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THE CHARACTER TABLE OF SYMMETRIC GROUPS

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Abstract

This paper gives two methods to construct simultaneously the character tables of the symmetric groups $\mathfrak{S}_{n-1}$, $\mathfrak{S}_n$ and $\mathfrak{S}_{n+1}$ with the aid of the character table of $\mathfrak{S}_{n-2}$ as well as the author's formulae cited in [9]. These methods require merely the rational operation of integers.

1. Introduction

Let $\chi_\rho^{(\lambda)}=\chi_{(1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n})}^{(\lambda_1,\lambda_2,\cdots,\lambda_m)}$ be the simple characteristic of the symmetric group $\mathfrak{S}_n$ on n letters, where $(\lambda)=(\lambda_1, \lambda_2, \cdots, \lambda_m)$ is a (general) partition of n, that is

$$\lambda_1+\lambda_2+\cdots+\lambda_m=n,\ \lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_m\geqslant 0,\ m\geqslant n,$$

and $\rho=(1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n})$ is a conjugate class with α_1 cycles of order 1, α_2 cycles of order 2, etc., and $\sum_{j=1}^{n} j\alpha_j=n$.

The numerical computation of $\chi_\rho^{(\lambda)}$ has been discussed by many authors (see [3]—[8]). In [9] and [10] we also introduced a practical method for calculating $\chi_\rho^{(\lambda)}$ by merely solving some systems of 2×2 nonhomogeneous linear equations that $\chi_\rho^{(\lambda)}$ must satisfy. The method is useful and easy to apply.

The purpose of this paper is to construct simultaneously the character tables for symmetric groups $\mathfrak{S}_{n-1}$, $\mathfrak{S}_n$, $\mathfrak{S}_{n+1}$ with the aid of the character table of $\mathfrak{S}_{n-2}$ and using the author's formulae cited in [9]. The problem for constructing the character table of $\mathfrak{S}_n$ by means of the character table of $\mathfrak{S}_{n-1}$ was considered by Littlewood, but he failed to solve it completely (cf. [1]).

The method presented in this paper requires merely the rational operations of integers, hence it is more convenient for use than the other methods given in [3]—[8].

2. Basic Theorems

Let $y_1, y_2, \cdots, y_m$ be m arbitrary real variables, and denote

$$S_k=\sum_{i=1}^{m} y_i^k,\ k=1,\ 2,\ \cdots,$$

$$V_m=V(y_1,\ y_2,\ \cdots,\ y_m)=\prod_{1\leqslant i<j\leqslant n}(y_j-y_i),$$

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$$D_t=\sum_{i=1}^{m} y_i^t \frac{\partial}{\partial y_i}$$

Lemma 1. $D_3V_m=(m-3/2)S_2+\left[\frac{1}{2}S_1^2\right]V_m.$

Proof Since V_m may be expressed as the following homogeneous anti-symmetric polynomial of degree $\frac{m(m-1)}{2}$

$$V_m=\sum_{(j_1j_2\cdots j_m)}(-1)^{N(j_1j_2\cdots j_m)}y_{j_1}^{m-1}y_{j_2}^{m-2}\cdots y_{j_{m-1}}^1y_{j_m}^0,$$

where

$$N(j_1j_2\cdots j_m)=\begin{cases}1, \text{ if } j_1j_2\cdots j_m \text{ is odd permutation,}\\ 0, \text{ if } j_1j_2\cdots j_m \text{ is even permutation,}\end{cases}$$

we have

$$D_3V_m=\sum_{i=1}^{u} y_{j_i}^3\frac{\partial}{\partial y_{j_i}}V_m=\sum_{i=1}^{m}(m-i)y_{j_i}^3\sum_{(j_1\cdots j_m)}(-1)^{N(j_1\cdots j_m)}y_{j_1}^{m-1}\cdots y_{j_i}^{m-i-1}\cdots y_{j_{m-1}}^1y_{j_m}^0$$

$$=\sum_{i=1}^{m}(m-i)\sum_{(j_1\cdots j_m)}(-1)^{N(j_1\cdots j_m)}y_{j_1}^{m-1}\cdots y_{j_i}^{m-i+2}\cdots y_{j_{m-1}}^1y_{j_m}^0. \quad (1)$$

By means of the homogeneity of the anti-symmetric polynomial, (1) may be simplified as

$$\begin{aligned}D_3V_m=&(m-1)\sum_{(j_1\cdots j_m)}(-1)^{N(j_1\cdots j_m)}y_{j_1}^{m+1}y_{j_2}^{m-2}\cdots y_{j_{m-1}}^1y_{j_m}^0\\&+(m-2)\sum_{(j_1\cdots j_m)}(-1)^{N(j_1\cdots j_m)}y_{j_1}^{m-1}y_{j_2}^{m}y_{j_3}^{m-3}\cdots y_{j_{m-1}}^1y_{j_m}^0\\=&(m-1)\sum_{(j_1\cdots j_m)}(-1)^{N(j_1\cdots j_m)}y_{j_1}^{m+1}y_{j_2}^{m-2}\cdots y_{j_{m-1}}^1y_{j_m}^0\\&-(m-2)\sum_{(j_1\cdots j_m)}(-1)^{N(j_1\cdots j_m)}y_{j_1}^{m}y_{j_2}^{m-1}y_{j_3}^{m-3}\cdots y_{j_{m-1}}^1y_{j_m}^0.\end{aligned} \quad (2)$$

It is easy to varify that every homogeneous anti-symmetric polynomial can be expressed as the product of V_m and a symmetric polynomial in $S_1, S_2, \cdots, S_m$. Hence we obtain

$$\sum_{(j_1\cdots j_m)}(-1)^{N(j_1\cdots j_m)}y_{j_1}^{m+1}y_{j_2}^{m-2}y_{j_{m-1}}^{1}y_{j_m}^0=S_2V_m+\sum_{(j_1\cdots j_m)}(-1)^{N(j_1\cdots j_m)}y_{j_1}^{m}y_{j_2}^{m-1}y_{j_3}^{m-3}\cdots y_{j_{m-1}}^1y_{j_m}^0. \quad (3)$$

On the other hand

$$\sum_{(j_1\cdots j_m)}(-1)^{N(j_1\cdots j_m)}y_{j_1}^{m}y_{j_2}^{m-1}y_{j_3}^{m-3}\cdots y_{j_{m-1}}^1y_{j_m}^0=\frac{1}{2}(S_1^2-S_2)V_m. \quad (4)$$

Lemma 1 follows from (2), (3) and (4).

For any positive integer n and every conjugate class

$$\rho=(1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}),\ \sum_{j=1}^{n}j\alpha_j=n,$$

let

$$S_\rho=S_1^{\alpha_1}S_2^{\alpha_2}\cdots S_n^{\alpha_n}.$$

Let $\sigma=(1^{\beta_1}2^{\beta_2}\cdots(n-2)^{\beta_{n-2}})$ be a conjugate class of $\mathfrak{S}_{n-2}$.

We define

$$\sigma^{[1]}=(1^{\beta_1-1}2^{\beta_2}3^{\beta_3+1}4^{\beta_4}\cdots(n-2)^{\beta_{n-2}}),$$
$$\sigma^{[2]}=(1^{\beta_1}2^{\beta_2-1}3^{\beta_3}4^{\beta_4+1}\cdots(n-2)^{\beta_{n-2}}),$$
$$\cdots\cdots\cdots\cdots,$$
$$\sigma^{[n-4]}=(1^{\beta_1}2^{\beta_2}\cdots(n-4)^{\beta_{n-4}-1}(n-3)^{\beta_{n-3}}(n-2)^{\beta_{n-2}+1}),$$
$$\sigma^{[n-3]}=(1^{\beta_1}2^{\beta_2}\cdots(n-4)^{\beta_{n-4}}(n-3)^{\beta_{n-3}-1}(n-2)^{\beta_{n-2}}(n-1)),$$
$$\sigma^{[n-2]}=(1^{\beta_1}2^{\beta_2}\cdots(n-3)^{\beta_{n-3}}(n-2)^{\beta_{n-2}-1}n).$$

It is easy to see that $\sigma^{[i]}$ is one of the conjugate classes of $\mathfrak{S}_n$ if $\beta_i\geqslant 1$, $1\leqslant i\leqslant n-2$. Put

$$\sigma^{(2)}=(1^{\beta_1}2^{\beta_2+1}3^{\beta}\cdots(n-2)^{\beta_{n-2}}),$$
$$\sigma^{(1^2)}=(1^{\beta_1+2}2^{\beta_2}3^{\beta_3}\cdots(n-2)^{\beta_{n-2}}).$$

Then $\sigma^{(2)}$ and $\sigma^{(1^2)}$ are also conjugate classes of $\mathfrak{S}_n$.

If we define $S_{\sigma^{(i)}}=0$ when $\beta_i=0$ for some i's, then we have

Lemma 2. $D_3S_\sigma=\sum_{k=1}^{n-2}k\beta_kS_{\sigma^{[k]}}$.

Proof Since

$$\frac{\partial}{\partial y_i}S_\sigma=\frac{\partial}{\partial y_i}(S_1^{\beta_1}S_2^{\beta_2}\cdots S_{n-2}^{\beta_{n-2}})=\sum_{k=1}^{n-2}k\beta_ky_i^{k-1}S_1^{\beta_1}\cdots S_{k-1}^{\beta_{k-1}}S_k^{\beta_k-1}S_{k+1}^{\beta_{k+1}}\cdots S_{n-2}^{\beta_{n-2}},$$

we have

$$D_3S_\sigma=\sum_{i=1}^{m}\sum_{k=1}^{n-2}k\beta_ky_i^{k+2}S_1^{\beta_1}\cdots S_k^{\beta_k-1}\cdots S_{n-2}^{\beta_{n-2}}$$
$$=\sum_{k=1}^{n-2}k_{\beta_k}\left(\sum_{j=1}^{m}y_j^{k+2}\right)S_1^{\beta_1}\cdots S_k^{\beta_k-1}\cdots S_{n-2}^{\beta_{n-2}}=\sum_{k=1}^{n-2}k\beta_kS_{\sigma^{[k]}}.$$

Let $(\lambda)=(\lambda_1, \lambda, \cdots, \lambda_m)$ be a (general) partition of $\mathfrak{S}_n$, for any positive integer t $(1\leqslant t\leqslant n)$, let

$$(\lambda_{[t]})_i=(\lambda_1, \lambda_2, \cdots, \lambda_i-t, \lambda_{i+1}, \cdots, \lambda_m),\ i=1, 2, \cdots, m.$$

The definition of the corresponding simple characteristic $\omega_\sigma^{(\mu)}$ is given in the same way as in [9], where σ is a conjugate class of $\mathfrak{S}_{n-2}$.

Theorem 1. *Let (μ) be a partition of $\mathfrak{S}_{n-2}$ and let $\omega_\sigma^{(\mu)}$ and $\chi_\rho^{(\lambda)}$ be the simple characteristics of $\mathfrak{S}_{n-2}$ and $\mathfrak{S}_n$ respectively. Then, for every conjugate class σ of $\mathfrak{S}_{n-2}$ and every partition (λ) of $\mathfrak{S}_n$, it holds that*

$$\chi_{\sigma^{(2)}}^{(\lambda)}=\sum_{i=1}^{m}\omega_\sigma^{(\lambda_{[2]})_i},\tag{5}$$

$$\frac{1}{2}\chi_{\sigma^{(1^2)}}^{(\lambda)}+\sum_{j=1}^{n-2}j\beta_j\chi_{\sigma^{[j]}}^{(\lambda)}=\sum_{j=1}^{m}\left(\lambda_i-i-\frac{1}{2}\right)\omega_\sigma^{(\lambda_{[2]})_i}.\tag{6}$$

Proof It is well known that the famous Frobenius formula

$$S_\sigma V_m=\sum_{(\mu)}\sum_{(j_1\cdots j_m)}(-1)^{N(j_1\cdots j_m)}\omega_\sigma^{(\mu)}y_{j_1}^{\mu_1+m-1}\cdots y_{j_i}^{\mu_i+m-i}\cdots y_{j_m}^{\mu_m}.\tag{7}$$

holds. From Lemma 1 and Lemma 2, we obtain

$$D_3(S_\sigma V_m)=S_\sigma\left[(m-3/2)S_2+\frac{1}{2}S_2^1\right]V_m+\left(\sum_{j=1}^{n-2}j\beta_jS_{\sigma^{[j]}}\right)V_m$$
$$=\left(m-\frac{3}{2}\right)S_{\sigma^{(2)}}V_m+\frac{1}{2}S_{\sigma^{(1^2)}}V_m+\sum_{j=1}^{n-2}j\beta_jS_{\sigma^{[j]}}V_m.\tag{8}$$

Applying the Frobenius formula to the right hand side of (8), (8) may be simplified as

$$D_3(S_\sigma V_m)=\sum_{(\lambda)}\sum_{(j_1\cdots j_m)}(-1)^{N(j_1\cdots j_m)}\left[\left(m-\frac{3}{2}\right)\chi^{(\lambda)}_{\sigma^{(2)}}+\frac{1}{2}\chi^{(\lambda)}_{\sigma^{(1^2)}}+\sum_{j=1}^{n-2}j\beta_j\chi^{(\lambda)}_{\sigma^{(j)}}\right]\times y_{j_1}^{\lambda_1+m-1}\cdots y_{j_i}^{\lambda_i+m-i}\cdots y_{j_m}^{\lambda_m}. \tag{9}$$

On the other hand, we have

$$D_3\Big(\sum_{(\mu)}\sum_{(j_1\cdots j_m)}(-1)^{N(j_1\cdots j_m)}\omega_\sigma^{(\mu)}y_{j_1}^{\mu_1+m-1}\cdots y_{j_i}^{\mu_i+m-1}\cdots y_{j_m}^{\mu_m}\Big)$$
$$=\sum_{i=1}^{m}\sum_{(\mu)}\sum_{(j_1\cdots j_m)}(-1)^{N(j_1\cdots j_m)}(\mu_i+m-i)\,\omega_\sigma^{(\mu)}y_{j_1}^{\mu_1+m-1}\cdots y_{j_i}^{\mu_i+m-i+2}\cdots y_{j_m}^{\mu_m}. \tag{10}$$

Hence, act the differential operator D_3 on both sides of (7), from (9) and (10), we have

$$\sum_{\mu}\sum_{(j_1\cdots j_m)}(-1)^{N(j_1\cdots j_m)}\left[\left(m-\frac{3}{2}\right)\chi^{(\lambda)}_{\sigma^{(2)}}+\frac{1}{2}\chi^{(\lambda)}_{\sigma^{(1^2)}}+\sum_{j=1}^{n-2}j\beta_j\chi^{(\lambda)}_{\sigma^{(j)}}\right]y_{j_1}^{\lambda_1+m-1}\cdots y_{j_i}^{\lambda_i+m-i}\cdots y_{j_m}^{\lambda_m}$$
$$=\sum_{(\mu)}\sum_{i=1}^{m}(\mu_i+m-i)\sum_{(j_1\cdots j_m)}(-1)^{N(j_1\cdots j_m)}\omega_\sigma^{(\mu)}y_{j_1}^{\mu_1+m-1}\cdots y_{j_i}^{\mu_i+m-i+2}\cdots y_{j_m}^{\mu_m}. \tag{11}$$

For any (λ), compare the coefficents of $y_{j_1}^{\lambda_1+m-1}\cdots y_{j_m}^{\lambda_m}$ in (11). Note that if

$$\mu_1=\lambda_1,\ \cdots,\ \mu_{i-1}=\lambda_{i-1},\ \mu_i+2=\lambda_i,\ \mu_{i+1}=\lambda_{i+1},\ \cdots,\ \mu_m=\lambda_m,$$

or

$$(\mu)=(\lambda_1,\ \cdots,\ \lambda_{i-1},\ \lambda_i-2,\ \lambda_{i+1},\ \cdots,\ \lambda_m)=(\lambda_{[2]})_i,\ i=1,\ 2,\ \cdots,\ m,$$

then (μ) possesses terms of the form $y_{j_1}^{\lambda_1+m-1}\cdots y_{j_m}^{\lambda_m}$. Thus, for any (λ), we have

$$(m-3/2)\chi^{(\lambda)}_{\sigma^{(2)}}+\frac{1}{2}\chi^{(\lambda)}_{\sigma^{(1^2)}}+\sum_{j=1}^{n-2}j\beta_j\chi^{(\lambda)}_{\sigma^{(j)}}=\sum_{i=1}^{m}(\lambda_i+m-i-2)\,\omega_\sigma^{(\lambda_{[2]})_i}$$
$$=\sum_{i=1}^{n-2}(\lambda_i+m-i-2)\,\omega_\sigma^{(\lambda_{[2]})_i}.$$

As m is arbitrary, we may compare the coefficents of $m-3/2$ of the above equation and Theorem 1 follows.

Let $\rho=(1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n})$ be a conjugate class of $\mathfrak{S}_n$. Denote

$$\rho_{[2]}=(1^{\alpha_1}2^{\alpha_2-1}3^{\alpha_3}\cdots n^{\alpha_n}),\ \text{if } \alpha_2\geqslant 1;$$

and

$$\rho_k=(1^{\alpha_1}\cdots(k-2)^{\alpha_{k-2}+1}(k-1)^{\alpha_{k-1}}k^{\alpha_k-1}\cdots n^{\alpha_n}),\ \text{if } k\geqslant 3 \text{ and } \alpha_k\geqslant 1.$$

Theorem 2. *Let* $\chi_\rho^{(\lambda)}$ and $\omega_\sigma^{(\mu)}$ *be simple characteristics of* $\mathfrak{S}_n$ *and* $\mathfrak{S}_{n-2}$ *respectively. Then*

$$\chi_\rho^{(\lambda)}\equiv\chi^{(\lambda)}_{\rho^{(2)}_{[2]}}=\sum_{i=1}^{m}\omega_{\rho_{[2]}}^{(\lambda_{[2]})_i},\ \textit{if } k=2 \text{ and } \alpha_2\geqslant 1; \tag{12}$$

$$\chi_\rho^{(\lambda)}=\frac{1}{(k-2)(\alpha_{k-2}+1)}\left[\sum_{i=1}^{m}\left(\lambda_i-i-\frac{1}{2}\right)\omega_{\rho_k}^{(\lambda_{[2]})_i}-k(\alpha_k-1)\chi^{(\lambda)}{}_{\rho_k^{(k)}}\right.$$
$$\left.-\frac{1}{2}\chi^{(\lambda)}{}_{\rho_k^{(1^2)}}-\sum_{\substack{j=1\\ j\neq k-2,k}}^{n-2}j\alpha_j\chi^{(\lambda)}_{\rho_k^{(j)}}\right],\ \text{if } k\geqslant 3 \textit{ and } \alpha_k\geqslant 1. \tag{13}$$

Proof When $k=2$ and $\alpha_2\geqslant 1$, we put $\sigma=\rho_{[2]}$ in (5). Then it is evident that $\rho_{[2]}^{(2)}=\rho$, and hence (12) follows.

When $k \geqslant 3$ and $\alpha_k \geqslant 1$, we put $\sigma = \rho_n$, i. e.

$$(1^{\beta_1}2^{\beta_2}\cdots(n-2)^{\beta_{n-2}}) = (1^{\alpha_1}2^{\alpha_2}\cdots(k-2)^{\alpha_{k-2}+1}(k-1)^{\alpha_{k-1}}k^{\alpha_k-1}\cdots n^{\alpha_n}).$$

Then we have

$$\beta_{k-2} = \alpha_{k-2}+1,\ \beta_k = \alpha_k - 1;\ \beta_j = \alpha_j, \text{ when } j \neq k-2,\ k,$$

and it is evident that $\rho_k^{[k-2]} = \rho$, therefore (6) leads to

$$\frac{1}{2}\chi^{(\lambda)}{}_{\rho_k^{(12)}} + (k-2)(\alpha_{k-2}+1)\chi_\rho^{(\lambda)} + k(\alpha_k - 1)\chi^{(\lambda)}{}_{\rho_k^{\langle k\rangle}} + \sum_{\substack{j=1 \\ j\neq k-2,k}}^{n-2} j\alpha_j \chi^{(\lambda)}{}_{\rho_k^{\langle j\rangle}}$$

$$= \sum_{i=1}^{m}\left(\lambda_i - i - \frac{1}{2}\right)\omega_{\rho_k}^{(\lambda_{(2i)})}.$$

Hence (13) follows.

Lemma 3. $D_4 V_m = \left[\left(m - \frac{7}{3}\right)S_3 + \frac{3}{2} S_1 S_2 - \frac{1}{6} S_1^3\right] V_m.$

Proof It is easy to show that

$$D_4 V_m = (m-1) S_3 V_m + \sum_{(j_1\cdots j_m)} (-1)^{N(j_1\cdots j_m)} y_{j_1}^{m+1} y_{j_2}^{m-1} y_{j_3}^{m-3}\cdots y'_{j_{m-1}} y_{j_m}^0$$

$$- \sum_{(j_1\cdots j_m)} (-1)^{N(j_1\cdots j_m)} y_{j_1}^{m} y_{j_2}^{m-1} y_{j_3}^{m-2} y_{j_4}^{m-4}\cdots y'_{j_{m-1}} y_{j_m}^0$$

$$= (m-1) S_3 V_m + \left(\sum_{i\neq j} y_i^2 y_j\right) V_m - \left(\sum_{i\neq j\neq k} y_i y_j y_k\right) V_m$$

$$= \left[(m-2) S_3 + S_1 S_2 - \frac{1}{6}(S_1^3 - 3 S_1 S_2 + 2 S_3)\right] V_m$$

$$= \left[\left(m - \frac{7}{3}\right)S_3 + \frac{3}{2} S_1 S_2 - \frac{1}{6} S_1^3\right] V_m.$$

Let $\sigma = (1^{\beta_1}2^{\beta_2}\cdots(n-2)^{\beta_{n-2}})$ be a conjugate class of $\mathfrak{S}_{n-2}$, and put

$$\sigma^{\langle 1\rangle} = (1^{\beta_1-1}2^{\beta_2}3^{\beta_3}4^{\beta_4+1}\cdots(n-2)^{\beta_{n-2}}),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\sigma^{\langle n-5\rangle} = (1^{\beta_1}\cdots(n-5)^{\beta_{n-5}-1}(n-4)^{\beta_{n-4}}(n-3)^{\beta_{n-3}}(n-2)^{\beta_{n-2}+1}),$$

$$\sigma^{\langle n-4\rangle} = (1^{\beta_1}\cdots(n-4)^{\beta_{n-4}-1}(n-3)^{\beta_{n-3}}(n-2^{\beta_{n-2}}(n-1)),$$

$$\sigma^{\langle n-3\rangle} = (1^{\beta_1}\cdots(n-3)^{\beta_{n-3}-1}(n-2)^{\beta_{n-2}}n),$$

$$\sigma^{\langle n-2\rangle} = (1^{\beta_1}\cdots(n-3)^{\beta_{n-3}}(n-2)^{\beta_{n-2}-1}(n+1)).$$

When $\beta_i \geqslant 1$, $\sigma^{\langle i\rangle}$ is one of the conjugate classes of $\mathfrak{S}_{n+1}$. Put

$$\sigma^{(3)} = (1^{\beta_1}2^{\beta_2}3^{\beta_3+1}4^{\beta_4}\cdots(n-2)^{\beta_{n-2}}),$$

$$\sigma^{(12)} = (1^{\beta_1+1}2^{\beta_2+1}3^{\beta_3}\cdots(n-2)^{\beta_{n-2}}),$$

$$\sigma^{(1^3)} = (1^{\beta_1+3}2^{\beta_2}\cdots(n-2)^{\beta_{n-2}}).$$

Then $\sigma^{(3)}$, $\sigma^{(12)}$ and $\sigma^{(1^3)}$ are conjugate classes of $\mathfrak{S}_{n+1}$.

If we define $S_{\sigma^{\langle i\rangle}} = 0$ when $\beta_i = 0$ for some i's, then we have.

Lemma 4. $D_4 S_\sigma = \sum_{j=1}^{n-2} j\beta_j S_{\sigma^{\langle j\rangle}}.$

The proof is similar to that of Lemma 2.

Theorem 3. *Suppose $(\nu) = (\nu_1, \nu_2, \cdots, \nu_m)$ is a (general) partition and τ is a conjugate class of $\mathfrak{S}_{n+1}$. Let $\varphi_\tau^{(\nu)}$ and $\omega_\sigma^{(\mu)}$ be the simple characteristics of $\mathfrak{S}_{n+1}$ and $\mathfrak{S}_{n+1}$ respectively. Then, for every (ν) and every σ, we have*

$$\varphi^{(\nu)}_{\sigma^{(3)}}=\sum_{i=1}^{m}\omega_{\sigma}^{(\nu_{[3]})_i}, \tag{14}$$

$$\frac{3}{2}\varphi^{(\nu)}_{\sigma^{(12)}}-\frac{1}{6}\varphi^{(\nu)}_{\sigma^{(13)}}+\sum_{j=1}^{n}j\beta_j\varphi^{(\nu)}_{\sigma^{(j)}}=\sum_{i=1}^{m}\left(\lambda_i-i-\frac{3}{2}\right)\omega^{(\nu_{[3]})_i}, \tag{15}$$

where

$$(\nu_{[3]})_i=(\nu_1,\ \cdots,\ \nu_{i-1},\ \nu_i^{-3},\ \nu_{i+1},\ \cdots,\ \nu_m),\ i=1,\ 2,\ \cdots,\ m.$$

The proof is the same as that of Theorem 1 by using Lemmas 3 and 4.

For a conjugate class $\tau=(1^{\delta_1}\cdots k^{\delta_k}\cdots(n+1)^{\delta_{n+1}})$ of $\mathfrak{S}_{n+1}$, $\sum\limits_{j=1}^{n+1}j\delta_j=n+1$, put

$$\tau_{[3]}=(1^{\delta_1}2^{\delta_2}3^{\delta_3-1}4^{\delta_4}\cdots(n+1)^{\delta_{n+1}}),\ \text{if } k=3 \text{ and } \delta_3\geqslant 1,$$

and

$$\tau_k=(1^{\delta_1}\cdots(k-4)^{\delta_{k-4}}(k-3)^{\delta_{k-3}+1}(k-2)^{\delta_k-2}k^{\delta_k-1}\cdots(n+1)^{\delta_{n+1}}),\ \text{if } k\geqslant 4 \text{ and } \alpha_k\geqslant 1.$$

Then $\tau_{[3]}$ and all of the τ_k $(k\geqslant 4)$ are conjugate classes of $\mathfrak{S}_{n-2}$, and it is easy to see that

$$\tau_{[3]}^{(3)}=\tau,\ \tau_k^{(k-3)}=\tau.$$

If

$$\begin{aligned}\sigma&=(1^{\beta_1}2^{\beta_2}\cdots(n-2)^{\beta_{n-2}})\\&=(1^{\delta_1}\cdots(k-3)^{\delta_{k-3}+1}(k-2)^{\delta_{k-2}}(k-1)^{\delta_{k-1}}k^{\delta_k-1}\cdots(n+1)^{\delta_{n+1}})=\tau_k,\end{aligned}$$

then it is evident that

$$\beta_{k-3}=\delta_{k-3}+1,\ \beta_k=\delta_{k-1};\ \beta_j=\delta_j,\ \text{when } j\neq k,\ k-2.$$

Now take $\sigma=\tau_k$ in (15), and then we obtain

$$\begin{aligned}&\frac{3}{2}\varphi^{(\nu)}{}_{\tau_k^{(12)}}-\frac{1}{6}\varphi^{(\nu)}{}_{\tau_k^{(13)}}+(k-3)(\delta_{k-3}+1)\varphi^{(\nu)}_{\tau}+k(\delta_{k-1})\varphi^{(\nu)}{}_{\tau_k^{(k)}}+\sum_{j\neq k-3,k}j\alpha_j\varphi^{(\nu)}{}_{\tau_k^{(j)}}\\&=\sum_{i=1}^{m}\left(\lambda_i-i-\frac{2}{3}\right)\omega_{\tau_k}^{(\nu_{[3]})_i}.\end{aligned} \tag{16}$$

Taking $\sigma=\tau_{[3]}$ in (14) and combining it with (16), we arrive at the following.

Theorem 4. *Let* $\varphi^{(\nu)}_{\tau}$ *be the simple characteristic of* $\mathfrak{S}_{n+1}$, $\omega^{(\mu)}_{\sigma}$ *be hte simple characteristic of* $\mathfrak{S}_{n-2}$. Then

$$\varphi^{(\nu)}_{\tau}=\sum_{i=1}^{m}\omega_{[3]}^{(\nu_{[3]})_i},\ \textit{when } k=3 \textit{ and } \alpha_3\geqslant 1; \tag{17}$$

and

$$\begin{aligned}\varphi^{(\nu)}_{\tau}=\frac{1}{(k-3)(\delta_{k-3}+1)}\Bigg[&\sum_{i=1}^{m}\left(\lambda_i-i-\frac{2}{3}\right)\omega_{\tau_k}^{(\nu_{[3]})_i}+\frac{1}{6}\varphi^{(\nu)}{}_{\tau_k^{(13)}}\\&-\frac{3}{2}\varphi^{(\nu)}{}_{\tau_k^{(12)}}-k(\delta_{k-1})\varphi^{(\nu)}{}_{\tau_k^{(k)}}-\sum_{j\neq k-3,k}j\delta_j\varphi^{(\nu)}{}_{\tau_k^{(j)}},\ \Bigg],\ \textit{when } k\geqslant 4 \textit{ and } \alpha_a\geqslant 1.\end{aligned}$$

3. Two processes in constructing the character tables of $\mathfrak{S}_{n-1}$, $\mathfrak{S}_n$ and $\mathfrak{S}_{n+1}$.

The actual steps in constructing the character tables with the aid of the character table of $\mathfrak{S}_{n-2}$ may be summarised as follws:

(i) It is well known that the characters $\chi^{(\lambda)}_{\rho}$, in which the conjugats class ρ contains the cycles of order 1, may be calculated by using the character table of $\mathfrak{S}_{n-2}$. We then use (5) and (6) to calculate $\chi^{(\lambda)}_{\rho}$ in which the conjugate class ρ contains the cycles of order 2, order 3, ⋯ up to the cycles of order n, thus the character table of $\mathfrak{S}_n$ is completely constructed.

(ii) By using the author's reciprocal formulae for constructing the character tables of the symmetric group $\mathfrak{S}_m$ with the aid of the character table of $\mathfrak{S}_{m+t}$, $t>1$ (cf. [9]), we can easily write out the character table of $\mathfrak{S}_{n-1}$.

(iii) It is evident that the character $\varphi_\tau^{(\nu)}$ of $\mathfrak{S}_{n+1}$, in which the conjugate class τ contains the cycles of order 1, can be found immediately by using the character table of $\mathfrak{S}_n$, while $\varphi_\tau^{(\nu)}$ of $\mathfrak{S}_{n+1}$, in which the conjugate class τ contains the cycles of order 2, can be calculated by using the character table of $\mathfrak{S}_{n-1}$. Then we may repeatedly use (17) and (18) to calculate $\varphi_\tau^{(\nu)}$ in which the conjugate classes τ contain the cycles of order 3, order 4, ⋯ up to order $n+1$, hence the character table of $\mathfrak{S}_{n+1}$ may be completed.

Of course, we may also replace step (iii) by the following

(iii)′ Use (5) and (6) as well as the character tables of $\mathfrak{S}_{n-1}$ and $\mathfrak{S}_n$ to calculate the character table of $\mathfrak{S}_{n+1}$.

References

[1] Littlewood, D. E., The Theory of Group Character and Matrix Representations of Groups, Oxford, 2nd (1950).

[2] Murnaghan, F. D., The Theory of Group Representations, Baltimore (1983).

[3] Murnaghan, F. D., The Charcters of $\mathfrak{S}_n$, *Annals Acad. Brasil, Ci.* **23** (1951), 141—154.

[4] Murnaghan, F. D., On the Characters of $\mathfrak{S}_n$, *Proc. Nat. Acad. Sci.*, **41** (1953), 396—398.

[5] Foulkes, H. O., Monomial Symmetric Funtions, *S*-functions and Group Characters, *Proc. London Math. Soc.*, **2**:3 (1952), 45—49.

[6] Gamba, A., Sui Garratteri delle representazioni del gruppo simmetaico, *Atti. Acad. Naz. Lincei. Rend* **12** (1952), 167—169.

[7] Ingram, R. E., Some Characters of $\mathfrak{S}_n$, *Proc. Amer. Math. Soe.*, **1** (1950), 358—369.

[8] Specht, W., Die Charaktere der Symmetrischen Gruppe, *Math. Zeit.*, Bd. **73** (1960), 312—329.

[9] Tu Boxun (Tu Bao-xhiang), The Characters of Symmetric Group, *Acta Sci. Nat. Univ. Fudan*, **9**:**3** (1964), 313—323.

[10] Tu Boxun (Tu Bao-xhiang), Calculatihn of the character of Symmetric Group, *ibid.* **9**:4 (1964) 449—457.