamma_{2t}} \frac{\partial u^c}{\partial n}\, ds\, dt. \tag{6}$$

Here $\int_{\Gamma_{2t}} \frac{\partial u^c}{\partial n} ds$ is positive for each t, and if $t_2>t_1$, we have $\int_{\Gamma_{2t_1}} \frac{\partial u^c}{\partial n} ds > \int_{\Gamma_{2t_2}} \frac{\partial u^c}{\partial n} ds$. If $\int_{\Gamma_2} \frac{\partial u^c}{\partial n} ds$ is bounded, then the right side of (6) will be bounded, so from $\int_{\Gamma_2} u^c\, ds \to \infty$ we will have $\int_{\Gamma_{2d}} u^c\, ds \to \infty$ and

$$\int_{\Omega_{2d}} (u^c)^p\, dx \geqslant k \int_{\Omega_{2d}} u^c\, dx = k \int_0^d \int_{\Gamma_{2t}} u^c\, ds\, dt \to \infty,$$

but

$$\int_{\Gamma_2} \frac{\partial u^c}{\partial n} ds \geqslant \int_\Omega \Delta u^c\, dx = \int_\Omega (u^c)^p\, dx \geqslant \int_{\Omega_{2d}} (u^c)^p\, dx,$$

this indicates that $\int_{\Gamma_2} \frac{\partial u^c}{\partial n} ds$ cannot be bounded.

Combining this fact with $u^c \geqslant 0$ and the monotonicity of $\int_{\Gamma_2} \frac{\partial u^c}{\partial n} ds$, we know there exists at least one constant c_0 such that the solution to (4) with $g=c_0$ satisfies

$$\int_{\Gamma_2} \frac{\partial u}{\partial n} ds = 1,$$

hence it is also the solution of (1).

The solution of (1) is non-negative. Otherwise, it will attain a negative minimum at a point in Ω or on Γ_2. For the same reason as that for solutions of (4), we know that the solution cannot attain a negative minimum in Ω. Furthermore, if it attains a negative minimum on Γ_2, then $\frac{\partial u}{\partial n} \leqslant 0$ on Γ_2 and $\int_{\Gamma_2} \frac{\partial u}{\partial n} ds \leqslant 0$, this contradicts the boundary condition on Γ_2.

From now on we can write the equation of (1) as $\Delta u = u^p$. By Lemma 1 we know the solution should be unique. Thus the proof of Proposition 1 is complete.

We are going to discuss the asymptotic behavior of the solution for problem (1) or (2), when Γ_2 shrinks to a point enclosed by Γ_2. For convenience we assume Γ_2 shrinks to the origin 0, the additional assumptions for Γ_2 are: Γ_2 is starshape with respect to 0 and the ratio of the measure of Γ_2 to d^{n-1} is bounded, where d is minimum distance from 0 to Γ_2. From now on, we use the notation $\Gamma_2 \to 0$ to denote Γ_2 shrinks to the point 0 under these assumptions.

Definition. *We call set of functions $\{u\}$ compact-uniformly bounded in $\Omega_1 \backslash 0$ if for any compact set K in $\Omega_1 \backslash 0$, elements u in this set are uniformly bounded on K (For the requirement of the definition we may often extend the domain of u to the whole Ω_1, preserving its bound and continuity).*

Proposition 2. *When $\Gamma_2 \to 0$, the set of solutions of (1), (or (2)) and their derivatives are compact-uniformly bounded in $\Omega_1 \backslash 0$.*

Proof The proof is very similar to that of Lemma 1 in [5]. For reader's conveniency we sketch it here.

Multiplying both side of (1) by u and then integrating on Ω, we get

$$a_\Omega(u, u) + \int_\Omega u^{p+1} dx = u|_{\Gamma_2}, \tag{7}$$

where

$$a_\Omega(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx.$$

Since Γ_2 is starshape, we can describe it as $r = r(\theta)$, where θ denotes the points on the unit sphere S^{n-1}. If we denote the distance from 0 to Γ_1 along direction θ by $R(\theta)$, then the following inequality holds

$$\int_{S^{n-1}} \int_{r(\theta)}^{R(\theta)} r^{n-1} \left(\frac{\partial u}{\partial r}\right)^2 dr \, d\omega_\theta \leqslant \int_\Omega \left(\frac{\partial u}{\partial r}\right)^2 dx \leqslant a_\Omega(u, u) \leqslant u|_{\Gamma_2}. \tag{8}$$

Since $u|_{\Gamma_1} = 0$ we have

$$u(r(\theta), \theta) = -\int_{r(\theta)}^{R(\theta)} \frac{\partial u}{\partial r} dr,$$

$$|u(r(\theta), \theta)| \leqslant b\int_{r(\theta)}^{R(\theta)} r^{n-1} \left(\frac{\partial u}{\partial r}\right)^2 dr + \frac{1}{b} \int_{r(\theta)}^{R)\theta)} \frac{1}{r^{n-1}} dr$$

$$\leqslant b\int_{r(\theta)}^{(\theta)} r^{n-1} \left(\frac{\partial u}{\partial r}\right)^2 dr + \frac{1}{(n-2)b} \frac{1}{d^{n-2}}, \tag{9}$$

where b is a constant and will be chosen below.

Integratinge (9) with respect to θ, using (8), we get

$$\omega_{n-1} u|_{\Gamma_2} \leqslant b u|_{\Gamma_2} + \frac{\omega_{n-1}}{(n-2)b} \frac{1}{d^{n-2}}. \tag{10}$$

Choosing b sufficiently small and using the assumptions about Γ_2, we have the estimation on Γ_2

$$u \leqslant C r^{2-n}, \tag{11}$$

where C is an appropriate constant, which is independent of u. Using Lemma 2 we conclude that the estimation (11) is also valid in the whole Ω.

The estimation (11) implies the compact-uniformly boundedness of solutions of (2), when $\Gamma_2 \to 0$. The compact-uniformly boundedness of the derivatives follows from Schauder estimates. This completes the proof.

Remark 1. Since the singularity of the solution G of (3) is the same as r^{2-n}, the similar inequality

$$u \leqslant C'G \quad \text{in } \Omega \tag{12}$$

also holds.

§ 3. Asymptotic properties

Now we are in a position to state the main results.

Theorem 1. *Suppose* $n \geqslant 3, p \geqslant \frac{n}{n-2}$. *When* $\Gamma_2 \to 0$, *the solution of* (1) *converges to* 0 *in any compact set of* $\Omega_1 \backslash 0$.

Proof According to Proposition 2, the set of solutions $\{u\}$ of (1) with their derivatives is uniformly bounded in any compact subset of $\Omega_1 \backslash 0$. Using Arzela-Ascoli theorem we can choose a subsequence $\{u_m\}$, which converges to v in any compact subset of $\Omega_1 \backslash 0$. Obviously, the function v satisfies $\Delta v = |v|^{p-1} v$ in $\Omega_1 \backslash 0$, $v \geqslant 0$ and $v|_{\Gamma_1} = 0$. Using Brézis and Veron's result in [3] on removable singularities we know that $\Delta v = |v|^{p-1} v$ holds in the entire Ω_1 (we may need to asssign the value of v in point 0). Using uniqueness of solution of this Dirichlet problem we get $v = 0$. By the same reason we can always choose a subsequence $\{u_{m_k}\}$ from any subsequence $\{u_m\}$ of set $\{u\}$, such that $u_{m_k} \to 0$ in any compact subset of $\Omega_1 \backslash 0$, this impliea Theorem I.

We notice that $\max\limits_{x \in \Omega} u = u|_{\Gamma_2}$ approaches infinity when $\Gamma_2 \to 0$. This can be explained as follows. Let Ω_0 be a domain containing 0. When $\Gamma_2 \to 0$, it will contain Ω_2 eventually. Obviously

$$1 = \int_{\Gamma_2} \frac{\partial u}{\partial n} ds = \int_{\Omega} |u|^{p-1} u \, dx = \int_{\Omega_0 \cap \Omega} u^p \, dx + \int_{\Omega \backslash \Omega_0} u^p \, dx. \tag{13}$$

If u is bounded, we can choose Ω_0 small enough such that $\int_{\Omega_0 \cap \Omega} u^p dx < \varepsilon$, by Theorem 1 we know $\int_{\Omega \backslash \Omega_0} u^p \, dx \to 0$, but this contradicts (13).

It may seem strange that $\max\limits_{x \in \Omega} u$ approaches infinity when $\Gamma_2 \to 0$, but u approaches 0 pointwise. We use the following example to explain this.

Assume Γ_1 and Γ_2 are spheres with center 0 and radii R and δ, and denote the solution of (1) by u_δ, which only depends on r. We claim that for any $\varepsilon > 0$ and $0 < r_0 < R$. The inequality $u_\delta(r) \leqslant \frac{\varepsilon}{r^{n-2}}$ will hold in $r_0 \leqslant r \leqslant R$, as long as δ is sufficiently small. Using Lemma 2 we only need to show $u_\delta(r) \leqslant \frac{\varepsilon}{r^{n-2}}$ holds for some $r = \delta_1 (\delta \leqslant \delta_1$

$\leqslant r_0$). Indeed, if this is not true, we can find numbers ε, r_0, such that for any small δ, the inequality $u_\delta(r)>\frac{\varepsilon}{r^{n-2}}$ holds in $[\delta, r_0]$, then

$$I=\int_{\Gamma_2}\frac{\partial u_\delta}{\partial n}ds>\int_{\Gamma_2}\frac{\partial u_\delta}{\partial n}ds+\int_{\Gamma_1}\frac{\partial u_\delta}{\partial n}ds=\int_\Omega \Delta u_\delta\,dx=\int_{\delta\leqslant r\leqslant r_0}u_\delta^p\,dx$$
$$>\int_\delta^{r_0}\varepsilon^p\omega_{n-1}r^{-p(n-2)+n-1}dr. \tag{14}$$

Because of $p\geqslant\frac{n}{n-2}$, the integration $\int_\delta^{r_0}\varepsilon^p\omega_{n-1}r^{-p(n-2)+n-1}dr$ will tend to ∞, when $\delta\to 0$; so (14) is impossible.

Theorem 2. *Suppose* $n\geqslant 3$, $\frac{n+1}{n-1}\leqslant p<\frac{n}{n-2}$. *Then as* $\Gamma_2\to 0$, *the solution of* (1) *converges to a non-zero limit in any relatively compact open set of* $\Omega_1\backslash 0$.

Proof As in Theorem 1, from any subsequence $\{u_m\}$ of the solution set $\{u\}$, we can choose a convergent subsequence $\{u_{m_k}\}$, such that $u_{m_k}\to v$, $Du_{m_k}\to Dv$, when $\Gamma_2\to 0$. Obviously, v satisfies elliptic equation $\Delta v=v^p$. Therefore, if $v=0$ holds in some open set in $\Omega_1\backslash 0$, it should hold in whole $\Omega_1\backslash 0$. But we can show this is impossible. In fact, if $v=0$ for sufficiently large k, $-\int_{\Gamma_1}\frac{\partial u_{m_k}}{\partial n}ds<\frac{1}{3}$. Furthermore, because of (10), we can choose δ_1, such that $\frac{c\omega_{n-1}\delta_1^{n-(n-2)p}}{n-(n-2)p}=\frac{1}{3}$, then for any Γ_2, which is sufficiently near 0, we have

$$\int_{D_{\delta_1}}u_{m_k}^p\,dx\leqslant c\omega_{n-1}\int_d^{\delta_1}r^{n-1-p(n-2)}dr\leqslant c\omega_{n-1}[n-p(n-2)]^{-1}\delta_1^{n-p(n-2)}=\frac{1}{3}$$

where D_{δ_1} denotes the domain between surface Γ_2 and $S_{\delta_1}^{n-1}$. Moreover, from $u_{m_k}\to v=0$ we can choose a sufficiently large k such that

$$\int_{\Omega\backslash D_{\delta_1}}u_{m_k}^p\,dx<\frac{1}{3}.$$

Summing up above estimates, we would get the contradiction

$$I=\int_{\Gamma_2}\frac{\partial u_{m_k}}{\partial n}ds=\int_\Omega \Delta u_{m_k}dx-\int_{\Gamma_1}\frac{\partial u_{m_k}}{\partial n}ds<\int_\Omega u_{m_k}^p\,dx+\frac{1}{3}$$
$$=\int_{D_{\delta_1}}u_{m_k}^p\,dx+\int_{\Omega\backslash D_{\delta_1}}u_{m_k}^p\,dx+\frac{1}{3}<1. \tag{15}$$

Now we are going to show that all limit functions of subsequence $\{u_{m_k}\}$ are same. Firstly, the limit function v satisfies equation $\Delta v=|v|^{p-1}v$ in the domain $\Omega_1\backslash 0$. Using (10) and Theorem 2 in [4], we know $v(x)\sim\alpha r^{2-n}$ near 0. If v_1, v_2 are limit functions of different subsequences, there are two cases. The first case is $v_i(x)\sim\alpha_i r^{2-n}$, $\alpha_1\neq\alpha_2$. Assume $\alpha_1>\alpha_2$. Then near 0 we have $v_1(x)>v_2(x)$, so between $\int_\Omega v_1^p\,dx-\int_{\Gamma_1}\frac{\partial v_1}{\partial n}ds$ and $\int_\Omega v_2^p\,dx-\int_{\Gamma_1}\frac{\partial v_2}{\partial n}ds$ at least one number does not equal to 1. If

$$\int_\Omega v_1^p\,dx-\int_{\Gamma_1}\frac{\partial v_1}{\partial n}ds>1,$$

then we can choose sufficiently small δ, such that

$$\int_{\Omega\setminus D_\delta} v_1^p\,dx - \int_{\Gamma_1} \frac{\partial v_1}{\partial n}\,ds > 1. \tag{16}$$

When Γ_2 is near to 0 enough, the corresponding u in the subsequence, which converges to v_1, will be sufficiently near to v_1, so

$$\int_{\Omega\setminus D_\delta} u^p\,dx - \int_{\Gamma_1} \frac{\partial u}{\partial n}\,ds > 1. \tag{17}$$

From (17) we will get $\int_{\Gamma_2} \frac{\partial u}{\partial n}\,ds > 1$, which contradicts the fact that u is a solution of (1). On the other hand, if $\int_\Omega v_2^p\,dx - \int_{\Gamma_1} \frac{\partial v_2}{\partial n}\,ds < 1$. we can use similar technique to lead to a contradiction.

Now let us turn to the case $\alpha_1 = \alpha_2$. Assume $\varepsilon > 0$ is sufficiently small. Then $v_1 + \varepsilon G$ satisfies

$$\begin{aligned} &\Delta(v_1+\varepsilon G) = v_1^p,\ x\in\Omega_1\setminus 0,\\ &v_1+\varepsilon G|_{\Gamma_1} = 0, \end{aligned} \tag{18}$$

and if Γ_2 is near to 0 enough, the following inequality will be valid

$$v_1 + \varepsilon G > v_2 \ \text{ on } \Gamma_2, \tag{19}$$

because of $v_2 \sim \alpha r^{2-n}$ near 0. For $v_1 + \varepsilon G - v_2$, we have

$$\begin{aligned} &\Delta(v_1+\varepsilon G - v_2) = v_1^p - v_2^p,\ x\in\Omega,\\ &v_1+\varepsilon G - v_2|_{\Gamma_1} = 0,\\ &v_1+\varepsilon G - v_2|_{\Gamma_2} > 0. \end{aligned} \tag{20}$$

By the maximum principle $v_1 + \varepsilon G - v_2 \geqslant 0$ in the whole Ω. Similarly, $v_1 - \varepsilon G - v_2 \leqslant 0$ holds in Ω. Therefore, we obtain the inequality

$$v_1 - \varepsilon G \leqslant v_2 \leqslant v_1 + \varepsilon G \ \text{ in } \Omega. \tag{21}$$

Because ε is an arbitrarily small number, v_1 must be equal to v_2 in any compact set of $\Omega_1\setminus 0$, it means $v_1 \equiv v_2$ in $\Omega_1\setminus 0$. Therefore, all limit functions of subsequence of solution set $\{u\}$ are the same and the whole set $\{u\}$ approaches a unique limit, which is not zero as we have proved.

Remark 2. If $p < \frac{n+1}{n-1}$, we only know that the solution of (1) can not converge to 0, but we don't know if it really converges.

Theorem 3. *When $\Gamma_2 \to 0$, the solution of (2) converges to G both in the sense of uniform convergence of compact subsets and in $L^q(\Omega_1)$ sense, where $q < \frac{n}{n-2}$.*

Proof It is known in [5] that the solution of (2) converges to G in the sense of uniform convergence of compact subsets of Ω_1 and in the sense of distribution. By (12) we know

$$|u-G| \leqslant CG \in L^q,$$

hence

$$|u-G|^q \leqslant C'G^q.$$

From the first conclusion of this theorem we know $u \to G$ almost everywhere in Ω_1, so by Lebesgue's dominate convergence theorem we get $u \xrightarrow{L^q} G$ immediately.

Readers can find the detailed description of Theorem 3 and the related consequences in [5].

References

[1] Li Taqian, Chen Shuxing, Zheng Sunmu, Tan Yungji, Boundary value problems with complementary boundary conditions for self-adjoint elliptic equation, *Fudan Journal (Natural Science)*, No. 2 (1978), 49—60 (in Chinese).

[2] Ladyzhenskaia O. A., and Ural'tseva N. N., Linear and quasilinear elliptic equations, (Translated from Russian) New York, Academic Press (1968).

[3] Brézis H., and Veron L., Removable singularities for some nonlinear elliptic equations, *Arch. Rat. Mech. Anal.*, **75** (1980), 1—6.

[4] Veron L., Solutions singulieres d'équations elliptiques semilinéaires, *C. R. Acad. Sc. Paris*, **288** (1979), 867—869.

[5] Li Taqian and Chen Shuxing, On the asymptotic behavior of solutions of equi-valued surface boundary value problems for the second order self-adjoint elliptic equation, *Fudan Journal (Natural Science)*, No.4 (1978), 6—14 (in Chinese).