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STRONGER DISTORTION THEOREMS OF UNIVALENT FUNCTIONS AND ITS APPLICATIONS

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Abstract

The purpose of this paper is to improve and to generalize the famous Fitz Gerald inequalities, the Bazilevic inequalities and the Hayman regular theorem, by means of the representation theorem of continuous linear functionals on the space of continuous functions.

I. Introduction

The research on the Bieberbach conjecture due to Fitz Gerald in 1971 is very important, for it not only improved the estimate for the coefficients of univalent functions, but enabled us to see more deeply the importance of applying the "exponentaited" Grunsky inequalities. Concretely speaking, he obtained the following "exponentaited" distortion theorems about the difference quotients and their reciprocals:

Let $f(z)=z+a_2z^2+a_3z^3+\cdots$ be in S, namely $f(z)$ is regular and univalent in the unit disk $\Delta=\{z:|z|<1\}$, and, let $\beta_\mu(\mu=1, 2, 3, \cdots, N)$ be arbitrary complex numbers which are not all zeros, and $z_\mu(\mu=1, 2, 3, \cdots, N)$ be arbitrary points in the unit disk, then

$$\left|\sum_{\mu,\nu=1}^{N}\beta_\mu\beta_\nu\left(\frac{f(z_\mu)-f(z_\nu)}{z_\mu-z_\nu}\frac{z_\mu z_\nu}{f(z_\mu)f(z_\nu)}\right)^{\varepsilon}\right|\leqslant\sum_{\mu,\nu=1}^{N}\beta_\mu\bar{\beta}_\nu(1-z_\mu\bar{z}_\nu)^{-1},\ (\varepsilon=1,\ -1)\quad(1.1)$$

$$\sum_{\mu,\nu=1}^{N}\beta_\mu\bar{\beta}_\nu\left|\frac{f(z_\mu)-f(z_\nu)}{z_\mu-z_\nu}\frac{1}{1-z_\mu\bar{z}_\nu}\right|^{l}\geqslant\left|\sum_{\mu=1}^{N}\beta_\mu\left|\frac{f(z_\mu)}{z_\mu}\right|^{l}\right|^{2},\quad(1.2)$$

for $l=1, 2$, where $(f(z_\mu)-f(z_\nu))/(z_\mu-z_\nu)$ is interpreted as $f'(z)$ when $z_\mu=z_\nu$.

In fact, Xia Daoxing already established a result stronger than (1.2) early in 1951. From (1.2) we have the following Fitz Gerald's coefficient inequalities:

Let $f(z)=z+a_2z^2+a_3z^3+\cdots$ be in S

$$a_{p,q}=\sum_{k=1}^{p+q-1}\beta_k(p,\ q)|a_k|^2-|a_pa_q|^2,\quad(1.3)$$

where

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$$\beta_k(p,\ q)=\begin{cases} p-|k-q|, & |k-q|<p,\ (p\leqslant q) \\ 0, & \text{otherwise} \end{cases} \tag{1.4}$$

then

$$(a_{p,q})_{2\leqslant p,q\leqslant N}\geqslant 0. \tag{1.5}$$

It means that the matrices $(a_{p,q})$ are positive semidefinite. We well know that from (1.5) a series of important results can be obtained. In 1979, Kung Seng first improved the Fitz Gerald's distortion theorems (1.2) and coefficient inequalities (1.5). In the same year, Hu Ke improved Kung Seng's results.

The aim of this paper is to make further improvements on the above mentioned Fitz Gerald inequalities, together with the Bazilevic inequaity and Hayman's regularity theorem by means of the representation theorem of continuous linear functionals on the space of continuous functions.

Suppose K is a bounded closed set in the complex plane. Denote by $c(K)$ the set of all bounded continuous functions on K. Endowed with the norm

$$\|f\|=\max_{z\in K}|f(z)|,\ f(z)\in c(K),$$

$c(K)$ is a Banach space.

Denote by $c^*(K)$ the conjugate space of $c(K)$. Let L_1, L_2 and $L\in c^*(K)$, $h(z,\zeta)$ be bounded continuous functions defined on $K\times K$. We define

$$L_1L_2(h(z,\ \zeta))=L_2(L_1(h(z,\ \zeta))),$$
$$|L|^2(h(z,\ \zeta))=L(\overline{L(h(z,\ \zeta))}),$$

where z is operated by the linear functionals ahead of ζ. Here and later we suppose that $\Phi(w)=\sum_{n=0}^{\infty}c_nw^n$ is an integal function and define $\Phi^+(w)=\sum_{n=0}^{\infty}|c_n|w^n$.

II. Strengthened Distortion Theorems

Is this section, we shall strengthen the Grunsky distortion theorem and the improved Fitz Gerald distiortion theorem due to Kung Seng, Hu Ke.

Lemma. *Suppose* $\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}x_\mu\bar{x}_\nu\geqslant 0$, $(\alpha_{\nu\mu}=\bar{\alpha}_{\mu\nu})$ *is non-negative Hermitian, then*

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}x_\mu\bar{y}_\nu\right|^2\leqslant\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}x_\mu\bar{x}_\nu\cdot\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}y_\mu\bar{y}_\nu \tag{2.1}$$

holds for any complex numbers $\{x_\mu\}$ *and* $\{y_\mu\}$ $(\mu=1,\ 2,\ \cdots,\ N)$.

Proof Since $\sum\alpha_{\mu\nu}\lambda_\mu\bar{\lambda}_\nu$ is non-negative Hermitian

$$0\leqslant\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}(e^{i\beta}x_\mu-\lambda y_\mu)\overline{(e^{i\beta}x_\nu-\lambda y_\nu)}$$
$$=\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}x_\mu\bar{x}_\nu-2\lambda\mathrm{Re}\left(e^{i\beta}\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}x_\mu\bar{y}_\nu\right)+\lambda^2\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}y_\mu\bar{y}_\nu$$

holds for any real numbers β and λ. This is a non-negative quadratic form of λ, so

$$\left[\operatorname{Re}\left(e^{i\beta}\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}x_{\mu}\bar{y}_{\nu}\right)\right]^{2}\leqslant\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}x_{\mu}\bar{x}_{\nu}\cdot\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}y_{\mu}\bar{y}_{\nu}.$$

Take $\beta=-\arg\left(\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}x_{\mu}\bar{y}_{\nu}\right)$), then (2.1) follows.

Theorem 1. *Suppose* $f\in S$, $\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}x_{\mu}\bar{x}_{\nu}\geqslant0\ (\alpha_{\mu\nu}=\bar{\alpha}_{\nu\mu},\ \mu,\ \nu=1,\ 2,\ \cdots\ N),\ l>0$, *m is a natural number. Denote by* $F_n(t)$ *the n-th Faber polynomial of* $f(z)$, $\varepsilon=1,\ -1$

$$g_n^{(\varepsilon)}(z)=F_n(1/f(z))-(z^{-n}+\bar{z}\varepsilon^{n}),\tag{2.2}$$

$$F(z,\ \zeta)=\frac{f(z)-f(\zeta)}{z-\zeta}\cdot\frac{z\zeta}{f(z)f(\zeta)},\ f(z,\ \zeta)=\frac{f(z)-f(\zeta)}{z-\zeta},\tag{2.3}$$

then

$$\begin{aligned}&\left|\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_1^{(\mu)}\overline{L_2^{(\nu)}}\left(\Phi\left[\left(\frac{l}{2}\sum_{n=1}^{\infty}\frac{1}{n}\,g_n^{(\varepsilon)}(z)\,\overline{g_n^{(\varepsilon)}(\zeta)}\right)^{m}\right]\right)\right|^{2}\\&\leqslant\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_1^{(\mu)}\overline{L_1^{(\nu)}}\left(\Phi^{+}\left[\left(\ln\frac{|F(z,\ \zeta)|^{l\varepsilon}}{|1-z\bar{\zeta}|^{l}}\right)^{m}\right]\right)\\&\times\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_2^{(\mu)}\overline{L_2^{(\nu)}}\left(\Phi^{+}\left[\left(\ln\frac{|F(z,\ \zeta)|^{l\varepsilon}}{|1-z\bar{\zeta}|^{l}}\right)^{m}\right]\right)\end{aligned}\tag{2.4}$$

holds for any integral function $\Phi(w)$ *and continuous linear functionals* $L_1^{(\mu)}$, $L_2^{(\mu)}$ *and* $L^{(\mu)}\in C^{*}(|z|\leqslant\rho<1)\ (\mu=1,\ 2,\ \cdots,\ N)$. *Particularly, we have*

$$\begin{aligned}&\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L^{(\mu)}\overline{L^{(\nu)}}\left(\left|\frac{f(z)f(\zeta)}{z\zeta}\right|^{l\varepsilon}\exp\left\{\frac{l}{2}\sum_{n=1}^{N}\frac{1}{n}\,g_n^{(\varepsilon)}(z)\overline{g_n^{(\varepsilon)}(\zeta)}\right\}\right)\\&\leqslant\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L^{(\mu)}\overline{L^{(\nu)}}\left(\frac{|f(z,\ \zeta)|^{l\varepsilon}}{|1-z\bar{\zeta}|^{l}}\right).\end{aligned}\tag{2.5}$$

It is obvious that (2.5) contains the result (1.1) in [5].

Proof We first consider the Löwner–Goluzin function[12, 13]

$$f(z)=\lim_{t\to+\infty}e^{t}f(z,\ t),$$

where $f(z,\ t)=e^{-t}(z+a_2(t)z^2+a_3(t)z^3+\cdots)$ satisfies the differential equation and initial condition

$$\frac{\partial}{\partial t}f(z,\ t)=-f(z,\ t)\frac{1+k(t)f(z,\ t)}{1-k(t)f(z,\ t)},\ f(z,\ 0)=z.\tag{2.6}$$

It is well known that thus obtained function set S^{*} is a dense subset of S. For $f(z)\in S^{*}$ and $F(\zeta')=1/f(\zeta'^{-1})$, we have[12]

$$\begin{cases}\ln\dfrac{F(z')-F(\zeta')}{z'-\zeta'}=\ln F(z,\ \zeta)=-2\displaystyle\int_0^{\infty}h(z,\ t)h(\zeta,\ t)dt,\\ \ln(1-1/z'\bar{\zeta}')=\ln(1-z\bar{\zeta})=-2\displaystyle\int_0^{\infty}h(z,\ t)\overline{h(\zeta,\ t)}dt,\end{cases}\tag{2.7}$$

where $z'=z^{-1}$, $\zeta'=\zeta^{-1}$, $|z|<1$, $|\zeta|<1$, $\ln1=0$ and

$$h(z,\ t)=k(t)f(z,\ t)/(1-k(t)f(z,\ t)).$$

It follows that

$$\ln\frac{|F(z,\ \zeta)|^{\varepsilon}}{|1-z\bar{\zeta}|}=\begin{cases}4\displaystyle\int_0^{\infty}\operatorname{Im}h(z,\ t)\operatorname{Im}h(\zeta,\ t)dt,&(\varepsilon=1),\\4\displaystyle\int_0^{\infty}\operatorname{Re}h(z,\ t)\operatorname{Re}h(\zeta,\ t)dt,&(\varepsilon=-1).\end{cases}\tag{2.8}$$

Suppose

$$\ln F(z, \zeta) = \sum_{m,n=1}^{\infty} \gamma_{mn} z^m \zeta^n. \tag{2.9}$$

By the properties of Faber polynomials[15], we have

$$g_n^{(\varepsilon)}(z) = -n \sum_{p=1}^{\infty} \gamma_{pn} z^p - \varepsilon \bar{z}^n. \tag{2.10}$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{n} g_n^{(\varepsilon)}(z) \overline{g_n^{(\varepsilon)}(\zeta)} = \sum_{p,q=1}^{\infty} \left(\sum_{n=1}^{\infty} n \gamma_{pn} \bar{\gamma}_{qn} \right) z^p \bar{\zeta}^q + \sum_{n=1}^{\infty} \frac{1}{n} (\bar{z}\zeta)^n$$
$$+ \varepsilon \left(\sum_{p,n=1}^{\infty} \gamma_{pn} z^p \zeta^n + \overline{\sum_{p,n=1}^{\infty} \gamma_{pn} \zeta^p z^n} \right).$$

From $\gamma_{np} = \gamma_{pn}$ and Milin's lemma[10]

$$\sum_{n=1}^{\infty} n \gamma_{pn} \bar{\gamma}_{qn} = \begin{cases} 0, & q \neq p, \\ p^{-1}, & q = p, \end{cases}$$

it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n} g_n^{(\varepsilon)}(z) \overline{g_n^{(\varepsilon)}(\zeta)} = 2 \operatorname{Re} \left\{ \sum_{p=1}^{\infty} \frac{1}{p} (z\bar{\zeta})^p + \varepsilon \sum_{p,n=1}^{\infty} \gamma_{pn} z^p \zeta^n \right\}$$
$$= 2[\ln(|F(z, \zeta)|^{\varepsilon} / |1 - z\bar{\zeta}|)],$$

comparing this expression with (2.8), we obtain

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} g_n^{(\varepsilon)}(z) \overline{g_n^{(\varepsilon)}(\zeta)} = 4 \int_0^{\infty} g(z, t) g(\zeta, t) dt, \tag{2.11}$$

where $g(z, t) = \operatorname{Im} h(z, t)$ when $\varepsilon = 1$ and $g(z, t) = \operatorname{Re} h(z, t)$ when $\varepsilon = -1$.

According to Riesz representation theorem, for any $L \in C^*(|z| \leqslant \rho < 1)$ there exists a complex measure $\mu(z)$ on $|z| \leqslant \rho$, which satisfies the condition $|\mu| = \|L\|$, so that[17]

$$L(\varphi) = \int_{|z| \leqslant \rho} \varphi(z) d\mu(z), \quad \varphi \in C(|z| \leqslant \rho < 1). \tag{2.12}$$

Then from (2.11), Fubini theorem, Cauchy inequality and (2.1), we have

$$\left| \sum_{\mu,\nu=1}^{N} \alpha_{\mu\nu} L_1^{(\mu)} \overline{L_2^{(\nu)}} \left(\left[\frac{l}{2} \sum_{n=1}^{\infty} \frac{1}{n} g_n^{(\varepsilon)}(z) \overline{g_n^{(\varepsilon)}(\zeta)} \right]^m \right) \right|^2$$
$$= \left| \sum_{\mu,\nu=1}^{N} \alpha_{\mu\nu} L_1^{(\mu)} \overline{L_2^{(\nu)}} \left(4l \int_0^{\infty} g(z, t) g(\zeta, t) dt \right)^m \right|^2$$
$$\leqslant \left(\int_0^{\infty} \cdots \int_0^{\infty} (4l)^m \left[\sum_{\mu,\nu=1}^{N} \alpha_{\mu\nu} L_1^{(\mu)} \left(\prod_{j=1}^{m} g(z, t_j) \right) \overline{L_1^{(\nu)} \left(\prod_{j=1}^{m} g(z, t_j) \right)} \right]^{1/2} \right.$$
$$\left. \times \left[\sum_{\mu,\nu=1}^{N} \alpha_{\mu\nu} L_2^{(\mu)} \left(\prod_{j=1}^{m} g(\zeta, t_j) \right) \overline{L_2^{(\nu)} \left(\prod_{j=1}^{m} g(\zeta, t_j) \right)} \right]^{1/2} \prod_{j=1}^{m} dt_j \right)^2$$
$$\leqslant \sum_{\mu,\nu=1}^{N} \alpha_{\mu\nu} L_1^{(\mu)} \overline{L_1^{(\nu)}} \left(4l \int_0^{\infty} g(z, t) g(\zeta, t) dt \right)^m$$
$$\times \sum_{\mu,\nu=1}^{N} \alpha_{\mu\nu} L_2^{(\mu)} \overline{L_2^{(\nu)}} \left(4l \int_0^{\infty} g(z, t) g(\zeta, t) dt \right)^m.$$

Using (2.8), we deduce

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_1^{(\mu)}\overline{L_2^{(\nu)}}\left(\frac{l}{2}\sum_{n=1}^{\infty}\frac{1}{n}\,g_n^{(\varepsilon)}(z)\overline{g_n^{(\varepsilon)}(\zeta)}\right)^m\right|^2$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_1^{(\mu)}\overline{L_1^{(\nu)}}\left(\ln\frac{|F(z,\zeta)|^{ls}}{|1-z\bar\zeta|^l}\right)^m\cdot\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_2^{(\mu)}\overline{L_2^{(\nu)}}\left(\ln\frac{|F(z,\zeta)|^{ls}}{|1-z\bar\zeta|^l}\right)^m. \quad (2.13)$$

Then for any integral function $\Phi(w)$, by Cauchy inequality

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_1^{(\mu)}\overline{L_2^{(\nu)}}\left(\Phi\left[\left(\frac{l}{2}\sum_{n=1}^{\infty}\frac{1}{n}\,g_n^{(\varepsilon)}(z)\overline{g_n^{(\varepsilon)}(\zeta)}\right)^m\right]\right)\right|^2$$
$$\leqslant\left(\sum_{p=1}^{\infty}|C_p|\left[\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_1^{(\mu)}\overline{L_1^{(\nu)}}\left(\ln\frac{|F(z,\zeta)|^{ls}}{|1-z\bar\zeta|^l}\right)^{mp}\right.\right.$$
$$\left.\left.\times\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_2^{(\mu)}\overline{L_2^{(\nu)}}\left(\ln\frac{|F(z,\zeta)|^{ls}}{|1-z\bar\zeta|^l}\right)^{mp}\right]^{1/2}\right)^2$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_1^{(\mu)}\overline{L_1^{(\nu)}}\Phi^+[(\ln|F(z,\zeta)|^{ls}/|1-z\bar\zeta|^l)^m])$$
$$\times\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_2^{(\mu)}\overline{L_2^{(\nu)}}(\Phi^+[(\ln|F(z,\zeta)|^{ls}/|1-z\bar\zeta|^l)^m])$$

holds. So (2.4) is true for any $f\in S^*$.

Especially, we choose $\Phi(w)=\exp\{w\}$, $m=1$, $L_1^{(\mu)}=L_2^{(\mu)}=L'^{(\mu)}$, then

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L'^{(\mu)}\overline{L'^{(\nu)}}\left(\exp\left\{\frac{l}{2}\sum_{n=1}^{\infty}\frac{1}{n}\,g_n^{(\varepsilon)}(z)\overline{g_n^{(\varepsilon)}(\zeta)}\right\}\right)\right|$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L'^{(\mu)}\overline{L'^{(\nu)}}(|F(z,\zeta)|^{ls}/|1-z\bar\zeta|^l. \quad (2.14)$$

The terms in the absolute value sign on the left hand side are

$$\sum_{p=1}^{\infty}\frac{1}{p!}\left(\frac{l}{2}\right)^p\sum_{n_1,\cdots,n_p=1}^{\infty}\left(\prod_{j=1}^{p}n_j^{-1}\right)\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L'^{(\mu)}\left(\prod_{j=1}^{p}g_{n_j}^{(\varepsilon)}(z)\right)\overline{L'^{(\nu)}\left(\prod_{j=1}^{p}g_{n_j}^{(\varepsilon)}(\zeta)\right)}.$$

It is obvious that they are non-negative. If we choose $L'^{(\mu)}(|z/f(z)|^{ls}\varphi)=L^{(\mu)}(\varphi)$, $\varphi\in C(|z|\leqslant\rho<1)$, then

$$\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L^{(\mu)}\overline{L^{(\nu)}}\left(\left|\frac{f(z)f(\zeta)}{z\zeta}\right|^{ls}\exp\left\{\frac{l}{2}\sum_{n=1}^{\infty}\frac{1}{n}\,g_n^{(\varepsilon)}(z)\overline{g_n^{(\varepsilon)}(\zeta)}\right\}\right)$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L^{(\mu)}\overline{L^{(\nu)}}(|f(z,\zeta)|^{ls}/|1-z\bar\zeta|^l).$$

So (2.5) is also true for any $f\in S^*$.

As S^* is dense in S, by the continuity of linear functionals, we know that (2.4) and (2.5) are true for any $f\in S$.

Especially, we choose $L^{(\mu)}(\varphi)=\lambda_\mu\varphi(z_\mu)$. From (2.5), we obtain[5]

$$\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda_\mu\bar\lambda_\nu\left|\frac{f(z_\mu)f(z_\nu)}{z_\mu z_\nu}\right|^{ls}\exp\left\{\frac{l}{2}\sum_{n=1}^{\infty}\frac{1}{n}\,g_n^{(\varepsilon)}(z_\mu)\overline{g_n^{(\varepsilon)}(z_\nu)}\right\}$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda_\mu\bar\lambda_\nu\frac{|f(z_\mu,z_\nu)|^{ls}}{|1-z_\mu\bar z_\nu|^l}. \quad (2.15)$$

Analogously, from (2.4) we can obtain

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda_\mu\overline{\lambda_2'}\left|\frac{f(z_\mu)f(z_\nu')}{z_\mu z_\nu'}\right|^{ls}\exp\left\{\frac{l}{2}\sum_{n=1}^{\infty}\frac{1}{n}g_n^{(\varepsilon)}(z_\mu)\overline{g_n^{(\varepsilon)}(z_\nu')}\right\}\right|^2$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda_\mu\bar\lambda_\nu\frac{|f(z_\mu,z_\nu)|^{ls}}{|1-z_\mu\bar z_\nu|^l}\cdot\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda_\mu'\overline{\lambda_\nu'}\frac{|f(z_\mu',z_\nu')|^{ls}}{|1-z_\mu'\bar z_\nu'|^l}, \quad (2.16)$$

which is stronger than (2.15). Thus Thorem 1 is proved.

Theorem 2. *Under the same hypotheses of Theorem 1, we have*

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_1^{(\mu)}\overline{L_2^{(\nu)}}\left(\Phi\left[\left(\frac{l}{n(1+n\varepsilon\mathrm{Re}\gamma_{nn})}\mathrm{Re}\,g_n^{(\varepsilon)}(z)\,\mathrm{Re}\,g_n^{(\varepsilon)}(\zeta)\right)^m\right]\right)\right|^2\leqslant A_1\cdot A_2, \quad (2.17)$$

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_1^{(\mu)}\overline{L_2^{(\nu)}}\left(\Phi\left[\left(\frac{l}{n(1-n\varepsilon\mathrm{Re}\gamma_{nn})}\mathrm{Im}\,g_n^{(\varepsilon)}(z)\mathrm{Im}\,g_n^{(\varepsilon)}(\zeta)\right)^m\right]\right)\right|^2\leqslant A_1\cdot A_2, \quad (2.18)$$

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_1^{(\mu)}\overline{L_2^{(\nu)}}\left(\Phi\left[\left(\frac{l}{n\sqrt{1-(n\mathrm{Re}\gamma_{nn})^2}}\mathrm{Re}\,g_n^{(\varepsilon)}(z)\,\mathrm{Im}\,g_n^{(\varepsilon)}(\zeta)\right)^m\right]\right)\right|^2\leqslant A_1\cdot A_2, \quad (2.19)$$

here

$$A_i=\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_i^{(\mu)}\overline{L_i^{(\nu)}}\left(\Phi^+\left[\left(\ln\frac{|F(z,\zeta)|^{l\varepsilon}}{\{1-z\bar{\zeta}|^l}\right)^m\right]\right)(i=1,\ 2).$$

Particularly, we have

$$\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L^{(\mu)}\overline{L^{(\nu)}}\left(\left|\frac{f(z)f(\zeta)}{z\zeta}\right|^{l\varepsilon}\exp\left\{\frac{l}{n(1+n\varepsilon\mathrm{Re}\gamma_{nn})}\mathrm{Re}\,g_n^{(\varepsilon)}(z)\,\mathrm{Re}\,g_n^{(\varepsilon)}(\zeta)\right\}\right)$$

$$\leqslant\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L^{(\mu)}\overline{L^{(\nu)}}\left(\frac{|f(z,\zeta)|^{l\varepsilon}}{|1-z\bar{\zeta}|^l}\right)=B, \quad (2.20)$$

$$\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L^{(\mu)}\overline{L^{(\nu)}}\left(\left|\frac{f(z)f(\zeta)}{z\zeta}\right|^{l\varepsilon}\exp\left\{\frac{l}{n(1-n\varepsilon\mathrm{Re}\gamma_{nn})}\mathrm{Im}\,g_n^{(\varepsilon)}(z)\mathrm{Im}\,g_n^{(\varepsilon)}(\zeta)\right\}\right)\leqslant B, \quad (2.21)$$

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L^{(\mu)}\overline{L^{(\nu)}}\left(\left|\frac{f(z)f(\zeta)}{z\zeta}\right|^{l\varepsilon}\exp\left\{\frac{l}{n\sqrt{1-(n\mathrm{Re}\gamma_{nn})^2}}\mathrm{Re}\,g_n^{(\varepsilon)}(z)\mathrm{Im}\,g_n^{(\varepsilon)}(\zeta)\right\}\right)\right|\leqslant B. \quad (2.22)$$

It is obvious that (2.20) and (2.21) contain (1.2) and (1.3) in [5].

Proof Suppose

$$h(z,\ t)=\sum_{p=1}^{\infty}b_p(t)z^p.$$

It was proved by Goluzin that[11]

$$\begin{cases}\gamma_{pq}=-2\int_0^\infty b_p(t)b_q(t)dt, & (p,\ q=1,\ 2,\ 3,\ \cdots),\\ \int_0^\infty b_p(t)\overline{b_q(t)}\ dt=\begin{cases}0, & q\neq p,\\ 1/2p, & q=p,\ (p,\ q=1,\ 2,\ \cdots).\end{cases}\end{cases} \quad (2.23)$$

Substituting (2.23) into (2.10), we obtain

$$\frac{1}{n}g_n^{(\varepsilon)}(z)=2\left(\int_0^\infty\sum_{p=1}^{\infty}z^pb_p(t)b_n(t)dt-\varepsilon\bar{z}^n\int_0^\infty b_n(t)\overline{b_n(t)}\ dt\right)$$

$$=2\int_0^\infty b_n(t)\left(\sum_{p=1}^{\infty}b_p(t)z^p-\varepsilon\sum_{p=1}^{\infty}\overline{b_p(t)z^p}\right)dt.$$

Hence

$$\frac{1}{n}\mathrm{Re}\,g_n^{(\varepsilon)}(z)=\begin{cases}4\int_0^\infty\mathrm{Im}\,b_n(t)\mathrm{Im}\,h(z,\ t)\,dt & (\varepsilon=1),\\ 4\int_0^\infty\mathrm{Re}\,b_n(t)\mathrm{Re}\,h(z,\ t)\,dt & (\varepsilon=-1),\end{cases} \quad (2.24)$$

$$\frac{1}{n}\operatorname{Im} g_n^{(\varepsilon)}(z)=\begin{cases}4\int_0^\infty \operatorname{Re} b_n(t)\operatorname{Im} h(z,\ t)dt & (\varepsilon=1),\\ 4\int_0^\infty \operatorname{Im} b_n(t)\operatorname{Re} h(z,\ t)dt & (\varepsilon=-1).\end{cases} \tag{2.25}$$

Define

$$b_n^{(\varepsilon)}(t)=\begin{cases}\operatorname{Im} b_n(t) & (\varepsilon=1)\\ \operatorname{Re} b_n(t) & (\varepsilon=-1),\end{cases}\quad g^{(\varepsilon)}(z,\ t)=\begin{cases}\operatorname{Im} h(z,\ t) & (\varepsilon=1),\\ \operatorname{Re} h(z,\ t) & (\varepsilon=-1).\end{cases}$$

From (2.24), (2.1) and Cauchy inequality, it follows that

$$\begin{aligned}&\left|\sum_{\mu,\nu=1}^N \alpha_{\mu\nu}L_1^{(\mu)}\overline{L_2^{(\nu)}}\left(\frac{l}{n(1+n\varepsilon\operatorname{Re}\gamma_{nn})}\operatorname{Re} g_n^{(\varepsilon)}(z)\operatorname{Re} g_n^{(\varepsilon)}(\zeta)\right)^m\right|\\
&=\left|\sum_{\mu,\nu=1}^N \alpha_{\mu\nu}L_1^{(\mu)}\overline{L_2^{(\nu)}}\left(\frac{16\,ln}{1+n\varepsilon\operatorname{Re}\gamma_{nn}}\int_0^\infty\int_0^\infty b_n^{(\varepsilon)}(t)b_n^{(\varepsilon)}(t')g^{(\varepsilon)}(z,\ t)g^{(\varepsilon)}(\zeta,\ t')dtdt'\right)^m\right|\\
&\leqslant\left(\frac{16\,nl}{1+n\varepsilon\operatorname{Re}\gamma_{nn}}\right)^m\int_0^\infty\cdots\int_0^\infty\left|\prod_{j=1}^m b_n^{(\varepsilon)}(t_j)b_n^{(\varepsilon)}(t'_j)\right|\\
&\quad\times\left(\sum_{\mu,\nu=1}^N \alpha_{\mu\nu}L_1^{(\mu)}\left(\prod_{j=1}^m g^{(\varepsilon)}(z,\ t_j)\right)\overline{L_1^{(\nu)}\left(\prod_{j=1}^m g^{(\varepsilon)}(\zeta,\ t_j)\right)}\right)^{1/2}\\
&\quad\times\left(\sum_{\mu,\nu=1}^N \alpha_{\mu\nu}L_2^{(\mu)}\left(\prod_{j=1}^m g^{(\varepsilon)}(z,\ t'_j)\right)\overline{L_2^{(\nu)}\left(\prod_{j=1}^m g^{(\varepsilon)}(\zeta,\ t'_j)\right)}\right)^{1/2}\\
&\quad\times\left(\prod_{j=1}^m dt_jdt'_j\right)\leqslant\left(\frac{16\,nl}{1+n\varepsilon\operatorname{Re}\gamma_{nn}}\right)^m\\
&\quad\times\left(\int_0^\infty\cdots\int_0^\infty\left|\prod_{j=1}^m b_n^{(\varepsilon)}(t'_j)\right|^2\cdot\sum_{\mu,\nu=1}^N \alpha_{\mu\nu}L_1^{(\mu)}\left(\prod_{j=1}^m g^{(\varepsilon)}(z,\ t_j)\right)\overline{L_1^{(\nu)}\left(\prod_{j=1}^m g^{(\varepsilon)}(\zeta,\ t_j)\right)}\prod_{j=1}^m dt_jdt'_j\right)^{1/2}\\
&\quad\times\left(\int_0^\infty\cdots\int_0^\infty\left|\prod_{j=1}^m b_n^{(\varepsilon)}(t_j)\right|^2\cdot\sum_{\mu,\nu=1}^N \alpha_{\mu\nu}L_2^{(\mu)}\left(\prod_{j=1}^m g^{(\varepsilon)}(z,\ t'_j)\right)\overline{L_2^{(\nu)}\left(\prod_{j=1}^m g^{(\varepsilon)}(\zeta,\ t'_j)\right)}\prod_{j=1}^m dt_jdt'_j\right)^{1/2}\\
&=\left(\frac{16\,nl}{1+n\varepsilon\operatorname{Re}\gamma_{nn}}\right)^m\left(\int_0^\infty|b_n^{(\varepsilon)}(t)|^2dt\right)^m\\
&\qquad\times\sum_{\mu,\nu=1}^N \alpha_{\mu\nu}L_1^{(\mu)}\overline{L_1^{(\nu)}}\left(\int_0^\infty g^{(\varepsilon)}(z,\ t)g^{(\varepsilon)}(\zeta,\ t)dt\right)^m\\
&\qquad\times\sum_{\mu,\nu=1}^N \alpha_{\mu\nu}L_2^{(\mu)}\overline{L_2^{(\nu)}}\left(\int_0^\infty g^{(\varepsilon)}(z,\ t)g^{(\varepsilon)}(\zeta,\ t)dt\right)^m.\end{aligned} \tag{2.26}$$

As we have

$$\begin{cases}\int_0^\infty|\operatorname{Im} b_n(t)|^2dt=\frac{1}{2}\int_0^\infty(|b_n(t)|^2-\operatorname{Re} b_n^2(t))dt=\frac{1}{4n}(1+n\varepsilon\operatorname{Re}\gamma_{nn}),\\ \int_0^\infty|\operatorname{Re} b_n(t)|^2dt=\frac{1}{2}\int_0^\infty(|b_n(t)|^2+\operatorname{Re} b_n^2(t))dt=\frac{1}{4n}(1-n\varepsilon\operatorname{Re}\gamma_{nn}),\end{cases} \tag{2.27}$$

we obtain from (2.26), (2.27) and (2.8)

$$\left|\sum_{\mu,\nu=1}^N \alpha_{\mu\nu}L_1^{(\mu)}\overline{L_2^{(\nu)}}\left(\frac{l}{n(1+n\varepsilon\operatorname{Re}\gamma_{nn})}\operatorname{Re} g_n^{(\varepsilon)}(z)\operatorname{Re} g_n^{(\varepsilon)}(\zeta)\right)^m\right|^2\leqslant D_1\cdot D_2, \tag{2.28}$$

here

$$D_i=\sum_{\mu,\nu=1}^N \alpha_{\mu\nu}L_i^{(\mu)}L_i^{(\nu)}\left(\ln\frac{|F(z,\ \zeta)|^{l\varepsilon}}{|1-z\bar\zeta|^l}\right)^m\quad(i=1,\ 2).$$

Analogously, we can obtain from (2.24), (2.25) and (2.27)

$$\left|\sum_{\mu,\nu=1}^N \alpha_{\mu\nu}L_1^{(\mu)}\overline{L_2^{(\nu)}}\left(\frac{l}{n(1-n\varepsilon\operatorname{Re}\gamma_{nn})}\operatorname{Im} g_n^{(\varepsilon)}(z)\operatorname{Im} g_n^{(\varepsilon)}(\zeta)\right)^m\right|^2\leqslant D_1D_2, \tag{2.29}$$

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_1^{(\mu)}\overline{L_2^{(\nu)}}\left(\frac{l}{n\sqrt{1-(n\mathrm{Re}\gamma_{nn})^2}}\mathrm{Re}\,g_n^{(s)}(z)\,\mathrm{Im}\,g_n^{(s)}(\zeta)\right)^m\right|^2\leqslant D_1\cdot D_2. \qquad (2.30)$$

Then, using the Cauchy inequality, (2.17), (2.18) and (2.19) follow from (2.28), (2.29) and (2.30). Moreover, we know that (2.20), (2.21) and (2.22) also hold.

Particularly, we choose $L^{(\mu)}(\varphi)=\lambda_\mu\varphi(z_\mu)$, then[5]

$$\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda_\mu\bar{\lambda}_\nu\left|\frac{f(z_\mu)f(z_\nu)}{z_\mu z_\nu}\right|^{ls}\exp\left\{\frac{l\mathrm{Re}\,g_n^{(s)}(z_\mu)\,\mathrm{Re}\,g_n^{(s)}(z_\nu)}{n(1+ns\mathrm{Re}\,\gamma_{nn})}\right\}$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda_\mu\bar{\lambda}_\nu\,|f(z_\mu,\ z_\nu)|^{ls}/|1-z_\mu\bar{z}_\nu|^l, \qquad (2.31)$$

$$\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda_\mu\bar{\lambda}_\nu\left|\frac{f(z_\mu)f(z_\nu)}{z_\mu z_\nu}\right|^{ls}\exp\left\{\frac{l\,\mathrm{Im}\,g_n^{(s)}(z_\mu)\,\mathrm{Im}\,g_n^{(s)}(z_\nu)}{n(1-ns\mathrm{Re}\gamma_{nn})}\right\}$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda_\mu\bar{\lambda}_\nu|\,f(z_\mu,\ z_\nu)\,|^{s}/|1-z_\mu\bar{z}_\nu|^l. \qquad (2.32)$$

Analogously, from (2.17), (2.18) and (2.22), we have

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda_\mu\overline{\lambda'_\nu}\left|\frac{f(z_\mu)f(z'_\nu)}{z_\mu z'_\nu}\right|^{ls}\exp\left\{\frac{l\mathrm{Re}\,g_n^{(s)}(z_\mu)\,\mathrm{Re}\,g_n^{(s)}(z'_\nu)}{n(1+ns\,\mathrm{Re}\,\gamma_{nn})}\right\}\right|^2\leqslant BB', \qquad (2.33)$$

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda_\mu\overline{\lambda'_\nu}\left|\frac{f(z_\mu)f(z'_\nu)}{z_\mu z'_\nu}\right|^{ls}\exp\left\{\frac{l\,\mathrm{Im}\,g_n^{(s)}(z_\mu)\,\mathrm{Im}\,g_n^{(s)}(z'_\nu)}{n(1-ns\mathrm{Re}\,\gamma_{nn})}\right\}\right|^2\leqslant BB', \qquad (2.34)$$

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda_\mu\bar{\lambda}_\nu\left|\frac{f(z_\mu)f(z'_\nu)}{z_\mu z'_\nu}\right|^{ls}\exp\left\{\frac{l\mathrm{Re}\,g_n^{(s)}(z_\mu)\,\mathrm{Im}\,g_n^{(s)}(z'_\nu)}{n\sqrt{1-(n\,\mathrm{Re}\,\gamma_{nn})^2}}\right\}\right|^2\leqslant BB', \qquad (2.35)$$

here

$$B=\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda_\mu\bar{\lambda}_\nu\frac{|f(z_\mu,\ z_\nu)|^{ls}}{|1-z_\mu\bar{z}_\nu|^l},\quad B'=\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda'_\mu\overline{\lambda'_\nu}\frac{|f(z'_\mu,\ z'_\nu)|^{ls}}{|1-z'_\mu\bar{z}'_\nu|^l}.$$

Apparently, (2.33), (2.34) are stronger than (2.31) and (2.32).

From (2.7) and (2.1), imitating the proof of Theorem 1, it is not difficult to prove the following

Theorem 3. *Suppose* $f\in S$, $\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}x_\mu\bar{x}_\nu\geqslant 0$ $(\bar{\alpha}_{\nu\mu}=\alpha_{\mu\nu},\ \mu,\ \nu=1,\ 2,\ \cdots,\ N)$, *$m$ is a natural number, p is a complex number, then for any integral function Φ and linear functionals* $L_1^{(\mu)}$, $L_2^{(\mu)}$ *and* $L^{(\mu)}\in c^*(D)$, *where D is an arbitrary closed subset of* $|z|<1$ *and* $\mu=1,\ 2,\ \cdots,\ N$, *we have*

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_1^{(\mu)}L_2^{(\nu)}(\Phi[(p\ln F(z,\ z'))^m])\right|^2$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_1^{(\mu)}\overline{L_1^{(\nu)}}(\Phi^+[(|p|\ln(1-z\bar{z}')^{-1})^m])$$
$$\times\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}L_2^{(\mu)}\overline{L_2^{(\nu)}}(\Phi^+[(|p|\ln(1-z\bar{z}')^{-1})^m]). \qquad (2.36)$$

Particularly, for $F(\zeta)\in\sum$, $|\zeta_\mu|>1$, $|\zeta'_\mu|>1$, *we have*

$$\left|\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda_\mu\lambda'_\nu\Phi([p\ln((F(\zeta_\mu)-F(\zeta'_\nu))/(\zeta_\mu-\zeta'_\nu))]^m)\right|^2$$
$$\leqslant\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda_\mu\bar{\lambda}_\nu\Phi^+([\ln(1-1/\zeta_\mu\zeta_\nu)^{-|p|}]^m)$$
$$\times\sum_{\mu,\nu=1}^{N}\alpha_{\mu\nu}\lambda'_\mu\lambda'_\nu\Phi^+([\ln(1-1/\zeta'_\mu\zeta'_\nu)^{-|p|}]^m). \qquad (2.37)$$

Apparently, (2.37) contains (1.1) and some results corresponding to (1.1) in [2, 8].

III. The Improvements of Bazilevic Inequality and Haynan's Regularity Theorem

Theorem 1. *Suppose $f\in S$, $\{\gamma_{kn}\}$ are the Grunsky coefficients of*

$$g(\zeta)=1/f(z)=\zeta+\sum_{n=0}^{\infty}b_n\zeta^{-n},$$

$z=1/\zeta$, $\theta_0=\theta_0(f)$ and α_f are the Hayman direction and Hayman constant of f respectively, $\ln(f(z)/z)=2\sum_{n=1}^{\infty}\gamma_n z^n$. If $\sum_{k=1}^{\infty}|\eta_k|^2/k<\infty$, then we have the sharp inequality

$$\left(\frac{1}{2}\ln\frac{1}{\alpha_f}-\sum_{n=1}^{\infty}n\left|\gamma_n-\frac{1}{n}e^{-in\theta_0}\right|^2\right)\left(\sum_{k=1}^{\infty}\frac{1}{k}|\eta_k|^2-\sum_{n=1}^{\infty}n\left|\sum_{k=1}^{\infty}\eta_k\gamma_{kn}\right|^2\right)$$
$$\geqslant\left|\sum_{k=1}^{\infty}\eta_k\left[\left(\frac{1}{k}e^{-ik\theta_0}-\gamma_k\right)+\sum_{n=1}^{\infty}n\overline{\left(\gamma_n-\frac{1}{n}e^{-jn\theta_0}\right)}\gamma_{kn}\right]\right|^2. \tag{3.1}$$

This improves the famous Bazilevic inequality[14] and Goluzin inequality. Particularly, it contains[6]

$$\left(\frac{1}{2}\ln\frac{1}{\alpha_f}-\sum_{n=1}^{\infty}n\left|\gamma_n-\frac{1}{n}e^{-n\theta_0}\right|^2\right)\left(1-\sum_{n=1}^{\infty}n|b_n|^2\right)$$
$$\geqslant\left|(\gamma_1-e^{-i\theta_0}+\sum_{n=1}^{\infty}n\overline{\left(\gamma_n-\frac{1}{n}e^{-in\theta_0}\right)}b_n\right|^2. \tag{3.2}$$

Proof From (2.4), it follows that for any $L\in C^*(|z|\leqslant\rho<1)$

$$|L|^2\left(\ln\frac{|F(z,\zeta)|^{l\varepsilon}}{|1-z\zeta|^l}-\frac{l}{2}\sum_{n=1}^{\infty}\frac{1}{n}g_n^{(\varepsilon)}(z)\overline{g_n^{(\varepsilon)}(\zeta)}\right)\geqslant 0. \tag{3.3}$$

Let $L(\varphi)=L'(\varphi)+x\varphi(z_0)$, where $\varphi\in C(|z|\leqslant\rho<1)$, $L'\in C^*(|z|\leqslant\rho<1)$, x is a real number, z_0 is an arbitrary fixed point in the unit disk. Assume $\varphi(z,\zeta)$ is a continuous function on $(|z|<1)\times(|\zeta|<1)$, which satisfies the condition

$$\overline{L'(\varphi(z,z_0))}=L'(\overline{\varphi(z_0,z)}).$$

Since $|L|^2\varphi(z,\zeta)\geqslant 0$,

$$|L'|^2\varphi(z,\zeta)+2x\mathrm{Re}[L'(\overline{\varphi(z_0,z)})]+x^2\overline{\varphi(z_0,z_0)}\geqslant 0. \tag{3.4}$$

Take $L=1$ (i. e. $L(\varphi)=\varphi$), It follows from $|L|^2\varphi(z,\zeta)\geqslant 0$ that $\varphi(z,\zeta)\geqslant 0$. So from (3.4), we obtain

$$\overline{\varphi(z_0,z)}\,|L'|^2(\varphi(z,\zeta))\geqslant(\mathrm{Re}[L'(\overline{\varphi(z_0,z)})])^2. \tag{3.5}$$

Substituting $\varphi(z,\zeta)=\ln(|F(z,\zeta)|^{l\varepsilon}/|1-z\zeta|^l)-\frac{l}{2}\sum_{n=1}^{\infty}\frac{1}{n}g_n^{(\varepsilon)}(z)\overline{g_n^{(\varepsilon)}(\zeta)}\Big)$ into (3.5), we have

$$\left(\ln\frac{|F(z_0,z_0)|^{l\varepsilon}}{(1-|z_0|^2)^l}-\frac{l}{2}\sum_{n=1}^{\infty}\frac{1}{n}|g_n^{(\varepsilon)}(z_0)|^2\right)$$
$$\times|L'|^2\left(\ln\frac{|F(z,\zeta)|^{l\varepsilon}}{|1-z\zeta|^l}-\frac{l}{2}\sum_{n=1}^{\infty}\frac{1}{n}g_n^{(\varepsilon)}(z)\overline{g_n^{(\varepsilon)}(\zeta)}\right)$$
$$\geqslant\left(\mathrm{Re}\left[L'\left(\ln\frac{|F(z_0,z)|^{l\varepsilon}}{|1-z_0\bar{z}|^l}-\frac{l}{2}\sum_{n=1}^{\infty}\frac{1}{n}\overline{g_n^{(\varepsilon)}(z_0)}g_n^{(\varepsilon)}(z)\right)\right]\right)^2. \tag{3.6}$$

Apparently, (3.6) is stronger than (2.7). In particular, taking $\varepsilon=1$, $l=2$, we have

$$\left(\ln\frac{|F(z_0,z_0)|^2}{(1-|z_0|^2)^2}-\sum_{n=1}^{\infty}\frac{1}{n}|g_n^{(1)}(z_0)|^2\right)\cdot|L'|^2\left(\ln\left|\frac{F(z,\zeta)}{1-z\bar\zeta}\right|^2-\sum_{n=1}^{\infty}\frac{1}{n}g_n^{(1)}(z)\overline{g_n^{(1)}(\zeta)}\right)$$
$$\geqslant\left(\operatorname{Re}\left[L'\left(\ln\left|\frac{F(z_0,z)}{1-z_0\bar z}\right|^2-\sum_{n=1}^{\infty}\frac{1}{n}\overline{g_n^{(1)}(z_0)}g_n^{(1)}(z)\right)\right]\right)^2. \tag{3.7}$$

Assume $z_0=re^{i\theta_0}$. As

$$0<\alpha_f=\lim_{r\to1}(1-r)^2|f(z_0)|/r,\quad \lim_{r\to1}1/f(z_0)=0,$$
$$\lim_{r\to1}g_n^{(1)}(z_0)=2n\left(\gamma_n-\frac{1}{n}e^{-in\theta_0}\right)^{[15]},\quad \left|\frac{zf'(z)}{f(z)}\right|\leqslant\frac{1+|z|}{1-|z|},$$

we deduce

$$\lim_{r\to1}\left(\ln\frac{|F(z_0,z_0|^2}{(1-r^2)^2}-\sum_{n=1}^{\infty}\frac{1}{n}|g_n^{(1)}(z_0)|^2\right)$$
$$\leqslant4\left(\frac{1}{2}\ln\frac{1}{\alpha_f}-\sum_{n=1}^{\infty}n\left|\gamma_n-\frac{1}{n}e^{-in\theta_0}\right|^2\right), \tag{3.8}$$
$$\lim_{r\to1}\left(\ln\left|\frac{F(z_0,z)}{1-z_0\bar z}\right|^2\right)=\ln\left|\frac{z}{f(z)(1-\bar z_0z)^2}\right|^2. \tag{3.9}$$

By the arbitrariness of z_0 and the continuity of L', it follows from (3.7), (3.8) and (3.9) that

$$\left(\frac{1}{2}\ln\frac{1}{\alpha_f}-\sum_{n=1}^{\infty}n\left|\gamma_n-\frac{1}{n}e^{-in\theta_0}\right|^2\right)|L'|^2\left(\ln\left|\frac{F(z,\zeta)}{1-z\bar\zeta}\right|^2-\sum_{n=1}^{\infty}\frac{1}{n}g_n^{(s)}(z)\overline{g_n^{(s)}(\zeta)}\right)$$
$$\geqslant\left(\operatorname{Re}\left[L'\left(\ln\left|\frac{z/f(z)}{(1-z_0\bar z)^2}\right|-\sum_{n=1}^{\infty}\overline{\left(\gamma_n-\frac{1}{n}e^{-in\theta_0}\right)}g_n^{(1)}(z)\right)\right]\right)^2. \tag{3.10}$$

Assume

$$L'(z^k)=\begin{cases}\eta_ke^{i\alpha}, & 1\leqslant k\leqslant N,\ 0\leqslant\alpha<+\infty,\\ 0, & k>N.\end{cases}$$

This can be done if we take $d\mu'(z)=e^{i\alpha}\sum_{p=1}^{N}\frac{p+1}{2\pi\rho^{p+1}}\eta_pe^{-ip\theta}drd\theta$, $z=re^{i\theta}$ in (2.12). Then

$$|L'|^2\left(\ln\left|\frac{F(z,\zeta)}{1-z\bar\zeta}\right|^2\right)=|L'|^2(\ln F(z,\zeta)+\overline{\ln F(z,\zeta)}-\ln(1-z\bar\zeta)(1-\bar z\zeta))$$
$$=\sum_{p,q=1}^{\infty}\bar\gamma_{pq}\overline{L'(z^p)}L'(\bar\zeta^q)+\sum_{p,q=1}^{\infty}\gamma_{pq}\overline{L'(\bar z^p)}L'(\zeta^q)+\sum_{p=1}^{\infty}\frac{1}{p}\overline{L'(z^p)}L'(\zeta^p)$$
$$+\sum_{p=1}^{\infty}\frac{1}{p}\overline{L'(\bar z^p)}L'(\bar\zeta^p)=\sum_{k=1}^{N}\frac{1}{k}|\eta_k|^2, \tag{3.11}$$
$$|L'|^2\left(\ln\left(\left|\frac{z}{f(z)}\right|\cdot|1-\bar z_0z|^{-2}\right)\right)=\sum_{p=1}^{\infty}L'\left(\frac{1}{p}((\bar z_0z)^p+(z_0\bar z)^p)-(\gamma_pz^p+\overline{\gamma_pz^p})\right)$$
$$=e^{i\alpha}\sum_{k=1}^{N}\left(\frac{1}{k}e^{-ik\theta_0}-\gamma_k\right)\eta_k. \tag{3.12}$$

$$L'(g_n^{(1)}(z))=-L'\left(n\sum_{p=1}^{\infty}\gamma_{pn}z^p+\bar z^n\right)=-ne^{i\alpha}\sum_{k=1}^{N}\eta_k\gamma_{kn},$$
$$|L'|^2\left(\sum_{n=1}^{\infty}\frac{1}{n}g_n^{(1)}(z)g_n^{(1)}(\zeta)\right)=\sum_{n=1}^{\infty}\frac{1}{n}|L'(g_n^{(1)}(z))|^2=\sum_{n=1}^{\infty}n\left|\sum_{k=1}^{N}\eta_k\gamma_{kn}\right|^2.$$

From (3.10) and the arbitrariness of α, we obtain

$$\left(\frac{1}{2}\ln\frac{1}{\alpha_f}-\sum_{n=1}^{\infty}n\left|\gamma_n-\frac{1}{n}e^{-in\theta_0}\right|^2\right)\left(\sum_{k=1}^{N}\frac{1}{k}|\eta_k|^2-\sum_{n=1}^{\infty}n\left|\sum_{k=1}^{N}\eta_k\gamma_k\right|^2\right)$$
$$\geqslant\sum_{k=1}^{N}\eta_k\left[\left(\frac{1}{k}e^{-ik\theta_0}-\gamma_k\right)+\sum_{n=1}^{\infty}n\overline{\left(\gamma_n-\frac{1}{n}e^{-in\theta_0}\right)}\gamma_{kn}\right]\Bigg|^2. \qquad (3.13)$$

Therefore (3.1) is true. Furthermore, when $\alpha_f=1$, it follows from

$$\sum_{n=1}^{\infty}n\left|\gamma_n-\frac{1}{n}e^{-in\theta_0}\right|^2\leqslant\frac{1}{2}\ln\frac{1}{\alpha_f}$$

that

$$\gamma_n=\frac{1}{n}e^{-in\theta_0}\quad(n=1,2,3,\cdots).$$

So the equality of (3.13) holds.

Finally, note that $\gamma_{1n}=-b_n$, so it is obvious that (3.1) contain (3.2). This completes the proof.

Theorem 2. *Suppose $f\in S$, $\theta_0(f)$ and α_f are the Hayman direction and Hayman constant of f respectively, $\{\gamma_{kn}\}$ are the Grunsky coefficients of f, $\ln(f(z)/z)=2\sum_{n=1}^{\infty}\gamma_n z^n$, then*

$$\left(\frac{1}{2}\ln\frac{1}{\alpha_f}-\frac{2n\left[\mathrm{Re}\left(\gamma_n-\frac{1}{n}e^{-in\theta_0}\right)\right]^2}{1+n\mathrm{Re}\gamma_{nn}}\right)\left(\sum_{k=1}^{N}\frac{1}{k}|\eta_k|^2-\frac{n\left|\sum_{k=1}^{N}\eta_k\gamma_{kn}+\frac{1}{n}\eta_n\right|^2}{4(1+n\mathrm{Re}\gamma_{nn})}\right)$$
$$\geqslant\left|\sum_{k=1}^{N}\eta_k\left(\frac{1}{k}e^{-ik\theta_0}-\gamma_k\right)+\frac{n\left(\sum_{k=1}^{N}\eta_k\gamma_{kn}+\frac{1}{n}\eta_n\right)\cdot\mathrm{Re}\left(\gamma_n-\frac{1}{n}e^{-in\theta_0}\right)}{1+n\mathrm{Re}\gamma_{nn}}\right|^2, \qquad (3.14)$$

$$\left(\frac{1}{2}\ln\frac{1}{\alpha_f}-\frac{2n\left[\mathrm{Im}\left(\gamma_n-\frac{1}{n}e^{-in\theta_0}\right)\right]^2}{1-n\mathrm{Re}\gamma_{nn}}\right)\left(\sum_{k=1}^{N}\frac{1}{k}|\eta_k|^2-\frac{n\left|\sum_{k=1}^{N}\eta_k\gamma_{kn}-\frac{1}{n}\eta_n\right|^2}{4(1-n\mathrm{Re}\gamma_{nn})}\right)$$
$$\geqslant\left|\sum_{k=1}^{N}\eta_k\left(\frac{1}{k}e^{-ik\theta_0}-\gamma_k\right)-i\frac{n\left(\sum_{k=1}^{N}\eta_k\gamma_{kn}-\frac{1}{n}\eta_n\right)\mathrm{Im}\left(\gamma_n-\frac{1}{n}e^{-in\theta_0}\right)}{1-n\mathrm{Re}\gamma_{nn}}\right|^2, \qquad (3.15)$$

$$1-\alpha_f\exp\left\{\left|\frac{4n\,\mathrm{Re}\left(\gamma_n-\frac{1}{n}e^{-in\theta_0}\right)\mathrm{Im}\left(\gamma_n-\frac{1}{n}e^{-in\theta_0}\right)}{\sqrt{1-(n\mathrm{Re}\gamma_{nn})^2}}\right|\right\}\geqslant 0, \qquad (3.16)$$

where $\eta_n=0$ when $n>N$. Especially, when $n=1$, (3.14), (3.15) *improve the corresponding results of* [9].

Proof Assume $\overline{L_2}=L_1=L$, $m=1$ in (2.17) and (2.18), then

$$|L|^2\left(\ln\frac{|F(z,\zeta)|^{ls}}{|1-z\bar{\zeta}|^l}-\frac{l}{n(1+ns\mathrm{Re}\gamma_{nn})}\mathrm{Re}\,g_n^{(s)}(z)\mathrm{Re}\,g_n^{(s)}(\zeta)\right)\geqslant 0, \qquad (3.17)$$

$$|L|^2\left(\ln\frac{|F(z,\zeta)|^{ls}}{|1-z\bar{\zeta}|^l}-\frac{l}{n(1-ns\mathrm{Re}\gamma_{nn})}\mathrm{Im}\,g_n^{(s)}(z)\mathrm{Im}\,g_n^{(s)}(\zeta)\right)\geqslant 0. \qquad (3.18)$$

Imitating the proof of Theorem 1, from (3.17) and (3.18), it is not difficult to prove

$$\left(\frac{1}{2}\ln\frac{1}{\alpha_f}-\frac{2n\left[\operatorname{Re}\left(\gamma_n-\frac{1}{n}e^{in\theta_0}\right)\right]^2}{1+n\operatorname{Re}\gamma_{nn}}\right)|L'|^2\left(\ln\frac{|F(z,\zeta)|^2}{|1-z\bar{\zeta}|^2}-\frac{2\operatorname{Re}g_n^{(1)}(z)\operatorname{Re}g_n^{(1)}(\zeta)}{n(1+n\operatorname{Re}\gamma_{nn})}\right)$$
$$\geqslant\left(\operatorname{Re}\left[L'\left(\ln\frac{|z/f(z)|}{|1-\bar{z}_0z|^2}-\frac{\operatorname{Re}g_n^{(1)}(z_0)\operatorname{Re}g_n^{(1)}(z)}{n(1+n\operatorname{Re}\gamma_{nn})}\right)\right]\right)^2, \tag{3.19}$$

$$\left(\frac{1}{2}\ln\frac{1}{\alpha_f}-\frac{2n\left[\operatorname{Im}\left(\gamma_n-\frac{1}{n}e^{-in\theta_0}\right)\right]^2}{1-n\operatorname{Re}\gamma_{nn}}\right)|L'|^2\left(\ln\left|\frac{F(z,\zeta)}{1-z\zeta}\right|^2-\frac{2\operatorname{Im}g_n^{(1)}(z)\operatorname{Im}g_n^{(1)}(\zeta)}{n(1-n\operatorname{Re}\gamma_{nn})}\right)$$
$$\geqslant\left(\operatorname{Re}\left[L'\left(\ln\frac{|z/f(z)|}{|1-z\bar{z}_0|^2}-\frac{\operatorname{Im}g_n^{(1)}(z_0)\operatorname{Im}g_n^{(1)}(z)}{n(1-n\operatorname{Re}\gamma_{nn})}\right)\right]\right)^2, \tag{3.20}$$

where $z_0=e^{i\theta_0}$. Assume

$$L'(z^k)=\begin{cases}\eta_k e^{i\alpha}, & 1\leqslant k\leqslant N,\ 0\leqslant\alpha<+\infty,\\ 0, & k>N,\end{cases}$$

it is easy to obtain

$$L'(\operatorname{Re}g_n^{(1)}(z))=\frac{1}{2}L'(g_n^{(1)}(z)+\overline{g_n^{(1)}(z)})=-\frac{n}{2}e^{i\alpha}\left(\sum_{k=1}^{N}\eta_k\gamma_{kn}+\frac{1}{n}\eta_n\right),$$
$$L'(\operatorname{Im}g_n^{(1)}(z))=\frac{1}{2i}L'(g_n^{(1)}(z)-\overline{g_n^{(1)}(z)})=-\frac{n}{2i}e^{i\alpha}\left(\sum_{k=1}^{N}\eta_k\gamma_{kn}-\frac{1}{n}\eta_n\right),$$

where $\eta_n=0$ when $n>N$. Then (3.14) and (3.15) can be easily deduced by substituting (3.11), (3.12) and the preceding two expressions into (3.19) and (3.20).

Finally, assume $L_2(\varphi)=L(\varphi)=\varphi(z_0)\ |z_0|<1$, from (2.19), we have

$$\left|\frac{\operatorname{Re}g_n^{(1)}(z_0)\cdot\operatorname{Im}g_n^{(1)}(z_0)}{n\sqrt{1-(n\operatorname{Re}\gamma_{nn})^2}}\right|\leqslant\ln\left|\frac{z_0^2f'(z_0)}{(1-|z_0|^2)f(z_0)^2}\right|.$$

Set $z_0=re^{i\theta_0}$, $\theta_0=\theta_0(f)$ and let $r\to1$, then (3.16) follows. Thus, Theorem 2 is proved.

IV. The Improvements of Fitz Gerald's Coefficients

Theorem 1. *Suppose* $f(z)=z+a_2z^2+a_3z^3+\cdots\in S$, $\{\gamma_{pq}\}(p,q=1,2,\cdots)$ *are the Grunsky coefficients of* $g(1/z)=f^{-1}(z)$, $\ln(f(z)/z)=2\sum_{n=1}^{\infty}\gamma_nz^n$, *then*

$$\sum_{p,q=2}^{N}a_{pq}\bar{x}_px_q\geqslant\sum_{t=1}^{\infty}\frac{1}{t!}\left(\frac{2}{n(1+n\operatorname{Re}\gamma_{nn})}\right)^t\left|\sum_{p=2}^{N}D_{p,n}^{(t)}x_p\right|^2$$
$$+P_n\left|\sum_{p=2}^{N}\left[p^2-|a_p|^2-\sum_{t=1}^{\infty}\frac{1}{t!}\left(\frac{4\operatorname{Re}\left(\gamma_n-\frac{1}{n}e^{-in\theta_0}\right)}{1+n\operatorname{Re}\gamma_{nn}}\right)^tD_{p,n}^{(t)}\right]x_p\right|^2, \tag{4.1}$$

$$\sum_{p,q=2}^{N}a_{pq}\bar{x}_px_q\geqslant\sum_{t=1}^{\infty}\frac{1}{t!}\left(\frac{2}{n(1-n\operatorname{Re}\gamma_{nn})}\right)^t\left|\sum_{p=2}^{N}E_{p,n}^{(t)}x_p\right|^2$$
$$+Q_n\left|\sum_{p=2}^{N}\left[p^2-|a_p|^2-\sum_{t=1}^{\infty}\frac{1}{t!}\left(\frac{4\operatorname{Im}\left(\gamma_n-\frac{1}{n}e^{-in\theta_0}\right)}{1-n\operatorname{Re}\gamma_{nn}}\right)^tE_{p,n}^{(t)}\right]x_p\right|^2, \tag{4.2}$$

where a_{pq}, $\beta_k(p, q)$ *are defined as in* (1.3), (1.4), *and besides, we define*

$$G_n(z)=F_n\left(\frac{1}{f(z)}\right)-\frac{1}{z^n}-z^n,\quad [G_n(z)]^t=\sum_{p=t}^{\infty}G_{p,n}^{(t)}z^p,$$

$$H_n(z)=F_n\left(\frac{1}{f(z)}\right)-\frac{1}{z^n}+z^n,\quad [H_n(z)]^t=\sum_{p=t}^{\infty}H_{p,n}^{(t)}z^p,$$

$$D_{p,n}^{(t)}=2^{-t}\sum_{r=0}^{t}\binom{t}{r}\sum_{k=1}^{N-t+r}a_kG_{p-k,n}^{(t-r)}\cdot\sum_{k=1}^{N-r}\overline{a_{k'}G_{p-k',n}^{(r)}},$$

$$E_{p,n}^{(t)}=(2i)^{-t}\sum_{r=0}^{t}\binom{t}{r}\sum_{k=1}^{N-t+r}a_kH_{p-k,n}^{(t-r)}\sum_{k'=1}^{N-r}\overline{a_{k'}H_{p-k',n}^{(r)}}$$

$$P_n=\alpha_f^2\left(1-\alpha_f^2\exp\left\{\frac{8n\left[\operatorname{Re}\left(\gamma_n-\frac{1}{n}e^{-in\theta_0}\right)\right]^2}{1+n\operatorname{Re}\gamma_{nn}}\right\}\right)^{-1},$$

$$Q_n=\alpha_f^2\left(1-\alpha_f^2\exp\left\{\frac{8n\left[\operatorname{Im}\left(\gamma_n-\frac{1}{n}e^{-in\theta_0}\right)\right]^2}{1-n\operatorname{Re}\gamma_{nn}}\right)\right\}^{-1}.$$

Particularly, we hava[7]

$$\sum_{p,q=2}^{N}a_{pq}\bar{x}_px_q\geqslant\frac{\alpha_f^2}{1-\alpha_f^2}\left|\sum_{p=1}^{N}(p^2-|a_p|^2)x_p\right|^2. \tag{4.3}$$

(4.1) and (4.2) extend and improve the corresponding results of [4, 9].

Proof From (2.20) and (2.21), we have

$$|L|^2\left(\left(\left|\frac{f(z)-f(\zeta)}{z-\zeta}\right|^2\cdot|1-z\bar{\zeta}|^{-2}-\left|\frac{f(z)f(\zeta)}{z\zeta}\right|^2\exp\left\{\frac{2\operatorname{Re}g_n^{(1)}(z)\operatorname{Re}g_n^{(1)}(\zeta)}{n(1-n\operatorname{Re}\gamma_{nn})}\right\}\right)\geqslant0, \tag{4.4}$$

$$|L|^2\left(\left|\frac{f(z)-f(\zeta)}{z-\zeta}\right|^2\cdot|1-z\bar{\zeta}|^{-2}-\left|\frac{f(z)f(\zeta)}{z\zeta}\right|^2\exp\left\{\frac{2\operatorname{Im}g_n^{(1)}(z)\operatorname{Im}g_n^{(1)}(\zeta)}{n(1-n\operatorname{Re}\gamma_{nn})}\right\}\right)\geqslant0. \tag{4.5}$$

Imitating the proof of Theorem 1 in section 3 we can deduce

$$|L'|^2\left(\left|\frac{f(z)-f(\zeta)}{z-\zeta}\right|^2\Big/|1-z\bar{\zeta}|^2-\left|\frac{f(z)f(\zeta)}{z\zeta}\right|^2\exp\left\{\frac{2\operatorname{Re}g_n^{(1)}(z)\operatorname{Re}g_n^{(1)}(\zeta)}{n(1+n\operatorname{Re}\gamma_{nn})}\right\}\right)$$
$$\geqslant P_n\left(\operatorname{Re}\left[L'\left(|1-\bar{z}_0z|^{-4}-\left|\frac{f(z)}{z}\right|^2\exp\left\{\frac{2\operatorname{Re}g_n^{(1)}(z_0)\operatorname{Re}g_n^{(1)}(z)}{n(1+n\operatorname{Re}\gamma_{nn})}\right\}\right)\right]\right)^2, \tag{4.6}$$

$$|L'|^2\left(\left|\frac{f(z)-f(\zeta)}{z-\zeta}\right|^2|1-z\bar{\zeta}|^{-2}-\left|\frac{f(z)f(\zeta)}{z\zeta}\right|^2\exp\left\{\frac{2\operatorname{Im}g_n^{(1)}(z)\operatorname{Im}g_n^{(1)}(\zeta)}{n(1-n\operatorname{Re}\gamma_{nn})}\right\}\right)$$
$$\geqslant Q_n\left(\operatorname{Re}\left[L'\left(|1-\bar{z}_0z|^{-4}-\left|\frac{f(z)}{z}\right|^2\exp\left\{\frac{2\operatorname{Im}g_n^{(1)}(z_0)\operatorname{Im}g_n^{(1)}(z)}{n(1-n\operatorname{Re}\gamma_{nn})}\right\}\right)\right]\right)^2, \tag{4.7}$$

here $z_0=e^{i\theta_0}$, $g_n^{(1)}(z_0)=2n\left(\gamma_n-\frac{1}{n}e^{-in\theta_0}\right)$.

It is not difficult to know that for any complex number sequence $x_1, x_2, \cdots, x_N$ and real number α, there exists

$$L'(z^{k-1}\bar{z}^{k'-1})=\begin{cases}x_ke^{i\alpha}, & 1\leqslant k'=k\leqslant N,\\ 0, & k'\neq k,\ k'=k>N.\end{cases}$$

So after some calculation, we have

$$L'(|1-\bar{z}_0 z|^{-4})=e^{i\alpha}\sum_{k=1}^{N}k^2 x_k, \tag{4.8}$$

$$L'\left(\left|\frac{f(z)}{z}\right|^2\right)=e^{i\alpha}\sum_{k=1}^{N}|a_k|^2 x_k, \tag{4.9}$$

$$|L'|^2\left(\left|\frac{f(z)-f(\zeta)}{(z-\zeta)(1-z\zeta)}\right|^2\right)$$
$$=\sum_{k,k'=1}^{\infty}\sum_{p,p'=0}^{\infty}\bar{a}_k a_{k'}\sum_{t=1}^{k}\sum_{t'=1}^{k'}\overline{L'(z^{k-t+p}\bar{z}^{k'-t'+p'})}L'(\zeta^{t'+p-1}\bar{\zeta}^{t+p'-1}).$$

The terms on the right hand side of the above expression don't vanish and $L'(z^{m-1}\bar{z}^{m-1})=x_m e^{i\alpha}$, $L'(\zeta^{n-1}\bar{\zeta}^{n-1})=x_n e^{i\alpha}$ if and only if

$$m=k-t+p+1=k'-t'+p'+1,\ n=t+p'=t'+p. \tag{*}$$

We can rewrite (*) as

$$k=k',\ p'=n-t,\ p'=m-k+t-1,\ t'=(n-m)+k-t+1,$$

then it follows that

$$|L'|^2\left(\left|\frac{f(z)-f(\zeta)}{(z-\zeta)(1-z\bar{\zeta})}\right|^2\right)=\sum_{m,n=1}^{N}\left(\sum_{k=1}^{n+m-1}K(k)\,|a_k|^2\right)\bar{x}_m x_n.$$

when $1\leqslant k\leqslant m-n$, as $1\leqslant t'\leqslant -t+1\leqslant 0$, this is impossible, so $K(k)=0$; when $m-n<k\leqslant m$, as $1\leqslant t'\leqslant n-m+k$, $K(k)=k+n-m$; when $m<k\leqslant n-m$, as $n\geqslant t'\geqslant 1+k-m$, $K(k)=n+m-k$. Therefore, when $n\leqslant m$

$$K(k)=\beta_k(n,\ m)=\begin{cases} n-|k-m|, & |k-m|<n,\\ 0, & \text{otherwise.}\end{cases}$$

Hence

$$|L'|^2\left(\left|\frac{f(z)-f(\zeta)}{(z-\zeta)(1-\bar{\zeta}z)}\right|^2\right)=\sum_{m,n=1}^{N}\left(\sum_{k=1}^{n+m-1}\beta_k(n,\ m)\,|a_k|^2\right)\bar{x}_m x_n, \tag{4.10}$$

$$|L'|^2\left(\left|\frac{f(z)f(\zeta)}{z\zeta}\right|^2\right)=\left|L'\left(\left|\frac{f(z)}{z}\right|^2\right)\right|^2=\left|\sum_{k=1}^{N}|a_k|^2 x_k\right|^2. \tag{4.11}$$

On the other hand, as

$$L'\left(\left|\frac{f(z)}{z}\right|^2[\operatorname{Re}g_n^{(1)}(z)]^t\right)=L'\left(\left|\frac{f(z)}{z}\right|^2[\operatorname{Re}G_n(z)]^t\right)$$
$$=\sum_{r=0}^{t}2^{-t}\binom{t}{r}\sum_{k,k'=1}^{\infty}\sum_{p=t-r}^{\infty}\sum_{p'=r}^{\infty}a_k\overline{a_{k'}}\,G_{p,n}^{(t-r)}\overline{G_{p',n}^{(r)}}\,L'(z^{k-1+p}\bar{z}^{k'-1+p'})=\sum_{m=2}^{N}D_{m,n}^{(t)}x_m e^{i\alpha},$$

we have

$$|L'|^2\left(\left|\frac{f(z)f(\zeta)}{z\zeta}\right|^2\exp\left\{\frac{2\operatorname{Re}g_n^{(1)}(z)\operatorname{Re}g_n^{(1)}(\zeta)}{n(1+n\operatorname{Re}\gamma_{nn})}\right\}\right)$$
$$=\sum_{t=0}^{\infty}\frac{1}{t!}\left(\frac{2}{n(1+n\operatorname{Re}\gamma_{nn})}\right)^t\left|L'\left(\left|\frac{f(z)}{z}\right|^2[\operatorname{Re}g_n^{(1)}(z)]^t\right)\right|$$
$$=\left|\sum_{k=1}^{N}|a_k|^2 x_k\right|^2+\sum_{t=1}^{\infty}\frac{1}{t!}\left(\frac{2}{n(1+n\operatorname{Re}\gamma_{nn})}\right)^t\left|\sum_{k=2}^{N}D_{k,n}^{(t)}x_k\right|^2, \tag{4.12}$$

$$L'\left(\left|\frac{f(z)}{z}\right|^2\exp\left\{\frac{2\operatorname{Re}g_n^{(1)}(z_0)\operatorname{Re}g_n^{(1)}(z)}{n(1+n\operatorname{Re}\gamma_{nn})}\right\}\right)$$
$$=\sum_{t=0}^{\infty}\frac{1}{t!}\left(\frac{2\operatorname{Re}g_n^{(1)}(z_0)}{n(1+n\operatorname{Re}\gamma_{nn})}\right)^t L'\left(\left|\frac{f(z)}{z}\right|^2[\operatorname{Re}g_n^{(1)}(z)]^t\right)$$
$$=e^{i\alpha}\left(\sum_{k=1}^{N}|a_k|^2 x_k+\sum_{t=0}^{\infty}\frac{1}{t!}\left(\frac{2\operatorname{Re}g_n^{(1)}(z_0)}{n(1+n\operatorname{Re}\gamma_{nn})}\right)^t\sum_{k=2}^{N}D_{kn}^{(t)}x_k\right). \tag{4.13}$$

Then (4.1) follows by substituting (4.8), (4.10), (4.12) and (4.13) into (4.6).

The existence of L' Still remains to be proved. For this purpose, we prove that for any $k(1\leqslant k\leqslant N)$, there exists $L_k\in C^*(|z|\leqslant\rho<1)$, such that

$$L_k(z^{p-1}\bar{z}^{q-1})=\begin{cases}1, & \text{if } p=k,\ q=k,\\ 0, & \text{otherwise.}\end{cases} \tag{4.14}$$

Assume that G_k is the linear subspace generated by $z^{p-1}\bar{z}^{q-1}(|p-k|+|q-k|=0)$, and $x_0=|z|^{2(k-1)}$, then

$$d(x_0,\ G_k)>0.$$

If it is not true, then there exists

$$\sum_{p,q=1}^{\infty}C_{pq}z^{p-1}\bar{z}^{q-1}\quad(C_{kk}=0),$$

which converges to x_0 uniformly in $|z|\leqslant\rho\leqslant1$, i. e.

$$x_0=\sum_{p,q=1}^{\infty}C_{pq}z^{p-1}\bar{z}^{q-1}.$$

Set $z=re^{i\theta}$ $(0\leqslant\theta<2\pi)$, and act $L(\varphi)=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\varphi(\theta)d\theta$ on both sides of the above expression, then it follows that

$$r^{2(k-1)}=\sum_{p=1}^{\infty}C_{pq}r^{2(p-1)}\quad(0\leqslant r\leqslant\rho).$$

Therefore we deduce $1=0$, this is impossible. So $d(x_0,\ G_k)>0$.

By the continuation theorem of functionals[16], there must exists linear functional L_k which satisfies condition (4.14). Then

$$L'=e^{i\alpha}\sum_{k=1}^{N}x_kL_k\in C^*(|z|\leqslant\rho<1)$$

is the linear functional we seek for. Therefore, (4.1) is true.

Imitating the proof of (4.12) and (4.13), we have

$$|L'|^2\left(\left|\frac{f(z)f(\zeta)}{z\zeta}\right|^2\exp\left\{\frac{2\operatorname{Im}g_n^{(1)}(z)\operatorname{Im}g_n^{(1)}(\zeta)}{n(1-n\operatorname{Re}\gamma_{nn})}\right\}\right)$$
$$=\left|\sum_{k=1}^{N}|a_k|^2x_k\right|^2+\sum_{t=1}^{\infty}\frac{1}{t!}\left(\frac{2}{n(1-n\operatorname{Re}\gamma_{nn})}\right)^t\left|\sum_{k=2}^{N}E_{k,n}^{(t)}x_k\right|^2, \tag{4.15}$$

$$L'\left(\left|\frac{f(z)}{z}\right|^2\exp\left\{\frac{2\operatorname{Im}g_n^{(1)}(z)\operatorname{Im}g_n^{(1)}(\zeta)}{n(1-\operatorname{Re}\gamma_{nn})}\right\}\right)$$
$$=e^{i\alpha}\left(\sum_{k=1}^{N}|a_k|^2x_k+\sum_{t=1}^{\infty}\frac{1}{t!}\left(\frac{2\operatorname{Im}g_n^{(1)}(z_0)}{n(1-n\operatorname{Re}\gamma_{nn})}\right)^t\sum_{k=2}^{N}E_{k,n}^{(t)}x_k\right). \tag{4.16}$$

Substitute (4.8), (4.10), (4.15) and (4.16) into (4.7), and note that α is arbitrary, then (4.2) can be easily obtained.

Finally, from (4.4) it follows that

$$|L|^2\left(\left|\frac{f(z)-f(\zeta)}{(z-\zeta)(1-z\bar{\zeta})}\right|^2-\left|\frac{f(z)f(\zeta)}{z\zeta}\right|^2\right)\geqslant0,$$

hence

$$|L'|^2\left(\left|\frac{f(z)-f(\zeta)}{(z-\zeta)(1-z\bar{\zeta})}\right|^2-\left|\frac{f(z)f(\zeta)}{z\zeta}\right|^2\right)\geqslant\frac{\alpha_f^2}{1-\alpha_f^2}\left(\operatorname{Re}\left[L'\left(|1-\bar{z}_0z|^{-4}-\left|\frac{f(z)}{z}\right|^2\right)\right]\right)^2. \tag{4.17}$$

Substitute (4.8), (4.9), (4.10) and (4.11) into (4.17), then (4.3) follows by the arbitrariness of α. Thus Theorem 1 is proved.

V. The Grunsky Inequalities of The Inveses of Meromorphic Univalent Functions

Theorem 1. *Suppose* $F(\zeta)=\zeta+b_1\zeta^{-1}+b_2\zeta^{-2}+\cdots\in\Sigma'$, $G(w)=F^{-1}(w)=w+B_1w^{-1}+B_2w^{-2}+\cdots$

$$\log\frac{G(w)-G(\zeta)}{w-\xi}=\sum_{p,q=1}^{\infty}\lambda_{p,q}w^{-p}\xi^{-q}. \tag{5.1}$$

Then we have

$$\left|\sum_{m,n=1}^{\infty}\lambda_{mn}x_mx_n\right|\leqslant\sum_{k=1}^{\infty}\frac{1}{n}\left|x_n+\sum_{m=n+2}^{\infty}B_m^{(-n)}x_m\right|^2, \tag{5.2}$$

where $B_n^{(-k)}$ *is defined as follows*

$$[G(w)]^{-k}=w^{-k}+\sum_{n=k+2}^{\infty}B_n^{(-k)}w^{-n}. \tag{5.3}$$

Proof By Theorem 3 of section 2, for $F\in\Sigma$, $L_\mu\in C^*(1<r\leqslant|\zeta|\leqslant R<\infty)$, we have

$$\left|\sum_{\mu,\nu=1}^{N}L_\mu L_\nu\ln\left(\frac{z-\zeta}{F(z)-F(\zeta)}\right)\right|\leqslant\sum_{\mu,\nu=1}^{N}L_\mu\overline{L}_\nu(-\ln(1-1/z\overline{\zeta})).$$

For $L'_\mu\in C^*(F(r\leqslant|\zeta|\leqslant R)$, therefore we have

$$\left|\sum_{\mu,\nu=1}^{N}L'_\mu L'_\nu\ln\left(\frac{G(w)-G(\xi)}{w-\xi}\right)\right|\leqslant B, \tag{5.4}$$

here

$$B=\sum_{k=1}^{\infty}\frac{1}{k}\sum_{\mu,\nu=1}^{N}L'_\mu\overline{L}'_\nu([G(w)\overline{G(\xi)}]^{-k}. \tag{5.5}$$

Suppose that G_μ is the linear subspace generated by $\{w^{-n}\}$ $(n\neq\mu,\ n=1,2,3,\cdots)$, then $d=d(w^{-\mu},G_\mu)>0$. If it is not true, then there exists a power series $\sum\limits_{n=1}^{\infty}C'_nw^{-n}$ $(C'_\mu=0)$ which converges to w^{-n} uniformly in $F(r\leqslant|\zeta|\leqslant R)$, i. e.

$$w^{-\mu}=\sum_{n=1}^{\infty}C'_nw^{-n}\quad(C'_\mu=0). \tag{5.6}$$

Assume $C_\rho=F(|\zeta|=\rho)$ $(r<\rho<R)$, $L(\varphi)=\frac{1}{2\pi}\oint_{C_\rho}w^\mu\varphi(w)dw$. Acting on both sides of (5.6) with operator $L(\varphi)$, and using Cauchy formula, we deduce $1=0$, this is impossible, so $d>0$.

Then, by the continuation theorem of functionals, we know that there exists $L'_\mu\in C^*(r\leqslant|\zeta|\leqslant R)$, such that

$$L'_\mu(w^{-n})=\begin{cases}x_\mu, & n=\mu,\\ 0, & n\neq\mu.\end{cases} \tag{5.7}$$

Substituting (5.7), (5.1) and (5.3) into (5.4), (5.5), after some simple calculation, we can easily obtain (5.2). This completes the proof of Theorem 1.

Corollary. *Suppose* $F(\zeta)=\zeta+b_1\zeta^{-1}+b_2\zeta^{-2}+\cdots\in\sum'$, $\zeta=F^{-1}(w)=w+B_1w^{-1}+B_2w^{-2}+\cdots$, $\{\gamma_{p,q}\}$ $(p, q=1, 2, 3, \cdots)$ *are the Grunsky coefficients of* $G(w)$, *then*

$$\left|a^2\left(B_3+\frac{1}{2}B_1^2\right)+2abB_2+b^2B_1\right|\leqslant|b|^2+\frac{1}{2}|a|^2, \tag{5.8}$$

$$\left|a^2\left(B_5+B_1B_2+B_2^2+\frac{1}{3}B_1^3+\right)2ab(B_4+B_1B_2)+b^2\left(B_3+\frac{1}{2}B_1^2\right)+C^2B_1\right.$$
$$\left.+2(acB_3+bcB_2)\right|\leqslant\left|c+aB_3^{(-1)}\right|^2+\frac{1}{2}|b|^2+\frac{1}{3}|a|^2. \tag{5.9}$$

We are convinced that these Grunsky inequalities will be of significance to the proof of Springer conjecture.

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