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MORSE THEORY ON BANACH SPACE AND ITS APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

The author introduces a definition about the nondegeneracy of critical points of a differentiable functional defined on a Banach space. Thus the Morse theory is estabished to these functionals on a Banach space with an equivalent differentiable norm. And by use of the Morse inequalities an exension of three critical point theorem due to Krasnoselski, Castro, Lazer and the author is provided. As an application, the multiple solutions of a quasilinear elliptic boundary value problems studied.

Morse theory has been extended by Palais[3] and Smale[5] to Hilbert manifolds, and by Uhlenbeck[7] and Tromba[6] to certain classes of functions on Banach Manifolds.

The main difficulty in developing a Morse theory on a Banach manifold is the lack of a proper definition of non-degenerate critical point. In a Hilbert space, a critical point x_0 of a C^2 function f is said to be nondegenerate if the Hessian $d^2f\ (x_0)$ (considered as a self adjoint operator) is invertible, i. e., the inverse operator exists and is bounded. This definition fits for Hilbert space, because it ensures that all nondegenerate critical points are isolated and that the Morse lemma holds. But for a Banach space $\mathscr{X}$, the above definition does not work, because $\mathscr{X}$ is not isomorphic to $\mathscr{X}^*$ in general.

Uhlenbeck and Tromba tried to give answers for an reasonable definition. In [7], a critical point x_0 of f is said to be weakly nondegenerate if there exists an hyperbolic operator L(cf. § 1, Def. 3), and a neighbourhood U of x_0, such that

$$\langle df(x),\ L(x-x_0)\rangle > 0 \text{ for } x\in U\setminus\{x_0\},$$

where $\langle\cdot,\ \cdot\rangle$ denotes the duality between $\mathscr{X}^*$ and $\mathscr{X}$.

In [6], the B-nondegeneracy is defined by a series of properties, which not only depends on the function f, but also depends upon the existence of a vector field, i. e., x_0 is B-nondegenerate, if there exist a neighbourhood U of x_0 and a C^1 vector field $V(x)$ on U, such that

(1) $\langle df(x),\ V(x)\rangle > 0$ for $x\in U\setminus\{x_0\}$,

(2) The Frechet derivative of V at x_0 is symmetric with respect to the Hessian

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$d^2f(x_0)$

$$d^2f(x_0)(dV(x_0)x,\ y)=d^2f(x_0)(x,\ dV(x_0)y),\ \forall(x,\ y)\in\mathscr{X}\times\mathscr{X},$$

(3) $dV(x_0)$: $\mathscr{X}\to\mathscr{X}$ is an hyperbolic operator,

(4) $d^2f(x_0)(dV(x_0)x,\ x)>0$ if $x\in\mathscr{X}\backslash\theta$.

In this paper, we slightly modify the definition of the nondegeneracy. Our definition reads as follows:

Definition 1. *A critical point x_0 of a C^2 function $f(x)$ is said to be s-nondegenerate, if*

(1) *It is isolated;*

(2) *there exists an hyperbolic operator $L=L_{x_0}$: $\mathscr{X}\to\mathscr{X}$ and a neighbourhood U of x_0 such that*

$$d^2f(x_0)(Lx,\ y)=d^2f(x_0)(x,\ Ly),\ \forall x,\ y\in\mathscr{X}, \tag{0.2}$$

$$d^2f(x_0)(Lx,\ x)>0,\ \forall x\in\mathscr{X}\backslash\theta, \tag{0.3}$$

$$d\langle f(x),\ L(x-x_0)\rangle>0,\ \forall x\in f_c\cap(U\backslash x_0), \tag{0.4}$$

where $c=f(x_0)$.

Definition 2. *The index of a s-nondegenerate critical point is defined to be the dimension of the maximal negative subspace of L.*

That the index is well defined, will be shown in § 1.

Obviously, a B-nondegenerate critical point is a s-nondegenerate critical point, and a weakly nondegenerate critical point satisfies all conditions of a s-nondegenerate critical point except (0.2). In fact, in [7], the proof of the Morse theory is incomplete. I convince that the condition (0.2) is needed.

In this paper, besides the Morse inequalities, a three critical point theorem, which is an extension of the work due to Castro, Lazer[1] and K. C. Chang[2], is proved. As an application, we prove that the following PDE:

$$\begin{cases} c\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)+\Delta u=g(u) & \text{in } \Omega\subset\mathbf{R}^n, \\ u|_{\partial\Omega}=0 \end{cases} \tag{0.5}$$

has at least three solutions, under certain assumptions on the function g (cf. § 4), where $c>0$, $p\geqslant 2$.

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§ 1. Preliminary

Most material in this section can be found in [6].

In the following, let $\mathscr{X}$ be a banach space:, L be a linear continuous operator from $\mathscr{X}$ into itself, and let $b(\cdot,\cdot)$ be a bilinear continuous form over $\mathscr{X}\times\mathscr{X}$. We say that b is commutative with L if

$$b(Lu,\ v)=b(u,\ Lv). \tag{1.1}$$

By definition, there is a linear continuous operator B: $\mathscr{X}\to\mathscr{X}^*$, induced by b, such that

$$b(u,\ v)=\langle Bu,\ v\rangle,\ \forall(u,\ v)\in\mathscr{X}\times\mathscr{X}. \tag{1.2}$$

Thus, for b commutes with L if and only if

$$BL=L^*B.$$

Definition 3. *A linear operator L is said to be a hyperbolic operator, if its spectrum $\sigma(L)$ is contained in two compact domains, one lies in the right hand half open plane H_+, and the other lies in the left hand half open plane H_-.*

If L ss hyperbolic, the resolvent $R_z=(zI-L)^{-1}$ is defined on the resolvent set $\rho(L)$, and also

$$R_z^*=(\bar{z}I-L^*)^{-1}.$$

According to Riesz, Nagy[4], the projection operator

$$P=-\frac{1}{2\pi i}\int_{\partial D}R_z\,dz$$

is well defined for any domain D with a rectificable boundary ∂D which does not intersect with the spectrum $\sigma(L)$. In particular, let

$$\sigma(L)=\sigma_+\cup\sigma_-,\ \text{where}\ \sigma_\pm=\sigma(L)\cap H_\pm,$$

then the projection operators P_+ and P_- are defined by

$$P_\pm=\frac{-1}{2\pi i}\int_{\partial D_\pm}R_z\,dz,$$

where $D_\pm$ is a compact domain containing $\sigma_\pm$. We have

$$\begin{cases}P_\pm^2=P_\pm,\\ P_+P_-=P_-P_+=0,\\ P_++P_-=I,\end{cases}$$

and the Banach space $\mathscr{X}$ is decomposed into $\mathscr{X}_+\oplus\mathscr{X}_-$, where $\mathscr{X}_\pm=P_\pm\mathscr{X}$.

So is $\mathscr{X}^*=\mathscr{X}_+^*\oplus\mathscr{X}_-^*$.

Let $L_\pm=L|_{\mathscr{X}_\pm}$. Provided by the functional calculus of the bounded linear operators, the operator

$$S_\pm=\frac{1}{2\pi i}\int_{\partial D_\pm}\sqrt{\pm z}\,R_z\,dz$$

is well defined, and commutes with $P_\pm$.

Since R_z can be expanded as a power series of L, we have

$$\begin{cases}BR_z=R_z^*B,\\ BP_\pm=P_\pm^*B,\\ BS_\pm=S_\pm^*B.\end{cases}$$

Proposition 1. *Let the bilinear form b be defined on $\mathscr{X}\times\mathscr{X}$. Assume that b commutes with an hyperbolic operator L, satisfying*

$$b(Lu, u)>0, \ \forall u\neq\theta,$$

Then b is positive on $\mathscr{X}_+$, and is negative on $\mathscr{X}_-$

Corollary 1. *The index of a s-nondegenerate critical point is well defined.*

Corollary 2. *In the proposition 1, if further, $\mathscr{X}_-$ is finite dimensional then b is negative definite on $\mathscr{X}_-$ i. e. there exists $\gamma>0$ such that*

$$b(u_-, u_-)\leqslant-\gamma\|u_-\|^2, \ \forall u_-\in\mathscr{X}_-.$$

Proposition 2. *Let b be the bilinear form defined in the Proposition 1, then $\forall u\in\mathscr{X}$, we have the decomposition*

$$b(u, u)=b(u_+, u_+)+b(u_-, u_-),$$

where $u=u_++u_-$, $u_\pm\in\mathscr{X}_\pm$.

Proposition 3. *Let L be an hyperbolic operator on $\mathscr{X}$ with $\mathscr{X}_\pm$ the positive/negative invariant subspace. Then there exists an equivalent norm $|\cdot|$, with the same smoothness as the original norm $\|\cdot\|$ of $\mathscr{X}$, and a constant $\rho>0$ such that*

and $$|x_++x_-|=|x_+|+|x_-|, \ \forall(x_+, x_-)\in\mathscr{T}_+\oplus\mathscr{X}_-,$$

$$|e^{Lt}x_\pm|\geqslant(1+\rho|t|)\,|x_\pm|, \ \textit{for} \ \pm t>0.$$

Proposition 4. *Suppose that the norm $|\cdot|$ on $\mathscr{X}$ is C^1-differentiable, then $\forall\mu>0$, we have*

$$\langle d|x_+|-\mu d|x_-|, Lx\rangle\geqslant\rho\cdot\min\{1, \mu\}\,|x|, \ \forall x\in\mathscr{X},$$

Proof Since $d|x_\pm|\in\mathscr{X}^*_\pm$, we have

$$\langle d|x_+|, Lx_+\rangle=\lim_{t\to+0}\frac{1}{t}(|e^{Lt}x_+|-|x_+|)\geqslant\rho|x_+|,$$

$$-\langle d|x_-|, Lx_-\rangle=\lim_{t\to+0}\frac{1}{t}(|e^{-Lt}x_-|-|x_-|)\geqslant\rho|x_-|,$$

and $$\langle d|x_\pm|, Lx_\mp\rangle=0.$$

Thus

$$\langle d|x_+|-\mu d|x_-|, Lx\rangle\geqslant\rho(|x_+|+\mu|x_-|)\geqslant\rho\min\{1, \mu\}\,|x|.$$

§ 2. Deformation lemma and the handle-body decomposition

The main step in extending the Morse theory is to prove the so-called handle-body decomposition theorem in Banach space. Once it has been established, the whole theory carries over. To this end, a deformation lemma is need.

Throughout this section, we make the assumptions on a function $f: \mathscr{X}\longrightarrow\mathbf{R}^1$, as follows:

(1) f is C^1-differentiable

(2) f satisfies the Palais Smale Condition (P. S.) i. e. for each $\{x_n\}\subset\mathscr{X}$, $f(x_n)$ is

bounded and $d\mathscr{f}(x_n)\to\theta$, implies

$\exists$ a convergent subsequence of $\{x_n\}$.

For each real number d, we denote by $\mathscr{f}_d$ the set $\{x\in\mathscr{X}\,|\,\mathscr{f}(x)\leqslant d\}$.

Deformation lemma. *Suppose that c is the only critical value of a C^{2-0}-function $f\colon \mathscr{X}\to\mathbf{R}^1$ in $\mathscr{f}^{-1}[a, b]$. Assume that $\mathscr{f}$ satisfies the P. S. condition, and has only isolated critical points in $\mathscr{f}^{-1}[a, b]$. Then $\mathscr{f}$ has only finite number of critical points in $\mathscr{f}^{-1}[a, b]$, and $\mathscr{f}_c$ is a deformation retract of $\mathscr{f}_b$.*

Proof The first conclusion follows from the P. S. condition directly. We shall prove the second.

1. Define a flow as follows

$$\begin{cases}\dot{\sigma}=-\left.\dfrac{(\mathscr{f}(x_0)-c)v(x)}{\langle d\mathscr{f}(x),\ v(x)\rangle}\right|_{x=\sigma(t,x_0)},\\ \sigma(0,\ x_0)=x_0\in\mathscr{f}_b\backslash\mathscr{f}_c,\end{cases}$$

where $v(x)$is the pseudo-gradient vector field of $\mathscr{f}$. It is easily seen

$$\frac{d}{dt}\mathscr{f}(\sigma(t,\ x_0))=-(\mathscr{f}(x_0)-c),$$

or

$$\mathscr{f}(\sigma(t,\ x_0))=(1-t)\mathscr{f}(x_0)+ct.$$

This implies that $\sigma(t, x_0)$ is defined for $[0, 1]\times(\mathscr{f}_b\backslash\mathscr{f}_c)$, and

$$\lim_{t\to1-0}\mathscr{f}(\sigma(t,\ x_0))=c.$$

However, the right hand side of the ODE approaches to 0 as σ approaches to a critical point, it is not obvious that $\lim\limits_{t\to1-0}\sigma(t, x_0)$ exists.

2. We shall prove that the limit $\lim\limits_{t\to1-0}\sigma(t, x_0)$ does exist. Let $\{z_i\}_1^N$ be the critical points in $\mathscr{f}^{-1}[a, b]$. Then either

(a) $\inf\limits_{t\in[0,1)}\|\sigma(t,\ x_0)-z_i\|>0,\ \forall i=1,\ \cdots,\ N$

(b) $\inf\limits_{t\in[0,1)}\|\sigma(t,\ x_0)-z_i\|=0$, for some i.

In case (a), we have $d>0$ such that $\inf\limits_{t\in[0,1)}\|d\mathscr{f}(\sigma(t,\ x_0))\|\geqslant d$, and

$$\|\sigma(t_2,\ x_0)-\sigma(t_1,\ x_0)\|\leqslant\int_{t_1}^{t_2}\left\|\frac{d\sigma}{dt}\right\|dt\leqslant(\mathscr{f}(x_0)-c)\int_{t_1}^{t_2}\left.\frac{\|v(x)\|}{\langle d\mathscr{f}(x),\ v(x)\rangle}\right|_{x=\sigma(t,x_0)}dt$$

$$\leqslant\frac{2}{d}(\mathscr{f}(x_0)-c)(t_2-t_1),$$

which implies the existence of the limit $\lim\limits_{t\to1-0}\sigma(t,\ x_0)$.

In case(b), we shall prove

$$\lim_{t\to1-0}\sigma(t,\ x_0)=z_i.$$

If not, there exists $\varepsilon>0$, such that for an infinite sequence of disjoint intervals $[t_j,\ t_{j+1}]\subset[0,\ 1)$ with $\sigma(t,\ x_0)\in B(z_i,\ 2\varepsilon)\backslash B(z_i,\ \varepsilon)$ for $t\in[t_j,\ t_{j+1}]$, where $B(x,\ \delta)$ is the ball centered at x with radius $\delta>0$. The ε can be chosen small enough such that

$$\inf_{t\in[t_j,t_{j+1}]}\|d f(\sigma(t, x_0))\|\geqslant d>0.$$

Then we get a contradiction

$$\varepsilon\leqslant\|\sigma(t_{j+1}, x_0)-\sigma(t_j, x_0)\|\leqslant\int_{t_j}^{t_{j+1}}\left\|\frac{d\sigma}{dt}\right\|dt$$
$$\leqslant\frac{2}{d}(f(x_0)-c)(t_{j+1}-t_j)\to 0.$$

3. Now we define a deformation retract by

$$\tau(t, x)=\begin{cases}\sigma(t, x), & \text{if}(t, x)\in[0, 1)\times(f_b\backslash f_c),\\ \lim\limits_{t\to 1-0}\sigma(t, x), & \text{if } (t, x)\in\{1\}\times(f_b\backslash f_c),\\ x, & \text{if } (t, x)\in[0, 1]\times f_c.\end{cases}$$

We only have to verify the continuity of τ. Four cases are distinguished:

(a) $(t, x)\in[0, 1]\times\mathring{f}_c$,

(b) $(t, x)\in[0, 1)\times(f_b\backslash f_c)$,

(c) $(t, x)\in\{1\}\times(f_b\backslash f_c)$,

(d) $(t, x)\in[0, 1]\times f^{-1}(c)$.

Case (a) is trivial; case (b) is easily verified by the fundamental theorem of ODE.

Verification for the case (c)

Let $x^*=\lim\limits_{t\to 1-0}\tau(t, x)$, we shall prove: $\forall\varepsilon>0$, $\exists\delta>0$, such that

$$\|x_0-y\|<\delta \text{ and } 1-\delta<t \text{ imply}$$
$$\|\tau(t, y)-x^*\|<\varepsilon.$$

We choose suitable $\delta_1>0$ and $\delta_2=\delta_2(\delta_1)>0$, such that

$$\|\tau(1-\delta_1, y)-x^*\|<\frac{\varepsilon}{2} \quad\text{for } \|y-x_0\|<\delta_2.$$

It is avaliable, because we have $\delta_1>0$ such that

$$\|\tau(t, x_0)-x\|^*\|<\frac{\varepsilon}{4} \quad\text{for } t\in[1-\delta_1, 1),$$

and we have $\delta_2>0$ such that

$$\|\tau(t, x_0)-\tau(t, y)\|<\frac{\varepsilon}{4} \text{ for } \|y-x_0\|<\delta_2,\ t\in[0, 1-\delta_1].$$

These imply that

$$\|\tau(1-\delta_1, y)-x^*\|<\frac{\varepsilon}{2}.$$

No loss of generality, we may assume that no critical points are in $\bar{B}(x^*, \varepsilon)\backslash B\left(x^*, \frac{\varepsilon}{2}\right)$. Thus

$$d'=\inf_{x\in\bar{B}(x^*,\varepsilon)/B\left(x^*,\frac{\varepsilon}{2}\right)}\|d f(x)\|>0.$$

Determine $\delta_1>0$ so small, such that

$$(b-c)\delta_1<\frac{1}{4}\varepsilon d'.$$

Then we shall prove

$$\|\tau(t, y)-x^*\|<\varepsilon \text{ for } t\in[1-\delta_1, 1), \ \|x_0-y\|<\delta_2.$$

If not, $\exists t'<t''$, and $y_0\in B(x_0, \delta_2)$, such that

$$\frac{\varepsilon}{2}\leqslant\|\tau(t, y_0)-x^*\|\leqslant\varepsilon \text{ for } t\in[t', t'']\subset[1-\delta_1, 1)$$

with
$$\|\tau(t', y_0)-x^*\|=\frac{\varepsilon}{2}, \ \|\tau(t', y_0)-x^*\|=\varepsilon.$$

This implies that

$$\frac{\varepsilon}{2}\leqslant\|\tau(t'', y_0)-\tau(t', y_0)\|\leqslant\int_{t'}^{t''}\left\|\frac{d\sigma}{dt}\right\|dt$$
$$\leqslant\frac{2}{d'}(b-c)\delta_1<\frac{\varepsilon}{2}.$$

This is a contradiction.

Case (d) will be verified in the same way. Since now $x^*=x_0$, y_0 is initially chosen in the ball $B\left(x_0, \frac{\varepsilon}{2}\right)$, only the last part of the above proof is applied.

Then we shall state the handle-body decomposition theorem and prove it.

Theorem 1. *Suppose that $\mathscr{X}$ is a Banach space with an equivalent differentiable norm. Suppose that f is a C^2-function, satisfying the P. S. condition on $\mathscr{X}$. Assume that c is the unique critical value in $f^{-1}[a, b]$, which corresponds only s-nondegenerate critical points with finite indices. Then there exists a handle-body decomposition. Namely, let $\{z_i\}_1^l$ be the critical points, $\{m_i\}_1^l$ be the corresponding indices; there exists $\varepsilon>0$ such that*

$$f_b\simeq f_{c-\varepsilon}\cup h_1(D^{m_1})\cup\cdots\cup h_l(D^{m_l}),$$

where h_i is a homeomorphism of m_i-dimensional disk into $\mathscr{X}_-^{m_i}\times\{z_{i+}\}$, ($\mathscr{X}_-^{m_i}$ is the negative space induced by L_{z_i} and $z_i=z_{i+}+z_{i-}$) with

$$h_i(\theta)=z_i$$

and
$$f_{c-\varepsilon}\cap h_i(D^{m_i})=f^{-1}(c-\varepsilon)\cap h_i(D^{m_i})=h_i(\partial D^{m_i}),$$

where D^k denotes the k-dim. disk.

Before starting the proof, we need a lemma

Lemma. *Suppose that f is a C^2 function on a Banach space $\mathscr{X}$. Suppose that θ is a s-nondegenerate, finite index critical point of f with a hyperbolic operator L. Then there is a cone neighbourhood*

$$C_{\mu,\delta}=\{(x_+, x_-)\in\mathscr{X}_+\oplus\mathscr{X}_-\mid\|x_+\|\leqslant\mu\|x_-\|, \ \|x_-\|\leqslant\delta\},$$

where μ, $\delta>0$, such that

(a) $\langle df(x), x_-\rangle<-\frac{\nu}{2}\|x_-\|^2$ *for some* $\nu>0$,

(b) $\langle df(x), Lx\rangle>\frac{\nu}{2}\|x_-\|^2$,

(c) $f(x)<f(\theta)-\frac{\nu}{4}\|x_-\|^2$,

in $C_{\mu,\delta}\backslash\theta$.

Proof The negative space induced by L is finite dimensional, $d^2f(\theta)$ is negative on $\mathscr{X}_-$, so we have $\nu>0$ such that

$$d^2f(\theta)(x_-, x_-)\leqslant -\nu\|x_-\|^2,$$

and

$$d^2f(\theta)(Lx_-, x_-)\geqslant \nu\|x_-\|^2.$$

Since $d^2f(\theta)$ is bounded, there is a constant $M>1$ such that

$$d^2f(\theta)(x_+, x_+)\leqslant M\|x_+\|^2.$$

No loss of generality, we may assume $\|L\|\leqslant 1$.

(a) For given positive $\varepsilon<\nu/8M(1+\mu)^2$, we have $\delta>0$ such that

$$\|df(x)-d^2f(\theta)x\|<\varepsilon\|x\| \text{ for } 0<\|x\|<\delta.$$

This implies

$$\langle df(x), x_-\rangle-d^2f(\theta)(x_-, x_-)\leqslant\varepsilon\|x\|\,\|x_-\|\leqslant(1+\mu)\varepsilon\|x_-\|^2 \text{ for } x\in C_{\mu,\delta},$$

Thus

$$\langle df(x), Lx\rangle>\frac{\nu}{2}\|x_-\|^2, \ \forall x\in C_{\mu,\delta}\backslash\theta.$$

(b)

$$\langle df(x), Lx\rangle-d^2f(\theta)(Lx, x)\geqslant-\varepsilon\|x\|^2\geqslant-\varepsilon(1+\mu)^2\|x_-\|^2, \ \forall x\in C_{\mu,\delta}.$$

Thus

$$\langle df(x), Lx\rangle>\frac{\nu}{2}\|x_-\|^2, \ \forall x\in C_{\mu,\delta}\backslash\theta.$$

(c) From

$$\left|f(x)-f(\theta)-\frac{1}{2}d^2f(\theta)(x, x)\right|\leqslant\varepsilon\|x\|^2 \text{ for } \|x\|<\delta,$$

it follows

$$f(x)\leqslant f(\theta)-\frac{\nu}{4}\|x_-\|^2, \ \forall x\in C_{\mu,\delta}\backslash\theta,$$

with

$$\mu<\min\left\{1, \sqrt{\frac{\nu}{4M}}\right\}.$$

The proof of the theorem 1

We may simply assume that θ is the unique critical point with critical value $c=0$, because the critical points of f are isolated; they can be decomposed individually. Suppose that L and $\mathscr{X}_\pm$ are understood for $d^2f(\theta)$. Let $\delta>0$ be so amall such that

$$\langle df(x), Lx\rangle>0, \ \forall x\in f_0\cap(D_\delta^-\times D_\delta^+\backslash\theta),$$

where $D_\delta^\pm$ is the disk with the radius δ in $\mathscr{X}_\pm$. Let the cone neighbourhood $C_{\mu,\delta}$ be defined as in the above lemma. The deformation retract will be constructedby the following three steps:

(1) Provided by the deformation Lemma, $f_b\simeq f_0$,

(2) We shall prove in the latter that

$$f_0\simeq f_{-\varepsilon}\cup C_{\mu,\delta},$$

(3) $f_{-\varepsilon}\cup D_\delta^-\times\{\theta_f\}$ is obviously a deformation retract of $f_{-\varepsilon}\cup C_{\mu,\delta}$, provided by the facts that $df \,\overline{\pitchfork}\, D_\delta^-\times\{\theta_+\}$, and that D_δ^- is finitely dimensional; if $\delta>0$ is sufficiently small.

Since dim $\mathscr{X}_-$ is finite, and $f|_{\mathscr{X}_-}$ has a local minimum at θ_-, we may choose $\varepsilon>0$ so small such that $\mathscr{X}_-\cap f^{-1}[-\varepsilon, 0)$ is radically homeomorphic to the disk D_1^- in $\mathscr{X}_-$.

Therefore the proof will be complete if (2) is proved.

Now we shall prove the existence of the deformation retract: $f_b\simeq f_{-\varepsilon}\cup C_{\mu,\delta}$, where

$$0<\varepsilon\leqslant\frac{\nu}{4}\delta^2.$$

According to (c) of the lemma, it is seen that

$$\sup\{f(x)\,|\,x\in C_{\mu,\delta},\ |x_-|=\delta\}<-\varepsilon$$

i. e. $C_{\mu,\delta}\cap\partial D_\delta^-\times\mathscr{X}_+\subset f_{-\varepsilon}$. Choosing $\delta_1\in(0, \delta)$ such that

$$\sup\{f(x)\,|\,x\in C_{\mu,\delta},\ |x_-|=\delta_1\}<-\varepsilon.$$

We define a C^{1-0} function with support outside $D_{\delta_1}^-\times D_{\mu\delta_1}^+$:

$$\rho(x)=\begin{cases}1, & |x_-|\geqslant\delta \text{ or } |x_+|\geqslant\mu\delta,\\ 0, & |x_-|\leqslant\delta_1 \text{ and } |x_+|\leqslant\mu\delta_1,\end{cases}$$

and linear in $|x_+|$ and $|x_-|$ in $D_\delta^-\times D_{\mu\delta}^+\backslash D_{\delta_1}^-\times D_{\mu\delta_1}^+$.

Let
$$\chi(x)=(1-\rho(x))Lx+\rho(x)v(x),$$
where $v(x)$ is the pseudo-gradient vector field of f in $f^{-1}[a, b]$. The following flow:

$$\begin{cases}\dot\sigma(t, x)=-\chi(\sigma(t, x)),\ t>0,\\ \sigma(0, x)=x\in f_0\backslash(f_a\cup\{\theta\})\end{cases}$$

is well defined, and will be applied to the deformation, because $f\circ\sigma(t)$ is decreasing.

Let $[\theta, \beta)$ be the maximal interval for the existence of the flow, where $\beta=\beta(x)$ is equal to ∞ or not. We shall prove after a finite time

$$\sigma(t, x)\in f_{-\varepsilon}\cup C_{\mu,\delta},\ \forall x\in f_0\backslash(f_a\cup\{\theta\}).$$

The proof is divided into the following cases:

Case 1.

$$\inf_{t\in[0,\beta)}\operatorname{dist}(D_{\delta_1}^-\times D_{\mu\delta_1}^+,\ \sigma(t, x))>0.$$

In this case, $\exists\varepsilon_1>0$ such that

$$\rho(\sigma(t, x))\geqslant\varepsilon_1,$$

and then

$$\begin{aligned}d(f\circ\sigma)(t)&=\langle df, \dot\sigma\rangle(t)=-\langle df, \chi\circ\sigma\rangle(t)\\ &\leqslant-\varepsilon_1\langle df, v\circ\sigma\rangle(t)\leqslant-\varepsilon_1\|df(\sigma(t))\|^2\\ &\leqslant-\varepsilon_1 d^2.\end{aligned}$$

It follows

$$\varepsilon_1 td^2\leqslant\varepsilon_1\int_0^t\|df(\sigma(\tau))\|^2d\tau\leqslant f(x)-f(\sigma(t)).$$

We conclude: either $\lim\limits_{t\to\beta}f\circ\sigma(t)=-\infty$ or β is finite and $\int_0^\beta\|df(\sigma(\tau))\|^2\,d\tau<+\infty$. If $\lim\limits_{t\to\beta}f\circ\sigma(t)=-\infty$, then after a finite time $T\in[0, \beta)$, we get $\sigma(T, x)\in f_{-\varepsilon}$.

Otherwise, from

$$\int_{\{\tau\in[0,\beta)\mid\rho\circ\sigma(\tau)=1\}}\|\chi(\sigma(\tau))\|\,d\tau=\int_{\{\tau\in[0,\beta)\mid\rho\circ\sigma(\tau)=1\}}\|v\circ\sigma(\tau)\|\,d\tau$$

$$\leqslant 2\int_0^\beta\|d\not{f}(\sigma(\tau))\|\,d\tau\leqslant 2\left\{\beta\int_0^\beta\|d\not{f}(\sigma(\tau))\|^2d\tau\right\}^{\frac{1}{2}}<+\infty.$$

We have
$$\int_0^\beta\|\dot{\sigma}(\tau)\|\,d\tau=\int_0^\beta\|\chi(\sigma(\tau))\|\quad d\tau<+\infty,$$
because $\|\chi(\sigma(\tau))\|$ is bounded for those $\tau\in[0,\beta)$, where $\rho\circ\sigma(\tau)<1$.

Then the limit $\lim\limits_{t\to\beta}\sigma(t,x)$ exists in $\not{f}_0\backslash D^-_{\delta_1}\times D^+_{\mu\delta_1}$ which condradicts with the maximality of β.

In one word, a fter afinite time, $\sigma(t,x)$ enters into $\not{f}_{-\varepsilon}$ in this case.

Case 2.

$$\inf_{t\in[0,\beta)}\operatorname{dist}(D^-_{\delta_1}\times D^+_{\mu\delta_1},\ \sigma(t,x))=0.$$

Logiclly, there are only two possibilities; either $\lim\limits_{t\to\beta}\operatorname{dist}(D^-_{\delta_1}\times D^+_{\mu\delta_1},\ \sigma(t,x))=0,$

or $\exists\,\delta_2\in\left(0,\ \frac{1}{2}(\delta-\delta_1)\right)$, and infinitely many disjoint intervals$[\alpha_j,\ \beta_j]$, $j=1,\ 2,\ \cdots,$ such that

$$\sigma(t,x)\in D^-_{\delta_1+2\delta_2}\times D^-_{\mu(\delta_1+2\delta_2)}\backslash D^-_{\delta_1+\delta_2}\times D^+_{\mu(\delta_1+\delta_2)}\ \text{for } t\in[\alpha_j,\ \beta_j].$$

But the latter can not happen except $\sigma(t,x)$ enters into $\not{f}_{-\varepsilon}$ after a finite time. In fact, $\exists\ d_1>0$ such that

$$\rho(\sigma(t,x))\geqslant d_1\ \text{ for } t\in[\alpha_j,\ \beta_j],\ j=1,\ 2,\ \cdots,$$

on one hand

$$\not{f}(x)-\lim_{t\to\beta}\not{f}(\sigma(t,x))=-\int_0^\beta d(\not{f}\circ\sigma)(t)$$

$$\geqslant\sum_j\int_{\alpha_j}^{\beta_j}\langle d\not{f},\ \chi\rangle|_{\sigma(t)}\,dt\geqslant d^2d_1\sum_j(\beta_j-\alpha_j),$$

i. e. $\sum\limits_j(\beta_j-\alpha_j)<+\infty$. on the other hand

$$\delta_2\leqslant\|\sigma(\beta_j,x)-\sigma(\alpha_j,x)\|\leqslant\int_{\alpha_j}^{\beta_j}\left\|\frac{d\sigma}{d\tau}\right\|d\tau\leqslant M(\beta_j-\alpha_j),$$

where $M=\sup\limits_{x\in D^-_\delta\times D^+_{\mu\delta}}\|\chi(x)\|$, This is a contradiction.

Now we turn to the former one

$$\lim_{t\to\beta}\operatorname{dist}(D^-_{\delta_1}\times D^+_{\mu\delta_1},\ \sigma(x,t))=0.$$

We shall prove: $\exists$ a finite $T\in[0,\beta)$ such that $\sigma(T,x)\in D^-_{\delta_1}\times D^+_{\mu\delta_1}$. Then, according to the Proposition 3, after a finite time, $\sigma(t,x)$ enters into $C_{\mu,\delta}$. We prove the existence of T by contradiction. Suppose that such a T does not exist. Since for each $\delta_3>0$, $\exists T_1\in[0,\beta)$ such that

$$\sigma(t,x)\in D^-_{\delta_1+\delta_3}\times D^+_{\mu(\delta_1+\delta_3)}\backslash D^-_{\delta_1}\times D^+_{\mu\delta_1}\ \text{ for } t>T_1$$

and
$$\|v(x)\|\leqslant\|d\not{f}(x)\|\leqslant\|d^2\not{f}(\theta)x\|+o(\|x\|),$$

it follows
$$|L\sigma-v(\sigma)|\leqslant M_1|\sigma(t)|,$$

where M_1 is a constant. Observing the equation

$$\dot{\sigma}(t) = -L\sigma + \rho(\sigma)[L\sigma - v(\sigma)],$$

we obtain

$$\sigma(t+\Delta t) = e^{-L\Delta t}\sigma(t) + \int_t^{t+\Delta t} e^{-L(t+\Delta t-\tau)}\rho(\sigma)[L\sigma - v(\sigma)]d\tau.$$

According to the Proposition 3

$$|P_-\sigma(t+\Delta t)| - |P_-\sigma(t)|$$
$$\geqslant |e^{-L\Delta t}P_-\sigma(t)| - |P_-\sigma(t)| - \varepsilon_3 M_1 \int_t^{t+\Delta t} \|e^{-L(t+\Delta t-\tau)}\| d\tau |\sigma(t)|,$$

where $$\varepsilon_3 = \sup_{x \in D^-_{\delta_1+\delta_3} \times D^+_{\mu(\delta_1+\delta_3)}} \rho(x).$$

Thus $$\frac{d}{dt}|P_-\sigma(t)| \geqslant \rho |P_-\sigma(t)| - \varepsilon_3 M_1 |\sigma(t)|.$$

Similarily, we have

$$\frac{d}{dt}|P_+\sigma(t)| \leqslant -\rho |P_+\sigma(t)| + \varepsilon_3 M_1 |\sigma(t)|.$$

Choosing $\delta_3 > 0$ sufficiently small, such that

$$\varepsilon_3 < \frac{\mu\rho}{2M_1},$$

we obtain $$\frac{d}{dt}(|P_-\sigma(t)| - |P_+\sigma(t)|) \geqslant \alpha |\sigma(t)|, \ \forall t > T_1,$$

where $$\alpha = \rho - 2M_1\varepsilon_3 > 0.$$

Then we arrive at the inequality

$$(|P_-\sigma(t)| - |P_+\sigma(t)|) - (|P_-\sigma(T_1)| - |P_+\sigma(T_1)|) \geqslant \alpha \int_{T_1}^t |\sigma(\tau)| d\tau. \quad (*)$$

But (*) cannot hold. Because the LHS of (*) is finite, and $|\sigma(t)| \geqslant \mu\delta_1$ for $t > T_1$ these imply $\beta < +\infty$. Then

$$\int_{T_1}^{\beta} |\dot{\sigma}(\tau)| d\tau = \int_{T_1}^{\beta} |\chi(\sigma(\tau))| d\tau \leqslant M \int_{T_1}^{\beta} |\sigma(\tau)| d\tau < +\infty.$$

This implies that $\lim_{t\to\beta} \sigma(t)$ exists in $f_0 \setminus (D^-_{\delta_1} \times D^+_{\mu\delta_1})^\circ$ which contradicts with the maximality of β.

In summary, we have proved that there is a finite $T \in [0, \beta)$ such that $\sigma(T, x) \in f_{-\varepsilon} \cup C_{\mu,\delta}$. Let us denote by γ_x the first time which makes $\sigma(t, x)$ entering into $f_{-\varepsilon} \cup C_{\mu,\delta}$.

The function $\chi \mapsto \gamma_x$ is continuous in $f_0 \setminus \theta$. Because

$$\langle df(x), \chi(x) \rangle > 0, \ \forall x \in f_0 \setminus \theta$$

implies $\chi \overline{\pitchfork} f^{-1}(\varepsilon)$, and

$$\langle d|x_+| - \mu d|x_-|, Lx \rangle \geqslant \rho \min\{1, \mu\} |x|, \ \forall x \in \mathscr{X},$$

implies $\chi \overline{\pitchfork} \partial C_{\mu,\delta} \setminus (f_{-\varepsilon} \cup \{\theta\})$ (the tangent space of $\partial C_{\mu,\delta} \setminus (f_{-\varepsilon} \cup \{\theta\}) = \ker (d|x_+| - \mu d|x_-|)$, and $\chi(x) = Lx$, $\forall x \in \partial C_{\mu,\varepsilon} \setminus (f_{-\varepsilon} \cup \{\theta\})$). And inside $D^-_{\delta_1} \times D^+_{\mu\delta_1}$, $\dot{\sigma} = -L\sigma$, which implies

$$\lim_{x\to\theta}\sigma(t\gamma_x, x)=\theta \text{ uniformly in } t\in[0, 1].$$

Hence the function

$$\tau(t, x)=\begin{cases}\sigma(t\gamma_x, x), & if\ x\in f_0\backslash\theta,\\ \theta, & if\ x=\theta\end{cases}$$

defines a deformation retract of f_0 into

$$f_{-\varepsilon}\cup C_{\mu,\delta}, \text{ where } (t, x)\in[0, 1]\times f^{-t}[a, 0].$$

§ 3. Morse inequalities and a three critical point theorem

In this section, we briefly sketch the outline of the proof of the Morse inequalities, and then turn to an extension of our three critical point theorem to Banach space.

Definition *Let f be a C^1 real function, satisfying the P. S. condition, defined on a Banach manifold. Let a, b be regular values of f. Suppose that f has only finite critical points, with critical values $C_1<C_2<\cdots<C_m$ in $f^{-1}[a, b]$. The number*

$$M_k=\sum_{i=1}^{m}\operatorname{rank} H_k(f_{C_i+\varepsilon}, f_{C_i-\varepsilon})$$

with $0<\varepsilon<\frac{1}{2}\min_{1\le i\le m-1}(C_{i+1}-C_i)$ is called the k-thMorse type number of f on $f^{-1}[a, b]$, $k=0, 1, \cdots$, where $H_*(A, B)$ is the relatively singular homology group.

According to the well known deformation lemma, the numbers M_k, $k=0, 1, \cdots$, are well defined, i. e. they do not depend on the special choice of the real number ε, If all critical points of a C^2-function, satisfying the P. S. condition, are S-nondegenerate with finite indices, then

$$\begin{aligned}H_k(f_{C_i+\varepsilon}, f_{C_i-\varepsilon})&=H_k(f_{C_i-\varepsilon}\cup h_1(D^{m_1})\cup\cdots\cup h_l(D^{m_l}), f_{C_i-\varepsilon})\\&=H_k\Big(\bigcup_{j=1}^{l}h_j(D^{m_j}), \bigcup_{j=1}^{l}h_j(\partial D^{m_j})\Big)\\&=\oplus\sum_{j=1}^{l}H_k(h_j(D^{m_j}; h_j(\partial D^{m_j}))\\&=\oplus\sum_{j=1}^{l}H_k(D^{m_j}, \partial D^{m_j})=\oplus\sum_{j=1}^{l}\delta_{km_j}G,\end{aligned}$$

where $(m_1, \cdots, m_l)$ are indices corresponding to the critical points on the level c_i; and G is the coefficient group. This provides a geometrical interpretation of the Morse type numbers. In this case, M_k is the number of critical points, whose indices are equal to k

Followed by the exactness of the homology sequence, one obtains the following Morse inequalities

$$M_0\geqslant\beta_0,$$
$$M_1-M_0\geqslant\beta_1-\beta_0.$$
$$\cdots \quad \cdots$$

$$M_n - M_{n-1} + \cdots + (-1)^n M_0 \geqslant \beta_n - \beta_{n-1} + \cdots + (-1)^n \beta_0,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

where $\beta_0, \beta_1, \cdots$, are the Betti numbers of the manifold $f^{-1}[a, b]$, i. e.

$$\beta_k = \operatorname{rank} H_k(f_b, f_a).$$

The three critical point theorem reads as follows:

Theorem 2. *Suppose that $\mathscr{X}$ is a Banach space with an equivalent differentiable norm. Suppose that f is a C^2 function, satisfying the P. S. condition. Assume that*

(1) *f is bounded from below,*

(2) *θ is a s-nondegenerate critical point, which is not a global minimum, but with finite index,*

then f has at least three critical points.

Proof The function f is bounded below, and satisfies the P. S. condition. The global minimum exists, say $m = \min\{f(x) \mid x \in \mathscr{X}\} = f(x_0)$. No loss of generality, we may assume that there is only one global minimum. For otherwise the proof is through.

Suppose that f has no critical points other than θ and x_0 Let $c = f(\theta)$.

Taking $b > c$ arbitrarily, there is no critical point outside f_b.

For any pair of topological spaces (X, Y) with $Y \subset X$, let $\chi(X, Y)$ be the Euler characteristic of the topological space pair, i. e.

$$\chi(X, Y) = \sum_{k=0}^{\infty} (-1)^k \operatorname{rank} H_k(X, Y).$$

It is known that for $Z \subset Y \subset X$, if both $\chi(Y, Z)$ and $\chi(X, Y)$ are finite, then

$$\chi(X, Z) = \chi(X, Y) + \chi(Y, Z). \qquad (**)$$

Now we take $(X, Y, Z) = (f_b, f_{c-\varepsilon}, \emptyset)$, where $c > m + \varepsilon$. According to the handle-body decomposition

$$\chi(f_b, f_{c-\varepsilon}) = \sum_{k=0}^{\infty} (-1)^k \operatorname{rank} H_k(f_b, f_{c-\varepsilon}) = (-1)^j,$$

where j is the index of f at θ. Since x_0 is a minimum, there is a small ball $B(x_0, \delta) \subset f_{c-\varepsilon}$. We have

$$\chi(f_{c-\varepsilon}, \emptyset) = \sum_{k=0}^{\infty} (-1)^k \operatorname{rank} H_k(f_{c-\varepsilon})$$

$$= \sum_{k=0}^{\infty} (-1)^k \operatorname{rank} H_k(B(x_0, \delta)) = 1.$$

But the left hand side of (**) equals to

$$\chi(f_b, \emptyset) = \sum_{k=0}^{\infty} (-1)^k \operatorname{rank} H_k(\mathscr{X}_b)$$

$$= \sum_{k=0}^{\infty} (-1)^k \operatorname{rank} H_k(\mathscr{X}) = 1,$$

because f_b is a deformation retract of $\mathscr{X}$.

§ 4. Application

In this section, we present an application of the above abstract theory to study the following PDE:

$$\begin{cases} c\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)+\Delta u=g(u) \text{ in } \Omega, \\ u|_{\partial\Omega}=0, \end{cases} \tag{*}$$

where Ω is a bounded open domain in $\mathbf{R}^n$, with sufficiently smooth boundary $\partial\Omega$, $c>0$ is a constant, and $p\geqslant 2$. The function g satisfies the following conditions:

(1) $g\in C^1(\mathbf{R}^1)$ with $g(0)=0$;

(2) $|g'(t)|\leqslant\begin{cases} C_2+C_1|t|^{\alpha-1}, \text{ if } p\leqslant n \\ \text{no restriction in growth, if } p>n, \end{cases}$

where
$$\alpha<\frac{np}{n-p}-1;$$

(3) Let $G(u)=\int_0^u g(t)dt$, we assume that

$$G(u)\geqslant -C_4|u|^p-C_3 \quad \text{with } C_4<\frac{C}{p}C_p^1,$$

where C_p^1 is the best constant such that

$$\int_\Omega|\nabla u|^p\geqslant C_p^1\int_\Omega|u|^p,\ \forall u\in C_0^\infty(\Omega);$$

(4) Let the eigenvalues of the$-\Delta$ with o–Dirichlet data on Ω be $\lambda_1<\lambda_2\leqslant\lambda_3\leqslant\cdots\cdots$. We assume that there is an integer $i\geqslant 1$ such that

$$\lambda_i<-g'(0)<\lambda_{i+1}.$$

Let us take the Sobolev space $\overset{\circ}{W}{}_p^1(\Omega)$, $p\geqslant 2$, and consider the functional on $\overset{\circ}{W}{}_p^1(\Omega)$

$$f(u)=\frac{c}{p}\int|\nabla u|^p+\frac{1}{2}\int(\nabla u)^2+\int G(u).$$

Obviously, θ is a critical point of f. In order to verify the s–nondegeneracy, we should check the following inequality

$$\langle df(u),\ Lu\rangle>0, \text{ for } u\in B(\theta,\ \delta)\backslash\theta,\ \delta>0 \text{ small},$$

where L is an hyperbolic operator from $\overset{\circ}{W}{}_p^1(\Omega)$ into itself. To this end, some estimates are needed. We devided them into the following lemmas.

Lemma 4.1. *There exist constants*

$$\beta=\beta(n,\ p)\in(0,\ 1],\ \text{and } M=M(n,\ p)>0,$$

such that

Note: Hereafter we write

$$|\nabla u|^p=\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^p$$

$$\left(\int_\Omega |u|^p \Big/ \int_\Omega |\nabla u|^p\right) \leqslant M\left\{\left(\int_\Omega |u|^2\right) \Big/ \left(\int_\Omega |\nabla u|^2\right)\right\}^{\frac{\beta p}{2}}$$

for all $u \in \overset{\circ}{W}{}_p^1(\Omega)\setminus\theta$, *where* $p \geqslant 2$.

Proof Provided by the well known Gagliardo-Nirenberg inequality, we have

$$\beta = \frac{2p}{np-2n+2p},$$

and a constant $C = C_{p,n}$ such that

$$\left(\int |u|^p\right)^{\frac{1}{p}} \leqslant C\left(\int |\nabla u|^p\right)^{\frac{1-\beta}{p}} \left(\int |u|^2\right)^{\frac{\beta}{2}}. \tag{1}$$

Combining (1) with the Hölder inequality

$$\left(\int |\nabla u|^2\right)^{\frac{1}{2}} \leqslant C'\left(\int |\nabla u|^p\right)^{\frac{1}{p}},$$

we obtain the desired inequality

Let E_k be the eigenspace of $-\Delta$ corresponding to the first k eigenvalues $\lambda_1, \cdots \lambda_k$. Then E_k is finitely dimensional. Let P_k be the orthogonal projection onto E_k in $L^2(\Omega)$, then it can be restricted on $\overset{\circ}{W}{}_p^1(\Omega)$, and maps onto $E_k \subset \overset{\circ}{W}{}_p^1(\Omega)$. Let K be the operator $(-\Delta)^{-1}$.

Lemma 4.2. *The functional*

$$J(u) = \int_\Omega \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^{p-1} \left|\frac{\partial}{\partial x_i} Ku\right|$$

is defined on $\overset{\circ}{W}{}_p^1(\Omega)$. *For each* $\varepsilon > 0$, *there is an integer* $N > 0$ *such that*

$$J(\widetilde{P}_n u) \leqslant \varepsilon \int |\nabla \widetilde{P}_n u|^p, \quad \forall u \in \overset{\circ}{W}{}_p^1(\Omega)$$

as $n > N$, *where* $\widetilde{P}_n = I - P_n$.

Proof According to the Hölder inequality and the L^p a priori estimates

$$J(u) \leqslant \sum_{i=1}^n \left(\int \left|\frac{\partial}{\partial x_i} u\right|^p\right)^{\frac{p-1}{p}} \left(\int \left|\frac{\partial}{\partial x_i} Ku\right|^p\right)^{\frac{1}{p}}$$

$$\leqslant C\left(\int |\nabla u|^p\right)^{\frac{p-1}{p}} \|Ku\|_{W_p^1} \leqslant C\|u\|_{W_p^1}^{p-1} \|u\|_{L^p},$$

where, and hereafter, we denote by C various constants. Hence $J(u)$ is defined on $\overset{\circ}{W}{}_p^1(\Omega)$. And we have

$$J(\widetilde{P}_n u) \leqslant C\left(\int |\nabla \widetilde{P}_n u|^p\right)^{\frac{p-1}{p}} \left(\int |\widetilde{P}_n u|^p\right)^{\frac{1}{p}}.$$

Due to the Lemma 4.1, we obtain

$$J(\widetilde{P}_n u) \leqslant C\int |\nabla \widetilde{P}_n u|^p \left\{\int |\widetilde{P}_n u|^2 \Big/ \int |\nabla \widetilde{P}_n u|^2\right\}^{\frac{\beta}{2}} \leqslant \frac{C}{\lambda_{n+1}^{\beta/2}} \int |\nabla \widetilde{P}_n u|^p.$$

Since $\lambda_n \to +\infty$ this proves the lemma

Lemma 4.3. *Let g be a real function, satisfying the conditions* (1) *and* (2). *Then for each* $\varepsilon > 0$, *there exists* $\delta>0$ *such that*

$$\left|\int_\Omega [g(u)-g'(0)u]v\right| \leqslant \varepsilon[\|u\|_{W_2^1}\|v\|_{W_2^1}+\|u\|_{W_p^1}^{p-1}\cdot\|v\|_{W_p^1}] \tag{2}$$

for all $u, v\in \overset{\circ}{W}{}_p^1(\Omega)$, *with* $\|u\|_{W_p^1}<\delta$, $p\geqslant 2$.

Proof If $p>n$ according to the Sobolev embedding theorem, then we have

$$\left|\int [g(u)-g'(0)u]v\right| \leqslant \varepsilon\|u\|_{L^2}\cdot\|v\|_{L^2} \quad \text{as} \|u\|_{W_p^1}<\delta,$$

(2) is obviousiy true. In the sequel, we assume $p\leqslant n$. Since $g\in C^1$, $\forall\varepsilon_1>0$ there is a $\eta=\eta(\varepsilon_1)>0$ such that

$$|g(u)-g'(0)u|<\varepsilon_1|u| \text{ for } |u|<\eta.$$

Let E_η be the set $\{x\in\Omega|\ |u(x)|<\eta\}$, and $E'_\eta=\Omega\backslash E_\eta$, it is easily seen that

$$\operatorname{mes}(E'_\eta)\leqslant (C\|u\|_{W_p^1}/\eta)^p$$

and

$$\left|\int_{E_\eta}[g(u)-g'(0)u]v\right|\leqslant\varepsilon_1\int_{E_\eta}|uv|\leqslant\varepsilon_1 C\|u\|_{W_2^1}\|v\|_{W_2^1}. \tag{3}$$

Applying the inequality

$$|g(u)-g'(0)u|\leqslant C(|1+|u|^\alpha),$$

where

$$\alpha+1<q:=\frac{np}{n-p},$$

we obtain

$$\left|\int_{E'_\eta}[g(u)-g'(0)u]v\right|\leqslant C\left[\int_{E'_\eta}|v|+\int_{E'_\eta}|u|^\alpha|v|\right].$$

However

$$\begin{aligned}\int_{E'_\eta}|u|^\alpha|v| &\leqslant\left(\int_{E'_\eta}|u|^{\alpha q'}\right)^{\frac{1}{q'}}\left(\int_{E'_\eta}|v|^q\right)^{\frac{1}{q}}\\ &\leqslant C\|u\|_{W_p^1}^\alpha\|v\|_{W_p^1}\operatorname{mes}(E'_\eta)^{\frac{1}{q'}-\frac{\alpha}{q}}\\ &\leqslant C\|u\|_{W_p^1}^{p-1}\|v\|_{W_p^1}\|u\|_{W_p^1}^{(\alpha+1)(1-\frac{p}{q})}/\eta^{p(1-\frac{\alpha+1}{q})}.\end{aligned} \tag{4}$$

Similarly, we have

$$\int_{E'_\eta}|v|\leqslant C\|u\|_{W_p^1}^{p-1}\|v\|_{W_p^1}\|u\|_{W_p^1}^{1-\frac{p}{q}}/\eta^{\frac{p}{q'}}.$$

For each $\varepsilon>0$ choose $\varepsilon_1<\frac{\varepsilon}{C}$ firstly determine η, and then fix it. Then we obtain $\delta>0$ such that (2) holds provided by (3) and (4).

Now we shall prove

Theorem 3. *Under the assumptions* (1)—(4) *on* g, *the equation* (*) *has at least three solutions for* $p>2$, $c>0$.

Proof Let $\mathscr{X}$ be the Banach space $\overset{\circ}{W}{}_p^1(\Omega)$, and let

$$f(u)=\frac{c}{p}\int|\nabla u|^p+\frac{1}{2}\int(\nabla u)^2+\int G(u),$$

we shall verify a series of conditions on f, which are needed in applying the Theorem2.

1. $\mathscr{f}\in C^2$. We denote by $\langle\ ,\ \rangle$ the duality between $\mathring{W}_p^1$ and its dual space. It is easily proved that

$$\langle d\mathscr{f}(u),\ v\rangle=\int\sum_{i=1}^{n}\left(c\left|\frac{\partial u}{\partial x_i}\right|^{p-2}+1\right)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}+g(u)v.$$

Applying the Lemma 4.3, we get

$$d^2\mathscr{f}(u)(v,\ w)=\int\sum_{i=1}^{n}\left(c(p-1)\left|\frac{\partial u}{\partial x_i}\right|^{p-2}+1\right)\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}+g'(u)vw.$$

2. $\mathscr{f}$ is bounded from below. This is due to the condition (3)

$$\mathscr{f}(u)\geqslant\frac{c}{p}\int|\nabla u|^p+\frac{1}{2}\int(\nabla u)^2-C_4\int|u|^p-C_3\ \mathrm{mes}(\Omega)$$
$$\geqslant\frac{1}{2}\int(\nabla u)^2-C_3\ \mathrm{mes}(\Omega)+\left(\frac{c}{p}-\frac{C_4}{C_p'}\right)\int\|\nabla u|^p. \qquad (5)$$

3. $\mathscr{f}$ satisfies the P. S. condition.

The operator $K=(-\Delta)^{-1}\in\mathscr{L}(W_r^{-1},\ \mathring{W}_r^1)$, and then is compact in $\mathscr{L}(\mathring{W}_r^1,\ \mathring{W}_r^1)$ for $1<r<\infty$, The eigenvalues of K are $\frac{1}{\lambda_1}>\frac{1}{\lambda_2}\geqslant\cdots\geqslant\frac{1}{\lambda_i}\geqslant\cdots$. Let

$$R(u)=cK\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)+u$$

and let

$$\mathscr{G}:\ u\mapsto g(u).$$

According to the sequences

$$\mathring{W}_p^1\longrightarrow L^{q_1}\xrightarrow{\mathscr{G}}L^{q_1'}\longrightarrow W_{p'}^{-1}\longrightarrow\mathring{W}_{p'}^1,$$

and

$$\mathring{W}_p^1\xrightarrow{\sum\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)}W_{p'}^{-1}\xrightarrow{K}\mathring{W}_{p'}^1,$$

where $q_1<q$, $\alpha=q_1/q_1'$, $\frac{1}{q_1}+\frac{1}{q_1'}=1$, and $\frac{1}{p}+\frac{1}{p'}=1$, we have

$$\langle d\mathscr{f}(u),\ v\rangle=\langle Ru+K\mathscr{G}(u),\ v\rangle.$$

Suppose that $\{u_n\}$ is a sequence in $\mathring{W}_p^1$ such that

$$\begin{cases}\mathscr{f}(u_n)\text{ is bounded, and}\\ R(u_n)+K\mathscr{G}(u_n)\longrightarrow\theta\ (\text{in }\mathring{W}_{p'}^1).\end{cases}$$

Then $\{u_n\}$ is bounded in $\mathring{W}_p^1$, provided by (5). We have a subsequence $\{u_{n_i}\}$ such that $K\mathscr{G}(u_{n_i})$ is strongly convergent in $\mathring{W}_{p'}^1$ say to w^*. This implies that

$$R(u_{n_i})\longrightarrow-w^*\quad(\text{in }\mathring{W}_{p'}^1).$$

However, the operator R is strongly monotone

$$\langle R(u)-R(v),\ u-v\rangle=\int_\Omega\sum_{i=1}^{n}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}-\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\right)\times\frac{\partial u}{\partial x_i}-\frac{\partial v}{\partial x_i}\Big)dx$$
$$\geqslant\frac{1}{2}p-2\int|\nabla(u-v)|^p,$$

we conclude that $\{u_{n_i}\}$ converges strongly in $\mathring{W}_p^1$. The P. S. condition is verified.

4. Now we define an operator

$$L=I+g'(0)K\in\mathscr{L}(\mathring{W}_p^1,\ \mathring{W}_p^1).$$

The spectra of L are as follows

$$1+\frac{g'(0)}{\lambda_1}<1+\frac{g'(0)}{\lambda_2}\leqslant\cdots\leqslant 1+\frac{g'(0)}{\lambda_i}\leqslant\cdots.$$

L is an hyperbolic operator. It is easily seen that

$$d^2f(\theta)(Lu,\ u)=\int[\nabla(Lu)\nabla u+g'(0)Lu\cdot u]$$

$$=\int|\nabla(Lu)|^2\geqslant\lambda_1\int(Lu)^2>0$$

for $u\neq\theta$, and that

$$d^2f(\theta)(Lu,\ v)=\int\nabla(Lu)\nabla(Lv)=d^2f(\theta)(u,\ Lv),$$

$\forall u,\ v\in\mathring{W}_p^1$. The index of $d^2f(\theta)$ then is equal to dim E_i.

5. The only thing remains to be proved is that $\langle df(u),\ Lu\rangle>0$ for $u\in B(\theta,\ \delta)\setminus\theta$, as $\delta>0$ small. In fact

$$\langle df(u),\ Lu\rangle=\int\Big[\sum_{i=1}^n\Big(c\Big|\frac{\partial u}{\partial x_i}\Big|^{p-2}+1\Big)\frac{\partial u}{\partial x_i}\frac{\partial(Lu)}{\partial x_i}+g(u)Lu\Big]$$

$$=\int|\nabla(Lu)|^2+c\int\sum_{i=1}^n\Big|\frac{\partial u}{\partial x_i}\Big|^{p-2}\frac{\partial u}{\partial x_i}$$

$$\times\frac{\partial}{\partial x_i}(Lu)+\int[g(u)-g'(0)u]Lu$$

$$=\int|\nabla(Lu)|^2+c\int|\nabla u|^p+cg'(0)J(u)+\int[g(u)-g'(0)u]Lu$$

$$\geqslant\nu\int(\nabla u)^2+c\int|\nabla u|^p-c|g'(0)|J(u)-\varepsilon[\|u\|^2_{W_2^1}+\|u\|^p_{W_p^1}]$$

as $\|u\|_{W_p^1}<\delta$, where

$$\nu=\min\Big\{1+\frac{g'(0)}{\lambda_{i+1}},\ -\Big(1+\frac{g'(0)}{\lambda_i}\Big)\Big\}.$$

$J(u)$ is defined in Lemma 4.2, and $\delta=\delta(\varepsilon)$ is defined in Lemma 4.3.

Choosing $\varepsilon\in\Big(0,\ \min\Big\{\frac{\nu}{2},\ \frac{c}{2}\Big\}\Big)$, we have

$$\langle df(u),\ Lu\rangle\geqslant\frac{\nu}{2}\|u\|^2_{W_2^1}+\frac{c}{2}\|u\|^p_{W_p^1}-c|g'(0)|J(u).$$

But

$$J(u)\leqslant 2^{p-1}\int\sum_{i=1}^n\Big[\Big|\frac{\partial P_Nu}{\partial x_i}\Big|^{p-1}+\Big|\frac{\partial\widetilde{P}_Nu}{\partial x_i}\Big|^{p-1}\Big]\Big[\Big|\frac{\partial KP_Nu}{\partial x_i}\Big|+\Big|\frac{\partial K\widetilde{P}_Nu}{\partial x_i}\Big|\Big]$$

$$\leqslant 2^{p-1}[J(P_Nu)+J(\widetilde{P}_Nu)+R_N(u)],$$

where $R_N(u)$ is the remainder. Let $u=\lambda v,\ \|v\|_{W_p^1}=1$, and λ is a scalar. It follows

$$J(u)\leqslant\lambda^p2^{p-1}[J(P_Nv)+J(\widetilde{P}_Nv)+R_N(v)].$$

and

$$|R_N(v)|\leqslant C(\|\widetilde{P}_Nv\|^{p-1}_{W^1_p}\|P_Nv\|_{W^1_p}+\|\widetilde{P}_Nv\|_{W^1_p}\|P_Nv\|^{p-1}_{W^1_p})$$
$$\leqslant C'(\|P_Nv\|^p_{W^1_p}+\|P_Nv\|_{W^1_p}).$$

Since
$$\|u\|^p_{W^1_p}\geqslant\frac{1}{2^p}\|\widetilde{P}_Nu\|^p_{W^1_p}-\|P_Nu\|^p_{W^1_p},$$
and
$$J(P_Nu)\leqslant C\|P_Nu\|^p_{W^1_p},$$
we obtain

$$\langle d f(u),\ Lu\rangle\geqslant\frac{\nu}{2}\|u\|^2_{W^1_2}+\frac{c}{2^{p+1}}\|\widetilde{P}_Nu\|^p_{W^1_p}-CJ(\widetilde{P}_Nu)$$
$$-\lambda^pC[\|P_Nv\|^p_{W^1_p}+\|P_Nv\|_{W^1_p}].$$

According to the Lemma 4.2, firstly, we choose N such that

$$\frac{c}{2^{p+1}}\|\widetilde{P}_Nu\|^p_{W^1_p}-CJ(\widetilde{P}_Nu)>0.$$

Fixing N, the norms $\|v\|_{W^1_2}$ and $\|v\|_{W^1_p}$ are equivalent in the finitely dimensional space $P_N\mathscr{X}$. At last, we arrive at

$$\langle d f(u),\ Lu\rangle\geqslant\frac{\nu}{2}\lambda^2\|P_Nv\|^2_{W^1_2}-\lambda^pC(\|P_Nv\|^p_{W^1_p}+\|P_Nv\|_{W^1_p})>0$$

for $\lambda=\|u\|_{W^1_p}$ small enough, say $\lambda<\delta$.

The proof is complete.

Remark. In case $p=2$, no loss of generality, we may assume $c=0$, because $\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right)=\Delta u$. In this case, the same result holds.

Since now $\mathring{W}^1_2$ is a Hilbert space, the verification is much easier than in Banach space, and only the theorem in [2] is applied.

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