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SOME RESULTS ON FIXED POINTS

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Abstract

In this paper, we first show several fixed point and or common fixed point theorems for point-valued and set-valued contractive type mappings in compact metric spaces, which satisfy very general contractive condition with strict inequality and may be discontinuous. Our theorems improve and generalize some main results in [1—4].

In the next place, we provide some new results for quite general orbitally contraction and quasi-contraction mappings. They improve, unify and extend some important results in [7—15].

§ 1. Introduction

Recently Chen and Shih[1] have proved the following

Theorem. *Let T be a self mapping on a compact metric space (X, d), such that for all $x, y \in X$, $x \neq y$,*

$$d(Tx, Ty) < \max\left\{d(x, y), \frac{1}{2}[d(x, Tx)+d(y, Ty)], \frac{1}{2}[d(x, Ty)+d(y, Tx)]\right\}. \tag{1}$$

Then T has a unique fixed point in X.

The condition (1) corresponds to the definition (20) in [2]. This theorem shows that the continuity of T is not necessary for the mapping defined by (20) in [2], provided that one adds the hypothesis of compactness to X.

Kasahara and Rhoades[3] generalized the theorem of [1] to a pair of mappings S and T which satisfy the definition (145) in [2]. Chen and Shih[4] extended the Theorem 1 of [3] to a pair of set-valued mappings.

In section 2 of this paper, we shall improve and generalize some main results obtained in [1—4].

In section 3 of this paper, we shall present some new fixed point theorems for orbitally contractions and quasi-contractions, which unify and improve a number of recent results obtained in [7—15].

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§ 2. Point-valued and set-valued contractive mappings

Let (X, d) be a nonempty metric space. $CL(X)$ denotes the set of all nonempty closed subsets of X. $D(A, x)=\inf\{d(y, x): y\in A\}$ is the distance from x to $A\subset X$. $H(\cdot, \cdot)$ denotes the Hausdorff distance on $CL(X)$ deduced by d.

A mapping $T: X\to X$ is said to be generalized point-valued contractive, if for all $x, y\in X$, $x\neq y$,

$$d(Tx, Ty)<\max\Big\{d(x, y),\ (x, Ty),\ d(y, Ty),\ \frac{1}{2}[d(x, Tx)+d(y, Ty)],\ \frac{1}{2}[d(x, Ty)+d(y, Tx)]\Big\}. \tag{2}$$

A mapping $F: X\to CL(X)$ is said to be generalized set-valued contractive, if for all $x, y\in X$, $x\neq y$,

$$H(Fx, Fy)<\max\Big\{d(x, y),\ D(Fy, x),\ D(Fy, y),\ \frac{1}{2}[D(Fx, x)+D(Fy, y)],\ \frac{1}{2}[D(Fy, x)+D(Fx, y)]\Big\}. \tag{3}$$

Remark 1. Clearly, any mapping satifying condition (1) is a generalized point-valued contractive mapping. The following example shows that a generalized point-valued contractive mapping need not satisfy condition (1).

Example[*] Let $X=\{(0, 2), (-1, 0), (0, 0), (1, 0)\}$. d denotes the usual metric in $\mathbf{R}^2$. Then (X, d) is a compact metric space. Let $T(0, 2)=(-1, 0)$, $T(-1, 0)=T(0, 0)=T(1, 0)=(1, 0)$. It is easy to check that T is a generalized contractive mapping and T doesn't satisfy condition (1).

Theorem 1. *Let (X, d) be a nonempty compact metric space. $F: X\to CL(X)$ is a generalized set-valued contractive mapping. Then F has a fixed point in X.*

Proof Consider the nonnegative function $D(Fx, x)$ on X. Let $\{x_n\}$ be a minimizing sequence for $D(Fx, x)$

$$\lim_{n\to\infty} D(Fx_n, x_n)=r=\inf\{D(Fx, x): x\in X\}.$$

Since Fx_n is compact, there exists $y_n\in Fx_n$ such that $d(y_n, x_n)=D(Fx_n, x_n)$. Since X is compact, we may assume that $\{y_n\}$ converges to a point $u\in X$. Let $A=\{n: x_n=u, n\in N\}$, where N denotes the set of all positive integers. If A is an infinite set, obviously u is a fixed point of F in X. If A is finite, we can assume $x_n\neq u$ for all $n\in N$.

Now suppose $D(Fu, u)>r$. Then there exists $\varepsilon>0$ such that $D(Fu, u)>r+5\varepsilon$. We can choose n so that

$$d(y_n, u)<\varepsilon \text{ and } D(Fx_n, x_n)<r+\varepsilon,$$

* I wish to thank Zhao Hanbin for supplying this example.

and hence we have

$$D(Fx_n, u) \leqslant d(y_n, u) < \varepsilon,$$

$$d(x_n, u) \leqslant d(x_n, y_n) + d(y_n, u) < D(Fx_n, x_n) + \varepsilon < r + 2\varepsilon,$$

and

$$D(Fu, x_n) \leqslant D(Fu, u) + d(x_n, u) < r + 2\varepsilon + D(Fu, u).$$

It follows from (3) that

$$\begin{aligned} D(Fu, u) &\leqslant D(Fx_n, u) + H(Fu, Fx_n) \\ &< \varepsilon + \max\Big\{d(u, x_n),\ D(Fx_n, u),\ D(Fx_n, x_n), \\ &\quad \frac{1}{2}[D(Fu, u) + D(Fx_n, x_n)],\ \frac{1}{2}[D(Fx_n, u) + D(Fu, x_n)]\Big\} \\ &< \varepsilon + \max\Big\{r + 2\varepsilon,\ \varepsilon,\ r + \varepsilon,\ \frac{1}{2}[r + \varepsilon + D(Fu, u)], \\ &\quad \frac{1}{2}[r + 3\varepsilon + D(Fu, u)]\Big\}, \end{aligned} \tag{4}$$

which yields $D(Fu, u) < r + 5\varepsilon$, a contradiction. Thus $D(Fu, u) \leqslant r$.

Now suppose $u \notin Fu$. By the compactness of Fu, there exists $z \in Fu$ such that

$$0 < D(Fu, u) = d(z, u).$$

It follows from (3) that

$$\begin{aligned} D(Fz, z) &\leqslant H(Fz, Fu) \\ &< \max\Big\{d(u, z),\ D(Fu, z),\ D(Fu, u),\ \frac{1}{2}[D(Fu, u) + D(Fz, z)], \\ &\quad \frac{1}{2}[D(Fz, u) + D(Fu, z)]\Big\} \\ &\leqslant \max\Big\{D(Fu, u),\ 0,\ D(Fu, u),\ \frac{1}{2}[D(Fu, u) + D(Fz, z)]\Big\}, \end{aligned}$$

which yields $r \leqslant D(Fz, z) < D(Fu, u) \leqslant r$, a contradiction. Therefore $u \in Fu$, i. e. u is a fixed point of F in X.

When F is a generalized point-valued contrative mapping in Theorem 1, we obtain the following.

Corollary 1. *Let (X, d) be a nonempty compact metric spcoe. T is a generalized point-valued contractive mapping on X. Then T has a unique fixed point u in X.*

Remark 2. Obviously, the theorem of [1] is a special case of Corollary 1, and the example shows that Corollary 1 is a proper generalization.

The following theorem is Theorem 5 in [4]. Here, we shall give a simpler proof.

Theorem 2. *Let (X, d) be a nonempty compact metric space. F and G are set-valued mappings of X in to $CL(X)$ such that for all $x, y \in X$ with $x \neq y$,*

$$H(Fx, Gy) < \max\Big\{d(x, y),\ \frac{1}{2}[D(Fx, x) + D(Gy, y)], \\ \frac{1}{2}[D(Gy, x) + D(Fx, y)]\Big\}. \tag{5}$$

Then F or G has a fixed point in X.

Proof Consider nonnegative function $D(Fx, x)$ on X. Let $\{x_n\}$ is a minimizing sequence for $D(Fx, x)$:

$$\lim_{n\to\infty} D(Fx_n, x_n)=r=\inf\{D(Fx, x): x\in X\}.$$

By the compactness of Fx_n, there exists $y_n\in Fx_n$ such that $d(y_n, x_n)=D(Fx_n, x_n)$. Since X is compact, we can assume $\{y_n\}\to u\in X$. If $A=\{n: x_n=u, n\in N\}$ is infinite, then $u\in Fu$. If A is finite, then we may assume $x_n\neq u$ for all $n\in N$. As in the proof of Theorem 1, we shows easily $D(Gu, u)\leqslant r$.

Now suppose $u\overline{\in} Gu$. By the compactness of Gu, there exists $z\in Gu$ such that $0<D(Gu, u)=d(z, u)$ and it follows from (5) that

$$\begin{aligned}D(Fz, z)&\leqslant H(Fz, Gu)\\&<\max\left\{d(z, u), \frac{1}{2}[D(Fz, z)+D(Gu, u)], \frac{1}{2}[D(Gu, z)+D(Fz, u)]\right\}\\&\leqslant\max\left\{D(Gu, u), \frac{1}{2}[D(Fz, z)+D(Gu, u)]\right\},\end{aligned}$$

which yields $r\leqslant D(Fz, z)<D(Gu, u)\leqslant r$, a contradiction. Hence $u\in Gu$, i. e. u is a fixed point of G in X.

Remark 3. Theorem 2 improves a lot of the known results. (e. g. see [4]).

Theorem 3. *Let (X, d) be a nonempty compact metric space without isolated points. $F, G: X\to CL(X)$ satisfy (5). If F or G is continuous from X to $CL(X)$, then F and G have a common fixed point in X.*

Proof Without loss of generality, we may assume that F is continuous. It is easy to prove that $D(Fx, x)$ is a nonnegative continuous function on X. By the compactness of X, there exists $u\in X$ such that

$$D(Fu, u)\leqslant D(Fx, x), \forall x\in X.$$

If $u\notin Fu$, by the compactness of Fu there exists $v\in Fu$ such that

$$0<D(Fu, u)=d(v, u).$$

By (5) we have

$$\begin{aligned}D(Gv, v)&\leqslant H(Gv, Fu)=H(Fu, Gv)\\&<\max\left\{d(u, v), \frac{1}{2}[D(Fu, u)+D(Gv, v)], \frac{1}{2}[D(Gv, u)+D(Fu, v)]\right\}\\&\leqslant\max\left\{D(Fu, u), \frac{1}{2}[D(Fu, u)+D(Gv, v)]\right\},\end{aligned}$$

which implies $D(Gv, v)<D(Fu, u)$.

If $v\notin Gv$, then using the same argument as in the above proof there exists $w\in Gv$ such that

$$D(Fu, u)\leqslant D(Fw, w)<D(Gv, v)<D(Fu, u).$$

This is a contradiction. Hence either $u\in Fu$ or $v\in Gv$.

1° If $u\in Fu$, there exists a sequence $\{x_n\}$ in $X\setminus\{u\}$ converging to u since u is not isolated. By the continuity of F we have

$$D(Fx_n, u) \leqslant H(Fx_n, Fu) \to 0, \; n \to \infty.$$

Hence for any given $\varepsilon > 0$ there exists n such that

$$d(x_n, u) < \varepsilon \text{ and } D(Fx_n, u) < \varepsilon.$$

From (5) we obtain

$$\begin{aligned} D(Gu, u) &\leqslant D(Fx_n, u) + H(Fx_n, Gu) \\ &< \varepsilon + \max\left\{d(x_n, u), \; \frac{1}{2}[D(Fx_n, x_n) + D(Gu, u)], \; \frac{1}{2}[D(Gu, x_n) + D(Fx_n, u)]\right\} \\ &< \varepsilon + \max\left\{\varepsilon, \; \frac{1}{2}[2\varepsilon + D(Gu, u)]\right\}, \end{aligned}$$

which implies $D(Gu, u) < 4\varepsilon$. Since ε is arbitrary, $D(Gu, u) = 0$, i. e. $u \in Gu$, and so u is a common fixed point of F and G in X.

2° If $v \in Gv$, there exists a sequence $\{x_n\}$ in $X \setminus \{v\}$ converging to v since v is not isolated. By the continuity of F we have

$$H(Fx_n, Fv) \to 0, \; n \to \infty.$$

By (5) we have

$$\begin{aligned} D(Fx_n, v) &\leqslant H(Fx_n, Gv) \\ &< \max\left\{d(x_n, v), \; \frac{1}{2}[D(Fx_n, x_n) + D(Gv, v)], \; \frac{1}{2}[D(Gv, x_n) + D(Fx_n, v)]\right\} \\ &\leqslant \max\left\{d(x_n, v), \; \frac{1}{2}[D(Fx_n, v) + d(x_n, v)]\right\}, \end{aligned}$$

which yields $D(Fx_n, v) < d(x_n, v) \to 0$, $n \to 0$. Then we have

$$\begin{aligned} D(Fv, v) &\leqslant D(Fx_n, v) + H(Fx_n, Fv) \\ &< d(x_n, v) + H(Fx_n, Fv) \to 0, \; n \to \infty. \end{aligned}$$

Hence $D(Fv, v) = 0$, i. e. $v \in Fv$ and so v is a common fixed point of F and G in X.

Corollary 2. *Let (X, d) be a nonempty compact metric space. $S, T: X \to X$ are point-valued self mappings on X. If for all $x, y \in X$ with $x \neq y$*

$$d(Sx, Ty) < \max\left\{d(x, y), \; \frac{1}{2}[d(x, Sx) + d(y, Ty)], \; \frac{1}{2}[d(x, Ty) + d(y, Sx)]\right\} \tag{6}$$

and either S or T is continuous, then each of S and T has a unique fixed point and these two points coincide.

Proof By Corollary 2 S and T have a common fixed point u in X. The uniqueness of u easily follows from (6).

Remark 4. Corollary 2 is Theorem 2 in [3], But it seems to us that the proof in [3] is not yet complete, since one cann't deduce a contradiction from $d(TSu, Su) < d(Su, u)$ and so cann't obtain $Su = u$.

In the following, we discuss the iteration of approximating fixed point.

Lemma 1.[6, p. 68] *Let (X, d) be a nonempty compact metric space. $\{x_n\}$ is a sequence in X. If u is the unique cluster point of $\{x_n\}$, then u is the limit point of $\{x_n\}$.*

Theorem 4. *Let S and T be continuous point-valued self mappings of a nonempty compact metric space (X, d). If for all $x, y \in X$ with $x \neq y$*

$$d(Sx, Ty) < \max\left\{d(x, y),\ d(x, Sx),\ d(y, Ty),\ \frac{1}{2}[d(x, Ty) + d(y, Sx)]\right\}, \tag{7}$$

then (i) *S and T have a unique common fixed point u.*

(ii) *for each $x \in X$ the sequence of iteration*

$$\{x_n\} = \{x_0 = x,\ x_1 = Sx_0,\ x_2 = Tx_1,\ \cdots,\ x_{2n} = Tx_{2n-1},\ x_{2n+1} = Sx_{2n},\ \cdots\}$$

converges to u.

Proof Since X is compact, for each $x \in X$ the sequence of iteration $\{x_n\}$ has at least a cluster point u. By Theorem 3 in [5] each of the cluster points of $[x_n\}$ is both a common fixed point of S and T. Hence u is a common fixed point of S and T. Using (7) we prove easily that u is the unique fixed point of S and T respectively and so u is the unique common fixed point of S and T.

On the other hand, since every cluster point of $\{x_n\}$ is both a common fixed point of S and T and the common fixed point u of S and T is unique, u is the unique cluster point of $\{x_n\}$. It follows from Lemma 1 that $\{x_n\}$ converges to u.

Remark 5. Obviously Theorem 4 improves and generalizes Theorem 6 of Chen and Shih in [4].

As an immediate consequence of Theorem 4, we have.

Theorem 5. *Let (X, d) be a nonempty compact metric space, $\mathscr{F}$ a family of continuous point-valued mappings of X into itself. If for all distinct $S, T \in \mathscr{F}$ and for all $x, y \in X$ with $x \neq y$ (7) holds, then*

(i) *$\mathscr{F}$ has a unique common fixed point u in X,*

(ii) *for each $x \in X$ and any distinct $S, T \in \mathscr{F}$ the sequence of iteration*

$$\{x_n\} = \{x = x_0,\ x_1 = Sx_0,\ x_2 = Tx_1,\ \cdots,\ x_{2n+1} = Sx_{2n},\ x_{2n+2} = Tx_{2n+1},\ \cdots\}$$

converges to u.

§ 3. Orbitally contractions and quasi-contractions

In this section ω denotes the set of all nonnegative integers, I^+ the set of all positive integers and R^+ the set of all nonnegative real numbers.

Let T be a mapping of a nonempty metric space (X, d) into itself. For each $x \in X$

$$O_T(x, 0, \infty) = \{T^n x : n \in \omega\}$$

denotes the orbit of T at x and for all $i, j \in \omega$, $j > i$, write

$$O_T(x, i, j) = \{T^i x,\ T^{i+1} x,\ \cdots,\ T^j x\}.$$

For any $A \subset X$, $D(A)$ denotes the diameter of A.

Lemma 2.[13] *Let $\Phi: R^+ \to R^+$ be a nondecreasing function, then the condition*

(Φ_1) $\lim_{n\to\infty}\Phi^n(t)=0$, $\forall t>0$, *where* Φ^n *denotes the n-th iteration of* Φ, *implies*

(Φ_2) $\Phi(t)<t$, $\forall t>0$.

Further, If Φ *is upper semicontinuous from the right, then* $(\Phi_1)\Leftrightarrow(\Phi_2)$.

A nondecreasing function $\Phi:R^+\to R^+$ *is said to be a contractive gauge function if* Φ *satisfies* (Φ_1) *and*

(Φ_3) $\lim_{t\to\infty}(t-\Phi(t))=\infty$.

Theorem 6. *Let* S *and* T *be continuous mappings of a complete metric space* (X, d) *into iteself. Suppose for each* $x\in X$ $D(O_S(x, 0, \infty)\cup O_T(x, 0, \infty))<\infty$. *If there exist* n, $m:X\to I^+$ *and a nondecreasing function* $\Phi:R^+\to R^+$ *satisfying* (Φ_1) *such that for each* $x\in X$

$$D(O_S(S^{n(x)}x, 0, \infty)\cup O_T(T^{m(x)}x, 0, \infty))\leqslant\Phi(D(O_S(x, 0, \infty)\cup O_T(x, 0, \infty))), \quad (8)$$

then for each $x\in X$, $\{S^nx\}_{n\in\omega}\to z_1\in X$, $\{T^nx\}_{n\in\omega}\to z_2\in X$ *and* $z_1=Sz_1$, $z_2=Tz_2$. *Further, if* (8) *is replaced by*

$$\begin{aligned}&D(O_S(S^{n(x)}x, 0, \infty)\cup O_T(T^{m(y)}y, 0, \infty))\\&\leqslant\Phi(D(O_S(x, 0, \infty)\cup O_T(y, 0, \infty))),\ \forall x, y\in X,\end{aligned} \quad (9)$$

then $z(=z_1=z_2)$ *is a unique common fixed point of* S *and* T.

Proof For any fixed $x\in X$ let $D(O_S(x,0,\infty)\cup O_T(x, 0, \infty))=M$. Consider the following subsequences of $\{S^nx\}_{n\in\omega}$ and $\{T^nx\}_{n\in\omega}$:

$$\{x_m^S\}=\{x_0=x,\ x_1^S=S^{n(x_0)}x_0,\ \cdots,\ x_{m+1}^S=S^{n(x_m^S)}x_m^S,\ \cdots\}$$
$$\{x_m^T\}=\{x_0=x,\ x_1^T=T^{m(x_0)}x_0,\ \cdots,\ x_{m+1}^T=T^{m(x_m^T)}x_m^T,\ \cdots\}.$$

By (8) we have

$$\begin{aligned}&D(O_S(x_m^S, 0, \infty)\cup O_T(x_m^T, 0, \infty))\\&\leqslant\Phi(D(O_S(x_{m-1}^S, 0, \infty)\cup O_T(x_{m-1}^T, 0, \infty)))\\&\ \vdots\\&\leqslant\Phi^m(D(O_S(x, 0, \infty)\cup O_T(x, 0, \infty))\\&=\Phi^m(M).\end{aligned} \quad (10)$$

Putting $m\to\infty$ in (10) we obtain

$$\lim_{m\to\infty}D(O_S(x_m^S, 0, \infty)\cup O_T(x_m^T, 0, \infty))=0. \quad (11)$$

Since $x_m^S\in\{S^nx\}_{n\in\omega}$, $x_m^T\in\{T^nx\}_{n\in\omega}$, $\forall m\in\omega$. Therefore (11) implies that $\{S^nx\}_{n\in\omega}$ and $\{T^nx\}_{n\in\omega}$ are both Cauchy sequences, and so $\{S^nx\}_{n\in\omega}\to z_1\in X$, $\{T^nx\}_{n\in\omega}\to z_2\in X$. It follows from the continuity of S and T that $z_1=Sz_1$ and $z_2=Tz_2$.

Now suppose (9) holds. Since (9)$\Rightarrow$(8), the conclusion as above is true. Assume $z_1\neq z_2$, Using (9) and Lemma 2 we obtain

$$\begin{aligned}d(z_1, z_2)&=D(O_S(S^{n(z_1)}z_1, 0, \infty)\cup O_T(T^{m(z_2)}z_2, 0, \infty))\\&\leqslant\Phi(D(O_S(z_1, 0, \infty)\cup O_T(z_2, 0, \infty)))\\&=\Phi(d(z_1, z_2))<d(z_2, z_2),\end{aligned}$$

a contradiction. Hence $z(=z_1=z_2)$ is a common fixed point of S and T. Similarly, we can prove that z is the unique fixed point of S and T respectively. Thus z is a unique common fixed point of S and T.

Corollary 3. *Let S and T be continuous mappings of a complete metric space (X, d) into itself. Suppose there exist $n, m: X \to I^+$ and a contractive gauge function Φ such that for all $x \in X$ and for all $r \in I^+$, $r \geqslant \max\{n(x), m(x)\}$*

$$D(O_S(S^{n(x)}x, 0, r) \cup (O_T(T^{m(x)}x, 0, r)) \leqslant \Phi(D(O_S(x, 0, r) \cup O_T(x, 0, r))). \tag{12}$$

Then for each $x \in X$, $\{S^n x\}_{n\in\omega} \to z_1$, $\{T^n x\}_{n\in\omega} \to z_2$, $Sz_1 = z_2$ and $Tz_2 = z_2$. Further, if (12) is replaced by

$$D(O_S(S^{n(x)}x, 0, r) \cup O_T(T^{m(y)}y, 0, r)) \leqslant \Phi(D(O_S(x, 0, r) \cup O_T(y, 0, r))),\ \forall x, y \in X;\ r \geqslant \max\{n(x), m(y)\}, \tag{13}$$

then $z(=z_1=z_2)$ is a unique common fixed point of S and T.

Proof By the assumption of this Corollary and Theorem 6, we only need to prove that $D(O_S(x, 0, \infty) \cup O_T(x, 0, \infty)) < \infty$, $\forall x \in X$. Suppose for some $x_0 \in X$ $D(O_S(x_0, 0, \infty) \cup O_T(x_0, 0, \infty)) = M = \infty$. Let $M_r = D(O_S(x_0, 0, r) \cup O_T(x_0, 0, r))$, $\forall r \in \omega$. Clearly $\{M_r\}_{r\in\omega}$ is a nondecreasing sequence and hence

$$\lim_{r\to\infty} M_r = M = \infty. \tag{14}$$

By (12) for any $r \in I^+$, $r \geqslant \max\{n(x_0), m(x_0)\}$, we have

$$\begin{aligned} M_r &= D(O_S(x_0, 0, r) \cup O_T(x_0, 0, r)) \\ &\leqslant D(O_S(x_0, 0, n(x_0)) \cup O_T(x_0, 0, m(x_0))) \\ &\quad + D(O_S(S^{n(x_0)}x_0, 0, r) \cup O_T(T^{m(x_0)}x_0, 0, r)) \\ &\leqslant D(O_S(x_0, 0, n(x_0)) \cup O_T(x_0, 0, m(x_0))) + \Phi(M_r). \end{aligned} \tag{15}$$

By (14), (15) and (Φ_3) we obtain

$$\infty = \lim_{r\to\infty}[m_r - \Phi(M_r)] \leqslant D(O_S(x_0, 0, n(x_0)) \cup O_T(x_0, 0, m(x_0))),$$

a contradiction. Hence we have $D(O_S(x, 0, \infty) \cup O_T(x, 0, \infty)) < \infty$, $\forall x \in X$. The conclusions of this Corollary follow from Theorem 6.

Corollary 4. *Let T be a continuous mapping of a complete metric space (X, d) into itself. Suppose there exist $n: X \to I^+$ and a contractive gauge function Φ such that for each $x \in X$ and for all $r \in I^+$, $r \geqslant n(x)$,*

$$D(O_T(T^{n(x)}x, 0, r)) \leqslant \Phi(D(O_T(x, 0, r))). \tag{16}$$

Then for each $x \in X$, $(T^n x)_{n\in\omega}$ converges to a fixed point z of T in X. Further, If (16) is replaced by

$$d(T^r x, T^r y) \leqslant \Phi(D(O_T(x, 0, r) \cup O_T(y, 0, r))) \tag{17}$$

for each $x \in X$ and for all $r \geqslant n(x)$, $y \in X$, then for each $x \in X$, $\{T^n x\}_{n\in\omega}$ converges to the unique fixed point z of T.

Proof Assume (16) holds. Letting $S = T$, $m(x) = n(x)$, $\forall x \in X$, in Corollary 3, we can come to the required conclusions. If (17) holds, Walter[10, p. 24] has proved that

$D(O_T(x, 0, \infty))<\infty$, $\forall x\in X$. It is easy to check that (17) implies

$$D(O_T(T^{n(x)}x, 0, \infty))\leqslant\Phi(D(O_T(x, 0, \infty))), \forall x\in X.$$

By Theorem 6 with $S=T, n(x)=m(x), \forall x\in X$, we show that for each $x\in X$ $\{T^n x\}_{n\in\omega}$ converges to a fixed point z of T. The uniqueness of z easily follows from (17).

Remark 6. Theorem 6 and Corollaries 3, 4 improve and unify a number of important results in [7, 8, 9, 10, 13].

Corollary 5. *Let S and T be continuous mappings of a complete metric space (X, d) into itself. Suppose for each $x\in X$, $D(O_S(x, 0, \infty)\cup O_T(x, 0, \infty))<\infty$. If there exist p, $q\in I^+$ and a nondecreasing function $\Phi:R^+\to R^+$ satisfying (Φ_1) such that for all x, $y\in X$*

$$D(O_S(S^p x, 0, \infty)\cup O_T(T^q y, 0, \infty))\leqslant\Phi(D(O_S(x, 0, \infty)\cup O_T(y, 0, \infty))). \quad (18)$$

Then S and T have a unique common fixed point z and for each $x\in X$, $\{S^n x\}_{n\in\omega}$ and $\{T^n x\}_{n\in\omega}$ both converge to z. Further, if $p=1$, then the continuity of S may be dropped.

Proof From Theorem 6 with $n(x)=p$, $m(x)=q$, $\forall x\in X$ the first conclusion of this theorem holds.

Now assume $p=1$ and S need not to be continuous. By the proof of Theorem 6 we see that for each $x\in X$, $\{T^n x\}_{n\in\omega}$ converges to a fixed point z of T. We shall show that z is also a fixed point of S. In fact, by (18) with $p=1$ we have

$$\begin{aligned} D(O_S(z, 0, \infty)) &= D(O_S(Sz, 0, \infty)\cup\{z\}) \\ &= D(O_S(Sz, 0, \infty)\cup O_T(T^q z, 0, \infty)) \\ &\leqslant\Phi(D(O_S(z, 0, \infty)\cup O_T(z, 0, \infty))) \\ &\leqslant\Phi(D(O_S(z, 0, \infty)). \end{aligned} \quad (19)$$

By Lemma 2, (19) implies $D(O_S(z, 0, \infty))=0$ and hence $z=Sz$. Therefore z is a common fixed point of S and T. The uniqueness of z easily follows from (18) with $p=1$.

Corollary 6. *Let T be a continuous mapping of a complete metric space (X, d) into itself. Suppose for each $x\in X$, $D(O_T(x,0,\infty))<\infty$. If there exist p, $q\in I^+$ and a nondecreasing function $\Phi:R^+\to R^+$ satisfying (Φ_1) such that for all x, $y\in X$ any one of the following conditions holds:*

(i) $D(O_T(T^p x, 0, \infty)\cup O_T(T^q y, 0, \infty))\leqslant\Phi(D(O_T(x, 0, \infty)\cup O_T(y, 0, \infty)))$.

(ii) $d(T^p x, T^q y)\leqslant\Phi(D(O_T(x, 0, \infty)\cup O_T(y, 0, \infty)))$,

then for each $x\in X$ $\{T^n x\}_{n\in\omega}$ converges to a unique fixed point z of T.

Proof If (i) holds, by Corollary 5 with $S=T$ the conclusion is true. If (ii) holcs, without loss of generality we may assume $p\geqslant q$. For any $\xi\in X$, i, $j\in\omega$, letting $x=T^i\xi$, $y=T^{p-q+j}\xi$ in (ii), we have

$$\begin{aligned} d(T^{p+i}\xi, T^{p+j}\xi) &\leqslant\Phi(D(O_T(T^i\xi, 0, \infty)\cup O_T(T^{p-q+j}\xi, 0, \infty))) \\ &\leqslant\Phi(D(O_T(\xi, 0, \infty))), \end{aligned}$$

which implies that

$$D(O_T(T^p x, 0, \infty)) \leqslant \Phi(D(O_T(x, 0, \infty))), \quad \forall x \in X.$$

Using Theorem 6 with $S=T$, $n(x)=m(x)=p$, $\forall x \in X$, we can show that for each $x \in X$, $\{T^n x\}_{n\in\omega}$ converges to a fixed point z of T. The uniqueness of z easily follows from (ii).

Corollary 7. *Let T be a continuous mapping of a complete metric space (X, d) into itself. If there exist $p, q \in I^+$ and a contractive gauge function Φ such that for all $x, y \in X$ and for all $r \geqslant \max\{p, q\}$ any one of the following conditions holds:*

(i) $D(O_T(T^p x, 0, r) \cup O_T(T^q y, 0, r)) \leqslant \Phi(D(O_T(x, 0, r) \cup O_T(y, 0, r)))$.

(ii) $d(T^p x, T^q y) \leqslant \Phi(D(O_T(x, 0, p) \cup O_T(y, 0, q)))$,

then for each $x \in X$, $\{T^n x\}_{n\in\omega}$ converges to a unique fixed point z of T.

Proof If (i) is true, by Corollary 3 with $S=T$, $n(x)=p$, $m(x)=q$, $\forall x \in X$ the conclusion of this corollary holds. If (ii) holds, without loss of generality we can assume $p \geqslant q$. For any $r \geqslant p$, $\xi \in X$, $0 \leqslant i, j \leqslant r-p$, letting $x=T^i\xi$, $y=T^{p-q+j}\xi$ in (ii) we have

$$\begin{aligned} d(T^{p+i}\xi, T^{p+j}\xi) &\leqslant \Phi(D(O_T(T^i\xi, 0, p) \cup O_T(T^{p-q+j}\xi, 0, q))) \\ &\leqslant \Phi(D(O_T(\xi, 0, r))), \ \forall 0 \leqslant i, j \leqslant r-p, \end{aligned}$$

which yields

$$D(O_T(T^p x, 0, r)) \leqslant \Phi(D(O_T(x, 0, r))), \ \forall x \in X.$$

By Corollary 4 with $n(x)=p$, $\forall x \in X$, we show that for each $x \in X$, $\{T^n x\}_{n\in\omega}$ converges to a fixed point z of T. The uniqueness of z easily follows from (ii).

Remark 7. Corollaries 5, 6 and 7 improve and generalize the main results in [2, 8, 11, 12].

Theorem 7. *Let T be a selfmap of a complete metric space (X, d) satisfying:*

(i) $D(O_T(x, 0, \infty)) < \infty$ *for each* $x \in X$,

(ii) *there exists a nondecreasing function $\Phi: R^+ \to R^+$ satisfying (Φ_1) such that for all $x, y \in X$*

$$d(Tx, T^2 y) \leqslant \Phi(D(O_T(x, 0, \infty) \cup O_T(y, 0, \infty))). \tag{20}$$

Then for each $x \in X$ $(T^n x)_{n\in\omega}$ converges to the unique fixed point z of T.

Proof By the proof of Theorem 6 with $S=T$, $n(x)=1$, $m(x)=2$, $\forall x \in X$, we see that for each $x \in X$ $\{T^n x\}_{n\in\omega} \to z \in X$. Then for each $\varepsilon > 0$ there exists $N \in I^+$ such that $n \geqslant N$ implies $d(T^n x, z) < \varepsilon$. For any $m \in I^+$ and $n \geqslant N$, from (20) it follows that

$$\begin{aligned} d(z, T^m z) &\leqslant d(z, T^{n+2}x) + d(T^m z, T^{n+2}x) \\ &\leqslant \varepsilon + \Phi(D(O_T(T^{m-1}z, 0, \infty) \cup O_T(T^n x, 0, \infty))) \\ &\leqslant \varepsilon + \Phi(\max\{2\varepsilon, D(O_T(z, 0, \infty)) + \varepsilon\}). \end{aligned} \tag{21}$$

Since ε is arbitrary, (21) implies

$$\sup\{d(z, T^m z) : m \in I^+\} \leqslant \Phi(D(O_T(z, 0, \infty))). \tag{22}$$

On the other hand, for any $1 \leqslant i < j < \infty$ by (20) we have

$$d(T^i z, T^j z) \leqslant \Phi(D(O_T(T^{i-1}z, 0, \infty) \cup O_T(T^{j-2}z, 0, \infty)))$$
$$\leqslant \Phi(D(O_T(z, 0, \infty))),$$

which implies

$$D(O_T(Tz, 0, \infty)) \leqslant \Phi(D(O_T(z, 0, \infty))). \tag{23}$$

By (22) and (23) we obtain

$$D(O_T(z, 0, \infty)) = \max\{\sup\{d(z, T^m z): m \in I^+\}, D(O_T(Tz, 0, \infty))\}$$
$$\leqslant \Phi(D(O_T(z, 0, \infty))). \tag{24}$$

By Lemma 2, (24) yields $D(O_T(z, 0, \infty)) = 0$ and hence $z = Tz$. The uniqueness of z easily follows from (20).

Remark 8. From Lemma 2 we see that Theorem 7 is an extension of Hegedüs[14] and Theorem 2 of Park and Rhoades[15].

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